Abstract

Bistable-fronts in discrete inhomogeneous media

Tony Humphries
Dept. of Mathematics and Statistics
McGill University

Brian E. Moore
Department of Mathematics
University of Central Florida

Erik S. Van Vleck
Department of Mathematics
University of Kansas

Bistable differential-difference equations with inhomogeneous diffusion are considered using McKean’s curvature of the cubic. Front solutions are constructed for essentially arbitrary inhomogeneous discrete diffusion and these solutions correspond, in the case of homogeneous diffusion, to monotone traveling front solutions or stationary front solutions in the case of propagation failure. Explicit conditions reveal relationships between zero wave speed and defects in the medium, and changes in wave speed and shape are analyzed as fronts propagate.


We seek solutions of \( u_t(t) = Lu_t(t) - f(u_t(t)) \).

\[ u_j(0) \text{ maps } \mathbb{R}^+ \times [0, T] \text{ into } \mathbb{R}, \quad f \in \mathbb{R} \]

\[ L_u(t) = u_j(t) + a_j u_{j-1}(t) - u_j(t) - a_j u_{j+1}(t) - a_j u_{j+1}(t) \]

and using \( a_j \in \mathbb{R} \) and \( a_j \in \mathbb{R} \) for \( -m \leq j \leq m \), the inhomogeneous medium is defined by

\[ f = \begin{cases} \alpha_j, & \text{for } j < m \text{ or } j > m, \\ \alpha_j, & \text{for } -m \leq j \leq m. \end{cases} \]

\[ f : R \rightarrow R \] is the derivative of a double-well potential. We use a piecewise linear \( f \) known as McKean’s curvature of the cubic

\[ f(u) = u - h(u-a) - h(u-a), \]

where \( h \) is the Heaviside function.

\( h(u) = \begin{cases} 0, & u < 0, \\ 1, & u > 0. \end{cases} \)

Problem Set-Up for Propagating Fronts

Make the traveling wave ansatz

\[ \phi_j(t) = \phi(x_j(t), \xi), \quad x_j(t) = j - a_j(t). \] (1)

We choose a particular wave form by setting

\[ a = \phi(x_j(t)). \] (2)

so that \( \xi \) is the spatial location of which \( \phi \) takes the value \( a \), and we seek solutions that satisfy

\[ \phi(x_j(t)) < a \quad \text{for } x_j < \xi, \]

\[ \phi(x_j(t)) > a \quad \text{for } x_j > \xi. \] (3)

Thus, the nonlinearity may be written as a linear inhomogeneous term

\[ f(\phi(x_j(t))) = \phi(x_j(t)) - h(a(x_j(t))). \] (4)

which is independent of \( a \), and it is natural to impose the boundary conditions

\[ \lim_{x \to +\infty} \phi(x, \xi) = 0, \quad \lim_{x \to -\infty} \phi(x, \xi) = 1. \] (5)

Solutions with Zero Wave Speed

Using a Fourier transform to solve

\[ -\partial_x \phi(x, \xi) = a_j \phi(x_{j-1}, \xi) - \phi(x, \xi) + \phi(x_{j+1}, \xi) - \phi(x, \xi) + b(\xi) - \phi(\xi) \] (6)

with \( \phi_j(t) = \phi_j(t) \), we obtain the solution

\[ \phi(x, \xi) = \psi(x, \xi) + \chi(x, \xi) \] (9)

\[ \chi(x, \xi) \] is the solution in the case \( a_j = a_j \), given by

\[ \chi(x, \xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(s) e^{-i s \xi} e^{-i s x} ds, \] (10)

where \( A(s) = 1 + 2 s \cos(s) \), and \( s \) is the wave speed in homogeneous media

\[ \phi(x, \xi) \] is the perturbation from \( \psi \) when \( a_j \neq a_j \). Defining \( \beta_j = \partial_s \phi(x_j, \xi) \), \( \gamma_j = \partial_s \phi(x_{j+1}, \xi) \), and \( \delta_j = \partial_s \phi(x_{j-1}, \xi) \)

\[ \beta_j = \psi(x_j, \xi) + \chi(x_j, \xi) + \chi(x_{j+1}, \xi) - \phi(x_{j+1}, \xi) - \phi(x_j, \xi) - \phi(x_j, \xi) + \phi(x_{j-1}, \xi) + b(\xi) - \psi(x_{j-1}, \xi) \]

\[ \gamma_j = \psi(x_{j+1}, \xi) + \chi(x_{j+1}, \xi) + \chi(x_j, \xi) - \phi(x_{j+1}, \xi) - \phi(x_{j+1}, \xi) - \phi(x_j, \xi) + \phi(x_{j-1}, \xi) + b(\xi) - \psi(x_{j-1}, \xi) \]

\[ \delta_j = \psi(x_{j-1}, \xi) + \chi(x_{j-1}, \xi) + \chi(x_j, \xi) - \phi(x_{j+1}, \xi) - \phi(x_{j+1}, \xi) - \phi(x_{j-1}, \xi) + \phi(x_j, \xi) + b(\xi) - \psi(x_{j-1}, \xi) \]

Changes in Wave Speeds and Wave Forms

\[ \text{Changes in wave speed and wave forms:} \]

\[ \phi(x_j(t)) = \psi(x_j(t)) + \chi(x_j(t)) \]

\[ \text{where} \]

\[ \chi(x_j(t)) = \text{perturbation from} \psi \text{when} a_j \neq a_j \]

Solutions with Non-Zero Wave Speeds

\[ \psi(x_j(t)) = \phi(x_j(t)) - h(a(x_j(t))) \]

\[ \text{Solutions with Non-Zero Wave Speeds} \]

\[ \phi(x, \xi) = \psi(x, \xi) + \chi(x, \xi) \]

\[ \chi(x, \xi) \]

\[ A(s) = 1 + 2 s \cos(s) \]

\[ \phi(x, \xi) \]

\[ \psi(x, \xi) \]

\[ \chi(x, \xi) \]

\[ \beta_j \]

\[ \gamma_j \]

\[ \delta_j \]

\[ \text{Changes in wave speed and wave forms:} \]

\[ \text{Changes in wave speed and wave forms:} \]

\[ \text{Changes in wave speed and wave forms:} \]

\[ \text{Changes in wave speed and wave forms:} \]