Bistable Waves in Discrete Inhomogeneous Media

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Bistable Waves in Discrete Inhomogeneous Media

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Our Nervous System

The nervous cells live inside the “Hot Dog Buns” which are called myelin sheath.

The inrush of sodium ($Na^+$) at the sodium channels causes the electric impulse to jump to the next cell.

Multiple Sclerosis causes the destruction of myelin, inhibiting conduction of electrical signals.
Electrical Impulse with Diseased Cells

Questions

- Where does the electrical impulse stop?
- How much destruction is required to make it stop?
- What happens to wave speed and shape as it passes through deteriorated region?
1952  Hodgkin and Huxley model pulse propagation in the nerve axon of a giant squid.

1961  FitzHugh proposed a simplified model
\[ u_t = \alpha u_{xx} - v - f(u), \quad v_t = bu - rv \]

1963  Hodgkin and Huxley win the Nobel Prize for their work.

1964  Nagumo et al. proposed a more simplified model
\[ u_t = \alpha u_{xx} - f(u) \]

1970  McKean proposed simplifications to the nonlinearity
\[ f(u) = u - h(u - a) \]

1980’s  Various researchers use discrete models instead
\[ u_{xx} \rightarrow u_{j+1} - 2u_j + u_{j-1} \]
\[ (\alpha u_x)_x \rightarrow \alpha_j(u_{j+1} - u_j) - \alpha_{j-1}(u_j - u_{j-1}) \]
Bistable Equation with Inhomogeneous Diffusion

\[ \dot{u}_j = \alpha_j(u_{j+1} - u_j) - \alpha_{j-1}(u_j - u_{j-1}) - f(u_j) \]

with

\[ \alpha_j = \begin{cases} 
\alpha & -m \leq j \leq n \\
\alpha & j < -m \text{ or } j > n 
\end{cases} \]

\[ m, n \in \{0\} \cup \mathbb{N} \]

The nonlinearity is the derivative of a double-well potential,

\[ f(u) = u(u - a)(u - 1) \quad \text{with} \quad a \in (0, 1). \]
McKean’s Caricature of the Cubic

Dashed red line:

\[ f(u) = u - h(u - a) \]

\[ h(x) = \begin{cases} 
1 & x > 0 \\
[0,1] & x = 0 \\
0 & x < 0 
\end{cases} \]
Recent History

1999  Cahn, Mallet-Paret, and Van Vleck derive traveling wave solutions on 2-D lattice with $\alpha_j = \alpha$ using McKean’s $f(u)$, and discuss relationship between $a$ and the wave speed.

2000  Lewis and Keener study steady states of the PDE model. Changing parameters $m$, $n$, and $\alpha_j$ leads to steady state solutions through a limit point bifurcation.

2001  Elmer and Van Vleck consider periodic diffusion and derive solutions using McKean’s $f(u)$, and discuss changes in the wave speed as the solution evolves.

2005  Elmer and Van Vleck derive traveling wave solutions for spatially discrete FitzHugh-Nagumo equation with McKean’s $f(u)$ and $\alpha_j = \alpha$. 
Numerical Simulations for the Evolution Equation

For the case of a single defect

\[ \alpha_j = \begin{cases} 
0.6 & j = 30 \\
1 & j \neq 30 
\end{cases} \]

A slightly slower wave is stopped by the defect.
Traveling Wave Ansatz

Define $\mathcal{R} = \{ j \in \mathbb{Z} : c_j(t) \neq ct \}$ and

$$u_j(t) = \varphi(\xi_j; \xi^*) \quad \text{for} \quad \xi_j = \begin{cases} j - c_j(t) & j \in \mathcal{R} \\ j - ct & j \notin \mathcal{R} \end{cases}$$

$\xi^* \in \mathbb{R}$ is a parameter that determines the position of the wave relative to the defect region. We seek solutions with

$$\varphi(-\infty; \xi^*) = 0, \quad \varphi(\infty; \xi^*) = 1, \quad \varphi(\xi^*; \xi^*) = a$$

$$\varphi(\xi; \xi^*) < a \quad \text{for} \quad \xi < \xi^* \quad \varphi(\xi; \xi^*) > a \quad \text{for} \quad \xi > \xi^*,$$

$$\implies h(\varphi(\xi_j; \xi^*) - a) = h(\xi_j - \xi^*)$$

$$\implies f(\varphi(\xi_j; \xi^*)) = \varphi(\xi_j; \xi^*) - h(\xi_j - \xi^*)$$
Solutions with Non-Zero Wave Speed

Solving

\[-d_j \varphi' (\xi_j) = \alpha_j \varphi (\xi_{j+1}) - (\alpha_j + \alpha_{j-1} + 1) \varphi (\xi_j) + \alpha_{j-1} \varphi (\xi_{j-1}) + h (\xi_j - \xi^*),\]

with \(d_j (t) = \dot{c}_j (t)\) by Fourier transform yields

\[\phi (\xi; \xi^*) = \psi (\xi; \xi^*) + \chi (\xi; \xi^*),\]

where \(\psi (\xi; \xi^*)\) is the solution for \(\alpha_j = \alpha \ \forall j\), with wave speed \(c\)

\[\psi (\xi; \xi^*) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{A(s) \sin (s (\xi - \xi^*))}{s (A(s)^2 + c^2 s^2)} ds + \frac{c}{\pi} \int_0^\infty \frac{\cos (s (\xi - \xi^*))}{A(s)^2 + c^2 s^2} ds\]

for \(A(s) = 1 + 2 \alpha (1 - \cos (s))\)

(Cahn, Mallet-Paret, Van Vleck 1999)
Solutions with Non-Zero Wave Speed

\( \chi(\xi; \xi^*) \) is the perturbation from \( \psi \) when \( \alpha_j \neq \alpha \).

Defining \( \beta_j = \frac{c}{d_j} - 1 \) and \( \gamma_j = \alpha_j - \alpha \),

\[
\chi(\xi; \xi^*) = \sum_{j \in \mathcal{R}} \beta_j F_j(\xi) B_j(\xi^*) + \alpha \sum_{j \in \mathcal{S}} F_j(\xi) C_j(\xi^*) \\
+ \sum_{j \in \mathcal{T}} \gamma_j (F_j(\xi) - F_{j+1}(\xi)) D_j(\xi^*)
\]

\( \mathcal{R} = \{ j \in \mathbb{Z} : c_j(t) \neq ct \}, \mathcal{S} = \{ j \in \mathbb{Z} : j \in \mathcal{R}, \text{ or } j \pm 1 \in \mathcal{R} \}, \mathcal{T} = \{ j \in \mathbb{Z} : \alpha_j \neq \alpha \}, \)

\[
B_j(\xi^*) = \alpha_j \varphi(\xi_{j+1}; \xi^*) - (1 + \alpha_j + \alpha_{j-1}) \varphi(\xi_j; \xi^*) + \alpha_{j-1} \varphi(\xi_{j-1}; \xi^*) + h(\xi_j - \xi^*), \\
C_j(\xi^*) = \varphi(\xi_{j+1}; \xi^*) - \varphi(\xi_j + 1; \xi^*) + \varphi(\xi_{j-1}; \xi^*) - \varphi(\xi_j - 1; \xi^*), \\
D_j(\xi^*) = \varphi(\xi_{j+1}; \xi^*) - \varphi(\xi_j; \xi^*), \quad F_j(\xi) = \frac{1}{\pi} \int_0^\infty \frac{A(s) \cos(s(\xi - \xi_j)) - cs \sin(s(\xi - \xi_j))}{A(s)^2 + c^2 s^2} ds.
\]
Change in Speed as Front Passes Through a Defect

\[ t \]

\[ d \]

\[ \alpha_0 = 0.9 \]

\[ \alpha_0 = 0.8 \]

\[ \alpha_0 = 0.7 \]
Change in Form as Front Passes Through a Defect

\[ d_0 = 0.5552 \text{ and } \xi^* = 0.4794 \]

Filon quadrature approx. compared to RK approx. of \( u_j(t) \).
Steady State Solutions

Definition: The range of $a$ values that yield standing waves is called the *interval of propagation failure*.

Standing waves are solutions of $\dot{u}_j = 0$ or equivalently

$$\alpha_j(u_{j+1} - u_j) - \alpha_{j-1}(u_j - u_{j-1}) = f(u_j)$$

$$\lim_{j \to \infty} u_j = 1 \quad \lim_{j \to -\infty} u_j = 0$$

where $f(u_j) = u_j - h(u_j - a) = u_j - h(j - \xi^*)$.

Note: Solutions are not translationally invariant.
Standing Waves for $\alpha_j = \alpha, \forall j$
Derivation of Solutions

To solve the difference equation

\[ \alpha_j (u_{j+1} - u_j) - \alpha_{j-1} (u_j - u_{j-1}) - u_j = -h_j, \]

with \( h_j = h(j - \xi^*) \), use method of undetermined coefficients. General Solution = Homogeneous Solution + Particular Solution

\[ u_j = u_j^* \rho_j + u_{j+1} \sigma_j + \begin{cases} 
\sum_{k=1}^{j-1} \frac{h_k}{\alpha_k} \sigma_{j-k} & j > j^* \\
0 & j = j^* \\
\frac{h_{j^*}}{\alpha_{j^*}} \sigma_{j-j^*} & j < j^*
\end{cases} \]

Fundamental solutions satisfy

\( (\rho_{j^*}, \rho_{j^*+1}) = (1, 0), \) and \( (\sigma_{j^*}, \sigma_{j^*+1}) = (0, 1). \)

The particular solution can be found in Teschl (2000).

The coefficients are determined by the boundary conditions.
Standing Waves for

$$\alpha_j(u_{j+1} - u_j) - \alpha_{j-1}(u_j - u_{j-1}) - u_j = -h_j$$

**Theorem**

Suppose

$$\alpha_j = \begin{cases} 
\alpha & -m \leq j \leq n \\
\alpha & j < -m \text{ or } j > n 
\end{cases}$$

and that $j^* \in [-m, n]$. Then the existence of solutions is guaranteed by the necessary and sufficient conditions:

1. $a \in (u_{j^*}, u_{j^*+1})$ with $h_{j^*} = 0$, if $j^* < \xi^* < j^* + 1$
2. $a \in u_{j^*}$ with $h_{j^*} = [0, 1]$, if $j^* = \xi^*$.

Explicit solutions are provided in the publication.
Interval of Propagation Failure

Theorem

If \( a \) yields a traveling wave for \( \alpha_0 = \alpha \), then

- Either \( a \in (0, 1/(\lambda + 2)) \) or \( a \in ((\lambda + 1)/(\lambda + 2), 1) \) with
  \[
  \lambda = \left(1 + \sqrt{1 + 4\alpha}\right)/2\alpha
  \]

- There are no corresponding standing waves for \( \alpha_0 < \alpha \) and
  \( \xi^* \notin (0, 1) \), nor for \( \alpha_0 > \alpha \) and \( \xi^* \in (0, 1) \).

- There exist standing waves for \( \alpha_0 < \alpha \) and \( \xi^* \in (0, 1) \), and
  for \( \alpha_0 > \alpha \) and \( \xi^* \notin (0, 1) \), provided
  \[
  a \in \left[\frac{\alpha_0/\alpha}{\lambda + 2(\alpha_0/\alpha)}, \frac{\lambda + \alpha_0/\alpha}{\lambda + 2(\alpha_0/\alpha)}\right].
  \]
Interval of Propagation Failure

$$\alpha = 1$$
Interval of Propagation Failure

**Blue:** $\alpha_0 = \alpha = \frac{1}{2}$  
**Black:** $\alpha_0 = \frac{1}{2}, \alpha = 2$
Interval of Propagation Failure

\[ \alpha = 1, \alpha_{defect} = 0.2 \]
Spatially Discrete FitzHugh-Nagumo Equations

\[
\begin{align*}
\dot{u}_j &= \alpha_j (u_{j+1} - u_j) - \alpha_{j-1} (u_j - u_{j-1}) - v_j - f(u_j), \\
\dot{v}_j &= b u_j - rv_j
\end{align*}
\]

The recovery term \( v_j \) gives pulse solutions.

Stationary Pulses \( \implies v_j = \frac{b}{r} u_j \)

\[ \implies \alpha_j (u_{j+1} - u_j) - \alpha_{j-1} (u_j - u_{j-1}) - \frac{b}{r} u_j = f(u_j) \]

with boundary conditions

\[ \lim_{j \to \pm \infty} u_j = 0 \]
We seek solutions that satisfy

\[ u_j < a \quad \text{for} \quad j < \xi^* \quad \text{and} \quad j > \xi^{**} \]

and

\[ u_j > a \quad \text{for} \quad \xi^* < j < \xi^{**}. \]

Rewriting the piecewise linear \( f \), yields the equation

\[
\alpha_j(u_{j+1} - u_j) - \alpha_{j-1}(u_j - u_{j-1}) - \left(1 + \frac{b}{r}\right)u_j = g_j
\]

where \( j^* = \lfloor \xi^* \rfloor \) and \( j^{**} = \lceil \xi^{**} \rceil \) implies

\[
g_j = h(j - j^*) - h(j - j^{**})
\]
Interval of Stationarity and Pulses
Stable 1-Pulse with $\alpha_k = \alpha$, $\forall k$
Stable 1-Pulse in the Presence of 3 Defects

\[ \alpha = 1; \alpha_2 = \alpha_3 = \alpha_5 = 1/4 \]
Stable Pulse Width Depending on $\alpha$
Dashed: No Defects; Solid: 3 Defects
Passing Traveling Pulses Through Defects
Numerical Results with $\alpha = 2$, $a \approx 0.1252$
Conclusions and Future Work

**Conclusions**
- Results on where an electrical impulse stops.
- Results on destruction required to make it stop.
- Results on wave speed and shape in the defect region.

**Future Work**
- Sufficient conditions for FHN stationary pulses
- Explicit propagating pulse solutions
- Rigorous analysis of pulse stability
- Other nonlinearities
- 2-dimensional lattice