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Controllability and observability of a heat equation with hyperbolic memory kernel

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\begin{abstract}
The exact controllability and observability for a heat equation with hyperbolic memory kernel in anisotropic and nonhomogeneous media are considered. Due to the appearance of such a kind of memory, the speed of propagation for solutions to the heat equation is finite and the corresponding controllability property has a certain nature similar to hyperbolic equations, and is significantly different from that of the usual parabolic equations. By means of Carleman estimate, we establish a positive controllability and observability result under some geometric condition. On the other hand, by a careful construction of highly concentrated approximate solutions to hyperbolic equations with memory, we present a negative controllability and observability result when the geometric condition is not satisfied.
\end{abstract}

\section{Introduction}
Given $T > 0$ and a bounded domain $\Omega$ of $\mathbb{R}^n$ ($n \in \mathbb{N}$) with a $C^2$ boundary $\Gamma = \partial \Omega$. Set $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \Gamma$. Let $\omega \subseteq \Omega$ be open and nonempty. Denote by $\chi_\omega$ the characteristic

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function of $\omega$, i.e., $\chi_\omega(x) = 1$ if $x \in \omega$, and $\chi_\omega(x) = 0$ if $x \notin \omega$. Throughout this paper, we denote $\sum_{i,j=1}^{n}$ and $\sum_{i=1}^{n}$ simply by $\sum_{i,j}$ and $\sum_{i}$, respectively; and denote the transpose of a vector (or matrix) $x$ by $x^\top$. In this paper, vector $x$ always means a column one.

To begin with, we recall the controllability theory for the classical heat equation:

$$
\begin{align*}
\begin{cases}
y_t - \sum_{i,j} (a^{ij}(x)y_{x_i})_{x_j} = u\chi_\omega & \text{in } Q, \\
y = 0 & \text{on } \Sigma, \\
y(0) = y_0 & \text{in } \Omega,
\end{cases}
\end{align*}
$$

where $x = (x_1, \ldots, x_n)^\top$, $(a^{ij})_{n \times n}$ is a given uniformly positive definite matrix, which represents the thermal conductivity of the material occupying $\Omega$, $y = y(t,x)$ is the state variable, $u = u(t,x)$ is the control variable. In system (1.1), the state space is chosen to be $L^2(\Omega)$, and the control space to be $L^2(\omega)$ (which is an abbreviation of $\{u \in L^2(\Omega) \mid \supp u \subseteq \omega\}$). It is well-known (see for example [12]) that for any given $T > 0$ and any given nonempty open subset $\omega$ of $\Omega$, system (1.1) is null controllable (resp. approximately controllable) in $L^2(\Omega)$, i.e., for any given $y_0 \in L^2(\Omega)$ (resp. for any given $\varepsilon > 0$, and $y_0, y_1 \in L^2(\omega)$), one can find a control $u \in L^2((0,T) \times \omega)$ (which is an abbreviation of $\{u \in L^2((0,T) \times \omega) \mid \supp u \subseteq (0,T) \times \omega\}$) such that the weak solution $y(\cdot) \in C([0,T]; L^2(\Omega)) \cap C((0,T); H^1_0(\Omega))$ of (1.1) satisfies $y(T) = 0$ (resp. $\|y(T) - y_1\|_{L^2(\Omega)} \leq \varepsilon$). On the other hand, due to the smoothing effect of solutions to the heat equation, exact controllability for (1.1) is impossible, i.e., the above $\varepsilon$ may not be taken to be zero.

It is notable that in the above, the controllability time $T$ and the controller $\omega$ can be chosen as small as one likes. This is due to the fact that the classical heat equation admits an infinite speed of propagation for a finite heat pulse. However, it has been known (e.g. [3,4]) for quite a long time that the property of instantaneous propagation for the heat equation is not really physical! To eliminate this paradox, a modified Fourier’s law was introduced [8], which results in a heat equation with memory. Among other things, this paradox, a modified Fourier’s law was introduced [8], which results in a heat equation with memory. We refer to [27] for an updated analysis on the well-posedness and the propagation speed of the heat equation with memory derived from a general modified Fourier’s law. Among other things, it was shown in [27] that, under certain conditions, the heat equation with a memory kernel admits a finite speed of propagation for finite heat pulses. Hence, heat equations with memory are more realistic for heat conduction.

Following [27], instead of (1.1), we consider the following controlled system:

$$
\begin{align*}
\begin{cases}
y_t - \sum_{i,j} (a^{ij}(x)b(t-s,x)y_{x_i}(s,x)ds)_{x_j} = u\chi_\omega & \text{in } Q, \\
y = 0 & \text{on } \Sigma, \\
y(0) = y_0 & \text{in } \Omega,
\end{cases}
\end{align*}
$$

with $b$ being referred to as a hyperbolic memory kernel. The same as in system (1.1), $y = y(t,x)$ and $u = u(t,x)$ are the state and control variables of system (1.2), and we choose its state space and the control space to be $L^2(\Omega)$ and $L^2((0,T) \times \omega)$, respectively. We refer to [27] for the well-posedness of (1.2).

The first purpose of this paper is to study the (instantaneous) exact controllability of (1.2), which means that, for any given $y_0, y_1 \in L^2(\Omega)$, there is a control $u \in L^2((0,T) \times \omega)$ such that the corresponding solution $y \in C([0,T]; L^2(\Omega))$ of (1.2) satisfies

$$
y(T) = y_1 \quad \text{in } \Omega.
$$

As we shall see later, the hyperbolic nature of system (1.2) allows us to show its exact controllability under suitable conditions on the waiting time $T$ and the controller $\omega$. 
By the classical duality argument \([14,15]\), the above controllability problem for \((1.2)\) can be reduced to the establishment of an observability estimate for its dual system:

\[
\begin{aligned}
  p_t + \sum_{i,j} \left( a^{ij}(x) \int_0^T b(s-t,x)p_{x_i}(s,x)\,ds \right)_{x_j} &= 0 & \text{in } Q, \\
  p &= 0 & \text{on } \Sigma, \\
  p(T) = p_0 & \text{in } \Omega,
\end{aligned}
\]

(1.4)

by which, we mean to find a constant \( C > 0 \), independent of \( p_0 \), such that the solution \( p \) of \((1.4)\) satisfies

\[
|p_0|_{L^2(\Omega)} \leq C |p|_{L^2((0,T) \times \omega)}, \quad \forall p_0 \in L^2(\Omega).
\]

(1.5)

There are numerous studies on observability estimate for partial differential equations (PDEs, for short), mainly for those without memory or at most with “small” memory (see \([28]\) and the rich references cited therein). The techniques that have been developed so far to obtain such estimates depend heavily on the nature of the equations. In the context of hyperbolic equations, one may use multipliers \([15]\) or microlocal analysis \([2]\); while, in the context of parabolic equations, one uses Carleman inequalities \([6]\). Carleman inequalities can also be used to obtain observability inequalities for hyperbolic equations \([7]\). However, the usual Carleman estimates do not seem to work directly for the observability problem of general parabolic and/or hyperbolic equations with large memory.

We refer to \([5,10,11,13,18,24,25]\) for some previous controllability and/or observability results for infinite-dimensional systems with memory. It is worthy of mentioning that, based on Laplace transform and cosine operator approach, respectively, \([1]\) and \([20]\) studied the controllability problem for \((1.2)\) when \((a^{ij})_{n \times n} = I\), the identity matrix, and the memory kernel \( b \) does not depend on \( x \). On the other hand, by means of Carleman estimate, exact controllability result for \((1.2)\) with \((a^{ij})_{n \times n} = l\) was given in \([26]\). Recently, further related results have been presented in \([9,21]\), especially an interesting negative controllability result can be found in \([9]\).

The key observation in \([26]\) is that, due to the special structure of system \((1.4)\) with \((a^{ij})_{n \times n} = l\), a modified Carleman inequality can be employed to derive the observability estimate for it. By combining and modifying carefully the Carleman estimate developed in \([26]\) and \([7]\), the first concern of this paper is to establish the observability estimate \((1.5)\) for system \((1.4)\) with general thermal conductivity matrix \((a^{ij})_{n \times n}\) under some further assumptions on the controller/observer \( \omega \) and the waiting time \( T \).

Note that, due to the finite propagation speed of solutions to system \((1.4)\) (see \([27]\)), it is clear that estimate \((1.5)\) is impossible unless \( T \) is large enough. On the other hand, we recall that, for the classical hyperbolic equations (without memory), one has to introduce some geometric conditions on \( \omega \) and \( T \), otherwise the expected observability inequality may fail to be true even if \( T \) is large \([2]\). These conditions show that the “position” rather than the “size” of \( \omega \) is crucial for the desired observability estimate. We shall show that the same phenomenon happens for the present nonlocal observability and controllability problems. This is exactly our second concern in this paper.

More precisely, the second goal of this paper is devoted to showing that, in view of its underlying hyperbolic nature, the observability estimate \((1.5)\) for system \((1.4)\) fails for the case that \( T \) and \( \omega \) do not satisfy a Geometric Optics Condition (see Assumption 2.1 in Section 2). For this purpose, we adapt the Gaussian Beam method developed in \([17,22,23]\) to construct a sequence of approximate solutions to \((1.4)\), with energies localized in \( \Omega \setminus \overline{\omega} \). To the best of our knowledge, this might be the first time to analyze directly the lack of observability of PDEs with memory. There are two difficulties in treating this problem for system \((1.4)\). The first one comes from the memory term. To overcome this difficulty, one needs to introduce more “corrected” terms into the usual Gaussian Beam approximate solutions (for the classical hyperbolic equations) to recover an accurate description. The second difficulty comes from the fact that system \((1.4)\) is equivalent to some hyperbolic equations with memory and with given null initial displacement (see system \((4.9)\)). This information is the key point that we shall use.
to derive the positive observability inequality (1.5) for system (1.4) via Carleman estimate. But now it
means that we have to show a negative result under a more restrictive assumption. To overcome this
difficulty, we need to superpose suitably two approximate solutions of the hyperbolic equations with
memory which are concentrated in a neighborhood of a given generalized ray (see Definition 2.2), one
of which is evolved forward and the other backward (see (7.10)). The key point is that, we need to
use a refined localization technique (see (5.34) and Theorem 5.2) to construct carefully the desired
backward approximate solutions so that their energies are concentrated near \( t = 0 \).

The rest of this paper is organized as follows. In Section 2, we state our main results. Some pre-
liminary results are collected in Section 3. We will prove our positive controllability and observability
result in Section 4. Sections 5–6 are devoted respectively to the constructions of highly concentrated
approximate solutions for hyperbolic equations with memory in the whole space \( \mathbb{R}^n \) and in any
bounded domain \( \Omega \), which have their independent interest. In Section 7, we shall construct suit-
able localized (exact) solutions for hyperbolic equations with memory in bounded domains (which
also has its independent interest), and via which we give a proof of our negative controllability and
observability result. Finally, Appendix A is devoted to the proofs of some technical results that are
used in the paper.

2. Statement of the main results

In the sequel, for any set \( M \subseteq \mathbb{R}^m \) (\( m \in \mathbb{N} \)), point \( z_0 \in \mathbb{R}^m \) and \( \delta > 0 \), we define
\[
O_\delta(M) = \left\{ x \in \mathbb{R}^m \mid |x - x'| < \delta \text{ for some } x' \in M \right\}, \quad O_\delta(z_0) = O_\delta(\{z_0\}).
\]

Throughout this paper, we will use \( C \) to denote a generic positive constant which may be different
from line to line.

First of all, we assume that the coefficients of systems (1.2) and (1.4) satisfy the following:
\[
\begin{align*}
& a^{ij}(\cdot) \in C^1(\overline{\Omega}), \quad a^{ij}(x) = a^{ji}(x), \quad \forall x \in \overline{\Omega}, \ i, j = 1, 2, \ldots, n, \\
& \frac{1}{C} |\xi|^2 \leq \sum_{i,j} a^{ij}(x)\xi_i\xi_j \leq C |\xi|^2, \quad \forall x \in \overline{\Omega}, \ \xi \equiv (\xi_1, \ldots, \xi_n)^\top \in \mathbb{R}^n.
\end{align*}
\]
and
\[
\begin{align*}
& b(\cdot, \cdot) \in C^3([0, +\infty) \times \overline{\Omega}), \\
& b(0, x) \equiv 1, \quad \forall x \in \overline{\Omega}.
\end{align*}
\]

Note that the second condition in (2.3) can be replaced by
\[
\frac{1}{C} \leq b(0, x) \leq C, \quad \forall x \in \overline{\Omega}.
\]

In fact, if (2.4) is assumed, by putting
\[
\tilde{a}^{ij}(x) \triangleq a^{ij}(x)b(0, x), \quad i, j = 1, 2, \ldots, n, \quad \tilde{b}(t, x) \equiv \frac{b(t, x)}{b(0, x)},
\]
we see that \( \tilde{a}^{ij}(\cdot) \) and \( \tilde{b}(\cdot, \cdot) \) satisfy (2.2)–(2.3). In what follows, we will keep (2.2)–(2.3) for simplicity of presentation. Next, suppose that there is a function \( d(\cdot) \in C^2(\overline{\Omega}) \) satisfying
\[
r_0 \triangleq \min_{x \in \overline{\Omega}} |\nabla d(x)| > 0,
\]
so that, for some constant $\mu_0 > 0$, it holds

$$
\sum_{i,j} \left\{ \sum_{i',j'} [2d^{ij}(a^{i'j'}d_{x_{i'}})_{x_{j'}} - (a^{ij})_{x_{j'}} a^{i'j'} d_{x_{i'}}] \right\} \xi_i \xi_j \geq \mu_0 \sum_{i,j} a^{ij} \xi_i \xi_j, \quad \forall (x, \xi) \in \overline{\Omega} \times \mathbb{R}^n. \tag{2.7}
$$

We refer to [7] for nontrivial examples satisfying conditions (2.6)--(2.7).

For the above function $d = d(\cdot)$ and a given (small) $\delta_0 > 0$, we introduce

$$
\Gamma_0 \triangleq \left\{ x \in \Gamma \mid \sum_{i,j} a^{ij} v_i v_j > 0 \right\}, \quad \omega = \mathcal{O}_{\delta_0}(\Gamma_0) \cap \Omega,
$$

where $\nu = \nu(x) = (\nu_1, \nu_2, \ldots, \nu_n)^\top$ is the outward normal vector of $\Omega$ at $x \in \Gamma$ so that

$$
\sum_{i,j} a^{ij}(x)v_i v_j = 1. \tag{2.9}
$$

One can check that, if $d(\cdot) \in C^2(\overline{\Omega})$ satisfies (2.7), then for any given constants $a_1 \geq 1$ and $a_2 \in \mathbb{R}$, the function

$$
\hat{d}(x) \triangleq a_1 d(x) + a_2 \tag{2.10}
$$

(scaling and translating $d(\cdot)$) still satisfies (2.7) with $\mu_0$ replaced by $a_1 \mu_0$; meanwhile, the scaling and translating $d(\cdot)$ do not change the set $\Gamma_0$. Hence, by scaling and translating $d(\cdot)$, if necessary, we may assume, without loss of generality, that

$$
\left\{ \begin{array}{l}
(2.7) \text{ holds with } \mu_0 \geq 4, \\
\frac{1}{4} \sum_{i,j} a^{ij}(x)d_{x_i}(x)d_{x_j}(x) \geq \max_{x \in \overline{\Omega}} d(x) \geq \min_{x \in \overline{\Omega}} d(x) > 0, \quad \forall x \in \overline{\Omega}. \tag{2.11}
\end{array} \right.
$$

In what follows, we put

$$
\left\{ \begin{array}{l}
R_1 \triangleq \max_{x \in \overline{\Omega}} \sqrt{d(x)}, \\
R_0 \triangleq \min_{x \in \overline{\Omega}} \sqrt{d(x)}, \\
T_0 \triangleq \inf \left\{ R_1 \mid d(\cdot) \text{ satisfies } (2.11) \right\} \equiv \inf \left\{ \max_{x \in \overline{\Omega}} \sqrt{d(x)} \mid d(\cdot) \text{ satisfies } (2.11) \right\}. \tag{2.12}
\end{array} \right.
$$

We have the following positive result of observability/controllability.

**Theorem 2.1.** Let (2.2), (2.3), and (2.11) hold, $\omega$ and $T_0$ be given respectively by (2.8) and (2.12), and $T > T_0$. Then

(i) There exists a constant $C > 0$, independent of $p_0$, such that the solution $p$ of system (1.4) satisfies (1.5);
(ii) System (1.2) is exactly controllable in $L^2(\Omega)$ at time $T$ by means of control $u \in L^2((0, T) \times \omega)$.

The proof of Theorem 2.1 will be given in Section 4.

In order to state our negative controllability/observability result, we need to introduce some notions. First, put

$$
A(x) = (a^{ij}(x))_{1 \leq i,j \leq n}, \quad g(x, \xi) \triangleq \sum_{i,j} a^{ij}(x) \xi_i \xi_j = \xi^\top A(x) \xi, \quad \xi = (\xi_1, \ldots, \xi_n)^\top. \tag{2.13}
$$
For any fixed $x$, $g(x, \cdot)$ induces a metric in $\mathbb{R}^n$. We now extend by continuity (up to the first order of partial derivatives) the matrix-valued function $A(x)$ defined on $\Omega$ (by (2.2)), to the whole $\mathbb{R}^n$. To simplify the notation, we still denote the extension by $A(x)$. Note however that the “new” $A(x)$ may fail to satisfy the elliptic condition in (2.2) in the whole $\mathbb{R}^n$. Therefore, we need to make a further modification. Noting that $A(x)$ satisfies the elliptic condition on $\overline{\Omega}$, we may choose a small $\delta_1 > 0$ and a cut-off function

$$
\rho \in C^\infty_0(\mathbb{R}^n), \quad \text{supp} \subseteq O_{\delta_1}(\Omega), \quad \text{satisfying} \quad \rho \equiv 1 \quad \text{on} \quad \overline{\Omega}, \quad 0 \leq \rho \leq 1 \quad \text{in} \quad \mathbb{R}^n, \quad (2.14)
$$

so that the matrix $\rho A(x) + (1 - \rho)I$ satisfies the elliptic condition in the whole $\mathbb{R}^n$. Hence, without loss of generality, we assume that

$$
\frac{1}{C} |\xi|^2 \leq \xi^T A(x) \xi \leq C |\xi|^2, \quad \forall x \in \mathbb{R}^n, \quad \xi \in \mathbb{R}^n, \quad (2.15)
$$

i.e., the matrix $A(x)$ is uniformly positive definite and bounded in $\mathbb{R}^n$.

For the above extended (and corrected) functions $a^{ij} \in C^1(\mathbb{R}^n)$, we define a formal differential operator on $\mathbb{R}^n$ as follows:

$$
W = \partial_{tt} - \sum_{i,j} a^{ij}(x) \partial_{xi} \partial_{xj}. \quad (2.16)
$$

We then introduce the following notions.

**Definition 2.1.** A null bicharacteristic of operator $W$ is defined to be a solution of the following (generally nonlinear) ordinary differential equations:

$$
\begin{align*}
\dot{x}(t) &= \nabla_x g(x(t), \xi(t)) \quad (\equiv 2A(x(t))\xi(t)), \\
\dot{\xi}(t) &= -\nabla_\xi g(x(t), \xi(t)), \\
x(0) &= x^0, \quad \xi(0) = \xi^0,
\end{align*} \quad (2.17)
$$

where the initial data $x^0$ and $\xi^0$ are chosen such that $g(x^0, \xi^0) = 1/4$ (here $\dot{x} = \frac{dx}{dt}$). The projection of the null bicharacteristic to the physical time-space, $(t, x(t))$, (which traces a curve in $\mathbb{R}^{1+n}$), is called a ray of operator $W$. Sometimes, one also refers to $(t, x(t), \xi(t))$ as a ray (starting from $x^0$ with initial direction $\xi^0$).

In the above, the choice of $1/4$ is only for convenience. Indeed, by scaling, one may replace it by any other nonzero real number. It is easy to check that

$$
g(x(t), \xi(t)) = \frac{1}{4}, \quad \forall t \in \mathbb{R}. \quad (2.18)
$$

When the matrix $A(x)$ is independent of $x$, say $A(x) \equiv I$, by (2.17) we see that rays of operator $W$ in $\mathbb{R}^n$ are simply straight lines. In this case, $\xi(t) \equiv \xi^0$ (for all $t \in \mathbb{R}$) is the direction of the ray.

**Remark 2.1.** The global existence of solutions to nonlinear system (2.17) is guaranteed by assumption (2.15). We refer to Proposition 3.2 (in Section 3) for a slightly more general existence result.

**Remark 2.2.** By (2.17), one can check that if $(t, x(t), \xi(t))$ is a ray of operator $W$ starting from $x^0$ with initial direction $\xi^0$, then $(t, x(-t), -\xi(-t))$ is also a ray of operator $W$ starting from $x^0$ but with opposite initial direction $-\xi^0$. 

Remark 2.3. The following regularity of solutions to (2.17) holds: If for some initial data \( x^0 \) and \( \xi^0 \), system (2.17) admits a solution \((x(t), \xi(t))\) in \((T_1, T_2)\) for some \( T_1 < 0 \) and \( T_2 > 0 \), then \((x(t), \xi(t)) \in C^{k+1}(T_1, T_2; \mathbb{R}^n) \times C^k(T_1, T_2; \mathbb{R}^n)\) provided that \((a^{ij})_{n \times n} \in C^k(\mathbb{R}^n)\) for some integer \( k \geq 1\).

With the above (corrected) \( A(x) \), we have defined the rays for operator \( W \) in Definition 2.1. A ray \((t, x(t), \xi(t))\) of operator \( W \) is said to start from \( \Omega \) at time \( t = 0 \) if \( x(0) \in \Omega \). When there exists a (minimal) finite time \( t_0 \) such that the ray reaches the boundary of \( \Omega \) at time \( t = t_0 \), i.e., \( x(t_0) \in \Gamma \), we say that the ray exits \( \Omega \) in finite time. Note that, since the ray \((t, x(t), \xi(t))\) is determined by (2.17), the above corrections on \( A(x) \) (outside \( \Omega \)) does not affect the section \( \{(t, x(t), \xi(t)) \mid 0 \leq t < t_0\} \) of the ray \((t, x(t), \xi(t))\) in \( \Omega \). This leads to the following notion.

Definition 2.2. A parametric curve: \([0, T] \ni t \mapsto (x(t), \xi(t)) \in \mathcal{S} \times \mathbb{R}^n\), with \( x(0) \in \Omega \) and \( x(T) \in \Omega \), is called a generalized ray of operator \( W \) in \( \mathcal{S} \) if there exists a finite partition \( 0 = s_0 < s_1 < \cdots < s_m = T \) \((m \in \mathbb{N})\), such that each \((t, x(t), \xi(t))|_{s_i \leq t \leq s_{i+1}} \equiv (x^i(t), \xi^i(t))\) is a ray of operator \( W \) (restricted to the time interval \([s_i, s_{i+1}]\), \(i = 0, 1, 2, \ldots, m - 1\)), which reaches \( \Gamma \) at time \( t = s_{i+1} \), and is reflected by \((t, x(t), \xi(t))|_{s_{i+1} \leq t \leq s_{i+2}} \equiv (x^{i+1}(t), \xi^{i+1}(t))\), following the law of geometric optics whenever \( i \in \{0, 1, 2, \ldots, m - 2\} \), i.e.,

\[
\xi^{i+1}(s_{i+1}) = \xi^i(s_{i+1}) - 2\left[\nu(x^i(s_{i+1}))^T A(x^i(s_{i+1})) \xi^i(s_{i+1})\right] v(x^i(s_{i+1})).
\]

(19.29)

In the sequel, we shall denote this generalized ray by \( \{(t, x^i(t), \xi^i(t)) \mid t \in [s_i, s_{i+1}]\}_{i=0}^{m-1} \), and call \( s_k \) the \( k \)-th reflection instant of this generalized ray \((k = 1, \ldots, m - 1)\).

Remark 2.4. In view of Remark 2.1, \((x^i(t), \xi^i(t))\) is well-defined for any \( t \in \mathbb{R} \) \((i = 0, 1, 2, \ldots, m - 1)\). In the above definition, we need only to use the restriction of \((x^i(t), \xi^i(t))\) on the time interval \([s_i, s_{i+1}]\). However, when we construct approximate solutions with energies concentrated along the generalized ray \( \{(t, x^i(t), \xi^i(t)) \mid t \in [s_i, s_{i+1}]\}_{i=0}^{m-1} \), we will need to use the information of \((x^i(t), \xi^i(t))\) for any \( t \in [-T, T] \). On the other hand, we regard any ray \( \{(t, x(t), \xi(t)) \mid t \in [0, T]\} \) \((\text{with } x(0) \in \Omega \text{ and } x(T) \in \Omega)\) of operator \( W \) in \( \Omega \) to be a (special) generalized ray of operator \( W \) in \( \overline{\Omega} \). Note that, in some situation, there may exist a ray \((t, x(t), \xi(t))\) of \( W \) starting from \( \Omega \) but never arrives at its boundary \( \Gamma \), i.e., \( x(t) \notin \Omega \) for each \( t \geq 0 \) \((\text{e.g., } [2])\).

Remark 2.5. Since \((t, x^i(t), \xi^i(t))\) satisfies system (2.17) and noting that matrix \( A(x) \) is invertible, one can check that (2.19) is equivalent to

\[
\dot{x}^{i+1}(s_{i+1}) = \dot{x}^i(s_{i+1}) - 2\left[\nu(x^i(s_{i+1}))^T A(x^i(s_{i+1})) \dot{x}^i(s_{i+1})\right] v(x^i(s_{i+1})).
\]

(20.20)

This means that the direction \( \dot{x}^{i+1}(s_{i+1}) \) of \( x^{i+1}(t) \) at \( t = s_{i+1} \) is obtained from that of the previous one, \( \dot{x}^i(s_{i+1}) \), by reflecting it with respect to \( \nu(x^i(s_{i+1})) \) (under the metric induced by \( g(x, \cdot) \), see (2.13)). When \( A(x) \equiv I \), formula (2.20) gives the classical law of geometric optics (see Fig. 1).

We introduce the following geometric assumption on the triple \((T, \Omega, \omega)\).

Assumption 2.1. There is a generalized ray: \( \{(t, x^i(t), \xi^i(t)) \mid t \in [s_i, s_{i+1}]\}_{i=0}^{m-1} \) of operator \( W \) in \( \mathcal{S} \) such that \( x^0(0) \in \Omega, x^m(T) \in \Omega, \) and \( x^i(t) \notin \overline{\Omega} \) for all \( t \in [s_i, s_{i+1}] \), \( i = 0, 1, 2, \ldots, m - 1 \).

Remark 2.6. For some domain \( \Omega \) and controller \( \omega \), the above geometric assumption may hold for any \( T > 0 \), see Fig. 2. In this case, we say that the ray is trapped.

We have the following negative result of observability and controllability.
Theorem 2.2. Let $\Omega$ be a bounded domain with boundary $\Gamma \in C^3$, (2.2)–(2.3) hold, and $a^{ij} \in C^3(\overline{\Omega})$ ($i, j = 1, 2, \ldots, n$). Let $(\Omega, \omega, T)$ satisfy Assumption 2.1. Then

(i) There is no constant $C > 0$, independent of $p_0$, such that the solution of system (1.4) satisfies (1.5);
(ii) System (1.2) is not exactly controllable in $L^2(\Omega)$ at time $T$ by means of control $u \in L^2((0, T) \times \omega)$.

The proof of Theorem 2.2 will be given in Section 7.

Remark 2.7. Similar to [19], one can show that conditions of Theorems 2.1 and 2.2 contradict each other. On the other hand, we recall that [2], $(\Omega, \omega, T)$ is said to satisfy the Geometric Control Condition (GCC for short) if $\partial\Omega$ is $C^\infty$ with no contact of infinite order with its tangent, and any generalized ray $\{(t, x^i(t), \xi^i(t)) \mid t \in [s_i, s_{i+1}], i = 0, 1, 2, \ldots, m-1\}$ of operator $W$ in $\overline{\Omega}$ with $x^0(0) \in \Omega$ and $x^m(T) \in \Omega$ satisfies $x^i(t) \in \omega$ for some $i \in \{0, 1, 2, \ldots, m-1\}$ and some $t \in [s_i, s_{i+1}]$. Clearly, GCC contradicts Assumption 2.1. It is shown in [2] that GCC is a sufficient condition for the observability estimate for time-independent hyperbolic equations without memory. It would be quite interesting to extend this result to the present nonlocal case, i.e., system (1.4) with large memory. The main difficulty, as we shall see later, is that system (1.4) is equivalent to system (4.9), which is a hyperbolic equation with both memory and time-dependent coefficients. Indeed, as far as we know, it is a longstanding open problem to extend the results in [2] to the case of hyperbolic equations with time-dependent coefficients, which is unsolved even for the case without memory.

3. Some preliminaries

In this section, we present some preliminary results.

First of all, similar to [7, Corollary 4.1], we have the following result.
Lemma 3.1. Let $b^{ij} (\cdot) = b^{ji} (\cdot) \in C^1 (\mathbb{R}^n)$ ($i, j = 1, 2, \ldots, n$) and $w (\cdot, \cdot) \in C^2 (\mathbb{R} \times \mathbb{R}^n)$. For any $\lambda, \delta, \varepsilon > 0$, set

$$
\phi = \phi (t, x) = d(x) - \frac{\delta}{2} t^2,
$$

$$
\psi = \lambda \left[ \sum_{i, j} (b^{ij} d_{xi})_{x_j} - \delta - \varepsilon \right],
$$

$$
\ell = \ell \phi, \quad v = \theta w, \quad \theta = e^\ell.
$$

Then

$$
\theta^2 \left| w_{tt} - \sum_{i, j} (b^{ij} w_{xi})_{x_j} \right|^2 + 2 \left[ \ell_t \left( v_t^2 + \sum_{i, j} b^{ij} v_{xi} v_{x_j} \right) - 2 \sum_{i, j} b^{ij} \ell_{xi} v_{x_j} v_t - \psi v v_t + (A + \psi) \ell_t v^2 \right]_t
$$

$$
+ 2 \sum_j \left\{ 2 \sum_{i, i', j'} b^{ij} b^{i' j'} \ell_{xi'} v_{xj'} v_{xj'} - \sum_{i, j, j'} b^{ij} b^{i' j'} \ell_{xi} v_{xj'} v_{xj'} + \psi v \sum_i b^{ij} v_{xi} \right\}
$$

$$
- 2 \ell_t v_t \sum_i b^{ij} v_{xi} + \sum_i b^{ij} \ell_{xi} v_t^2 - \sum_i b^{ij} \left[ (A + \psi) \ell_{xi} + \frac{\psi_{xi}}{2} \right] v^2 \right\}_{x_j}
$$

$$
\geq 2 \lambda \varepsilon v_t^2 + 2 \lambda \sum_i \sum_{i', j'} \left\{ 2 b^{ij} \left( b^{i' j'} d_{xi'} \right)_{xj'} - \left( b^{ij} \right)_{xj'} b^{i' j'} d_{xi'} \right\} v_{xi} v_{xj} + B v^2, \quad \text{(3.2)}
$$

where

$$
A = \lambda^2 \left( \delta^2 t^2 - \sum_{i, j} b^{ij} d_{xi} d_{xj} \right) + \lambda (2 \delta + \varepsilon),
$$

$$
B = 2 \lambda^3 \left[ (2 \delta + \varepsilon) \sum_{i, j} b^{ij} d_{xi} d_{xj} + \sum_{i, j} b^{ij} d_{xi} \left( \sum_{i', j'} b^{i' j'} d_{xi'} d_{xj'} \right)_{xj} - (3 \delta + \varepsilon) \delta^2 t^2 \right] + O (\lambda^4). \quad \text{(3.3)}
$$

Next, using the standard theory on Volterra integral equations, one can show the following result.

Lemma 3.2. Let $b (\cdot, \cdot) \in C^m (\overline{Q})$ for some $m \geq 1$. Then for any $w (\cdot, \cdot) \in C ([0, T]; L^2 (\Omega))$, there exists a unique solution $z (\cdot, \cdot) \in C ([0, T]; L^2 (\Omega))$ to the following equation:

$$
z(t, x) = w(t, x) + \int_0^t b(t - s, x) z(s, x) \, ds, \quad \text{a.e.} \quad (t, x) \in Q. \quad \text{(3.4)}
$$

Moreover, there exists a $\beta (\cdot, \cdot) \in C^m (\overline{Q})$, such that the solution $z (\cdot, \cdot)$ of (3.4) admits the following representation:

$$
z(t, x) = w(t, x) + \int_0^t \beta(t - s, x) w(s, x) \, ds, \quad \forall (t, x) \in Q. \quad \text{(3.5)}
$$
Further, we recall the following known result [7, Lemma 3.2].

**Lemma 3.3.** Let \( b^{ij} = b^{ji} \in C^1(\mathbb{R}_x^n) \) (\( i, j = 1, 2, \ldots, n \)), and \( h \triangleq (h^1, \ldots, h^n) : \mathbb{R}_t \times \mathbb{R}_x^n \to \mathbb{R}^n \) be a vector field of class \( C^1 \). Then, for any \( q(\cdot, \cdot) \in C^2(\mathbb{R}_t \times \mathbb{R}_x^n) \), it holds

\[
- \sum_j \left[ 2(h \cdot \nabla q) \sum_i b^{ij} q_{x_i} + h^j \left( q_t^2 - \sum_{i,k} b^{ik} q_{x_i} q_{x_k} \right) \right]_{x_j} \\
= 2 \left[ \left( q_{tt} - \sum_{i,j} (b^{ij} q_{x_i})_{x_j} \right) h \cdot \nabla q - \left( q_t h \cdot \nabla q \right)_t + q_t h_t \cdot \nabla q - \sum_{i,j,k} b^{ij} q_{x_i} q_{x_k} \frac{\partial h^k}{\partial x_j} \right] \\
- \left( \nabla \cdot h \right) q_t^2 + \sum_{i,j} q_{x_i} q_{x_j} \nabla \cdot (b^{ij} q).
\]

(3.6)

The following technical lemma can be found in [17].

**Lemma 3.4.** Let \( b(\cdot) \in L^\infty(\mathbb{R}^n) \) be a function satisfying \( |x| - x_0|^{-\alpha} b(\cdot) \in L^\infty(\mathbb{R}^n) \) for some \( x_0 \in \mathbb{R}^n \) and some \( \alpha \geq 0 \). Let \( H \) be a symmetric, positive definite, real \((n \times n)\) matrix. Then there exists a constant \( C > 0 \), independent of \( \varepsilon > 0 \) such that

\[
\int_{\mathbb{R}^n} \left| b(x) e^{-\varepsilon x^T Hx / \varepsilon} \right|^2 \, dx \leq C \varepsilon^{\frac{n}{2} + \alpha}.
\]

Further, we show the following simple result.

**Proposition 3.1.** Let \( b(\cdot) \in C(\Omega) \) and \( x_0 \in \mathbb{R}^n \). Then, for any symmetric, positive definite, real \((n \times n)\) matrix \( H \), it holds

\[
\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^{n/2}} \int_{\Omega} \left| b(x) e^{-\varepsilon (x-x_0)^T H (x-x_0)/\varepsilon} \right|^2 \, dx = \begin{cases} 
\frac{|b(x_0)|^2}{(\det H)^{\frac{1}{2}}} \left( \frac{\pi}{2} \right)^{\frac{n}{2}}, & \text{if } x_0 \in \Omega, \\
\frac{|b(x_0)|^2}{2(\det H)^{\frac{1}{2}}} \left( \frac{\pi}{2} \right)^{\frac{n}{2}}, & \text{if } x_0 \in \Omega \setminus \Omega_{\delta}(x_0), \\
0, & \text{if } x_0 \notin \Omega \setminus \Omega_{\delta}(x_0).
\end{cases}
\]

(3.7)

**Proof.** The case \( x_0 \in \mathbb{R}^n \setminus \Omega \) is obvious because of the exponential decay of the integrand with respect to \( \varepsilon \). We now consider the case \( x_0 \in \Omega \). Let \( \delta \in (0, 1) \) be small enough so that \( \Omega_{\delta}(x_0) \subseteq \Omega \) (recall (2.1) for the definition of \( \Omega_{\delta}(x_0) \)). For any such a fixed \( \delta > 0 \), we have

\[
\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^{n/2}} \int_{\Omega \setminus \Omega_{\delta}(x_0)} \left| b(x) e^{-\varepsilon (x-x_0)^T H (x-x_0)/\varepsilon} \right|^2 \, dx = 0.
\]

Also, by the continuity of \( b(\cdot) \), one has

\[
\frac{1}{\varepsilon^{n/2}} \int_{\Omega_{\delta}(x_0)} \left| \frac{|b(x) - b(x_0)|^2}{\varepsilon^{n/2}} \right| \, dx \leq \max_{x \in \Omega_{\delta}(x_0)} \left| b(x) - b(x_0) \right|^2 \frac{1}{\varepsilon^{n/2}} \int_{\mathbb{R}^n} e^{-2x^T Hx / \varepsilon} \, dx
\]
\[ \max_{x \in \mathcal{O}(x_0)} |b(x) - b(x_0)|^2 \frac{1}{\epsilon^n/2} \frac{1}{(\det H)^{1/2}} \left( \frac{\pi \epsilon}{2} \right)^n \rightarrow 0, \quad \text{as } \delta \rightarrow 0, \text{ uniformly in } \epsilon > 0. \]

Finally, for any fixed \( \delta > 0 \),

\[
\frac{1}{\epsilon^{n/2}} \int_{\mathcal{O}(x_0) \cap \Omega} \left| e^{-\langle x - x_0, H(x - x_0) \rangle/\epsilon} \right|^2 \, dx = \frac{1}{\epsilon^{n/2}} \int_{\mathcal{O}(0) \cap \Omega} \left| e^{-x^T H x/\epsilon} \right|^2 \, dx
\]

\[= \frac{1}{\epsilon^{n/2}} \int_{\mathbb{R}^n} e^{-2x^T H x/\epsilon} \, dx - \int_{\mathbb{R}^n \setminus \mathcal{O}(0) \cap \Omega} e^{-2x^T H x/\epsilon} \, dx \rightarrow \frac{1}{(\det H)^{1/2}} \left( \frac{\pi}{2} \right)^n, \quad \text{as } \epsilon \rightarrow 0. \quad (3.9)\]

Here, we use the fact that the integrand in the integral \( \int_{\mathbb{R}^n \setminus \mathcal{O}(0)} e^{-2x^T H x/\epsilon} \, dx \) decays exponentially in \( \epsilon \). Combining (3.7), (3.8) and (3.9), we obtain the desired conclusion for the case \( x_0 \in \Omega \).

Finally, we consider the case \( x_0 \in \Gamma \). For any \( \gamma > 0 \), put

\[ G_\gamma^+ \triangleq \{ \hat{x} = (\hat{x}_1, \ldots, \hat{x}_n) \in \mathbb{R}^n \mid |\hat{x}| < \gamma, \ \hat{x}_n \geq 0 \}. \]

Since \( \Gamma \in C^2 \), there exist a neighborhood \( \mathcal{O}(x_0) \) of \( x_0 \) and an \( C^2 \)-diffeomorphism from \( G_1^+ \) onto \( \mathcal{O}(x_0) \cap \Omega \) such that \( \sigma(0) = 0 \). Similar to the case \( x_0 \in \Omega \), it suffices to show that

\[
\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{n/2}} \int_{\mathcal{O}(x_0) \cap \Omega} \left| e^{-\langle x - x_0, H(x - x_0) \rangle/\epsilon} \right|^2 \, dx = \frac{1}{2(\det H)^{1/2}} \left( \frac{\pi}{2} \right)^n. \quad (3.10)
\]

We now show (3.10). First of all, for any fixed \( \gamma \in (0, 1] \), it follows that

\[
\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{n/2}} \int_{\mathcal{O}(x_0) \cap \Omega} \left| e^{-\langle x - x_0, H(x - x_0) \rangle/\epsilon} \right|^2 \, dx = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{n/2}} \int_{G_1^+} \left| e^{-\sigma(\hat{x})^T H \sigma(\hat{x})/\epsilon} \right|^2 \left| \det \frac{\partial \sigma}{\partial \hat{x}} \right| \, d\hat{x}
\]

\[= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{n/2}} \int_{G_\gamma^+} \left| e^{-\sigma(\hat{x})^T H \sigma(\hat{x})/\epsilon} \right|^2 \left| \det \frac{\partial \sigma}{\partial \hat{x}} \right| \, d\hat{x}. \quad (3.11)\]

Here, we use the fact that the integrand in the integral \( \int_{G_1^+} \left| e^{-\sigma(\hat{x})^T H \sigma(\hat{x})/\epsilon} \right|^2 \left| \det \frac{\partial \sigma}{\partial \hat{x}} \right| \, d\hat{x} \) decays exponentially with respect to \( \epsilon \). Next, putting \( \tilde{H} \triangleq \frac{\partial \sigma}{\partial \hat{x}} \big|_{\hat{x}=0} \), similar to (3.8), we conclude that

\[
\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{n/2}} \int_{G_\gamma^+} \left| e^{-\sigma(\hat{x})^T H \sigma(\hat{x})/\epsilon} \right|^2 \left| \det \frac{\partial \sigma}{\partial \hat{x}} \right| \, d\hat{x} = |\det \tilde{H}| \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{n/2}} \int_{G_\gamma^+} \left| e^{-\sigma(\hat{x})^T H \sigma(\hat{x})/\epsilon} \right|^2 \, d\hat{x}. \quad (3.12)\]

It remains to compute the limit in the right hand side of (3.12). For this purpose, using Taylor’s formula and noting \( \sigma(0) = 0 \), for sufficiently small \( \gamma \in (0, 1) \), one has

\[ \sigma(\hat{x}) = \tilde{H}\hat{x} + O(|\hat{x}|^2), \quad \forall \hat{x} \text{ satisfying } |\hat{x}| < \gamma. \]
Hence, for any fixed (small) \( \beta \in (0, 1) \), there is a sufficiently small \( \gamma \in (0, 1) \) such that
\[
(1 - \beta)(\hat{H} \hat{x})^{T} H \hat{x} \leq \sigma(\hat{x})^{T} H \sigma(\hat{x}) \leq (1 + \beta)(\hat{H} \hat{x})^{T} H \hat{x}, \quad \forall \hat{x} \in G_{\gamma}^{+}. 
\]

Therefore,
\[
\int_{G_{\gamma}^{+}} \left| e^{-(1 - \beta)x^{T} \hat{H}^{T} H \hat{x}/\varepsilon} \right| d\hat{x} \geq \int_{G_{\gamma}^{+}} \left| e^{-\sigma(\hat{x})^{T} H \sigma(\hat{x})/\varepsilon} \right| d\hat{x} \geq \int_{G_{\gamma}^{+}} \left| e^{-(1 + \beta)x^{T} \hat{H}^{T} H \hat{x}/\varepsilon} \right| d\hat{x}. \tag{3.13} 
\]

However, by (3.9) (recall (2.1) for the definition of \( O_{\gamma}(0) \)),
\[
\frac{1}{e^{n/2}} \int_{G_{\gamma}^{+}} \left| e^{-(1 \pm \beta)x^{T} \hat{H}^{T} H \hat{x}/\varepsilon} \right| d\hat{x} = \frac{1}{2e^{n/2}} \int_{O_{\gamma}(0)} \left| e^{-(1 \pm \beta)x^{T} \hat{H}^{T} H \hat{x}/\varepsilon} \right| d\hat{x} 
\to \frac{1}{2((1 \pm \beta)^{n} \det(\hat{H}^{T} H \hat{H}))^{1/2}} \left( \frac{\pi}{2} \right)^{n/2}, \quad \text{as} \ \varepsilon \to 0. \tag{3.14} 
\]

Noting the arbitrariness of \( \beta \), combining (3.13) and (3.14), we end up with
\[
\lim_{\varepsilon \to 0} \frac{1}{e^{n/2}} \int_{G_{\gamma}^{+}} \left| e^{-\sigma(\hat{x})^{T} H \sigma(\hat{x})/\varepsilon} \right| d\hat{x} = \frac{1}{2(\det(\hat{H}^{T} H \hat{H}))^{1/2}} \left( \frac{\pi}{2} \right)^{n/2}. \tag{3.15} 
\]

Combining (3.11)–(3.12) and (3.15), and noting \( \frac{|\det \hat{H}|}{(\det(\hat{H}^{T} H \hat{H}))^{1/2}} = \frac{1}{(\det H)^{1/2}} \), we arrive at (3.10). Hence, the desired conclusion for the case \( x_{0} \in \Gamma \) follows. This completes the proof of Proposition 3.1. \( \square \)

Finally, we show the following technical result on the global existence of solutions to system (2.17), which is crucial for the definition of the ray of operator \( W \) given by (2.16).

**Proposition 3.2.** System (2.17) admits a global solution for any initial data, provided that \( A(x) \) grows linearly at infinity, i.e.,
\[
|A(x)| \leq C(1 + |x|), \quad \forall x \in \mathbb{R}^{n}. \tag{3.16} 
\]

**Proof.** By (2.18) and the strictly positive definite condition of \( A(x) \), we deduce that
\[
|\xi(t)| \leq C, \quad \forall t \in \mathbb{R}. \tag{3.17} 
\]

Now, the first equation in (2.17) along with (3.16)–(3.17) yields
\[
\frac{d}{dt} |x(t)|^{2} = 2\dot{x}(t) \cdot x(t) = 4(A(x(t))\xi(t)) \cdot x(t) \leq C(1 + |x(t)|^{2}), \quad \forall t \in \mathbb{R}. \tag{3.18} 
\]

Hence,
\[
|x(t)| \leq (1 + |x_{0}|)e^{C|t|}, \quad \forall t \in \mathbb{R}. \tag{3.19} 
\]

Now, (3.17) and (3.19) gives the global existence of solution to (2.17). \( \square \)
4. Proof of the positive observability/controllability result

This section is devoted to a proof of our positive observability/controllability result for heat equations with hyperbolic memory kernel, i.e., Theorem 2.1. By the standard duality argument [14,15], it suffices to show the first assertion in Theorem 2.1. We borrow some ideas from [7] and [26]. The proof is divided into several steps.

**Step 1.** First, put

\[ z(t, x) \triangleq p(T - t, x), \]  

(4.1)

where \( p \) solves system (1.4). Then, by (1.4), we see that \( z \) satisfies

\[
\begin{cases}
  z_t(t, x) - \sum_{i,j} (a_{ij}(x)) \int_0^t b(t - s, x) z_{x_i}(s, x) \, ds \bigg|_{x_j} = 0 & \text{in } Q, \\
  z = 0 & \text{on } \Sigma, \\
  z(0) = z_0 \triangleq p_0 & \text{in } \Omega.
\end{cases}
\]  

(4.2)

One can easily see that inequality (1.5) is equivalent to the following:

\[
\int_{\Omega} |z_0(x)|^2 \, dx \leq C \int_0^T \int_{\omega} |z(t, x)|^2 \, dt \, dx.
\]  

(4.3)

(Moreover, system (1.4) is equivalent to system (4.2).)

Next, set

\[
q(t, x) \triangleq \int_0^t b(t - s, x) z(s, x) \, ds,
\]  

(4.4)

Hence, by the condition \( b(0, x) \equiv 1 \) in (2.3), we see that

\[
q_t(t, x) = z(t, x) + \int_0^t b_t(t - s, x) z(s, x) \, ds,
\]  

(4.5)

By the second conclusion in Lemma 3.2 and noting \( q(0, x) \equiv 0 \), one can find a function \( \beta \in C^1(Q) \) such that

\[
z(t, x) = q_t(t, x) + \int_0^t \beta(t - s, x) q(s, x) \, ds
\]  

\[
= q_t(t, x) + \beta(0, x) q(t, x) + \int_0^t \beta_t(t - s, x) q(s, x) \, ds,
\]  

(4.6)

By (4.4) and (4.6), and noting again \( q(0, x) \equiv 0 \), it follows that
\[ \begin{align*}
zt(t, x) - \sum_{i,j} (a^{ij} q_{x_i})_{x_j} & = q_{tt} - \sum_{i,j} (a^{ij} q_{x_i})_{x_j} + \left[ \beta(0, x) q(t, x) + \int_0^t \beta_t(t - s, x) q(s, x) \, ds \right]_t \\
& + \sum_{i,j} \left\{ a^{ij}(x) \left[ b_{xi}(0, x) q(t, x) + \int_0^t \left( b_{txi}(t - s, x) + b_{xi}(t - s, x) \beta(0, x) \right. \right. \right. \\
& \left. \left. \left. + \int_s^t b_{xi}(t - \tau, x) \beta_t(\tau - s, x) \, d\tau \right) \right] \right\}_{x_j}. 
\end{align*} \] (4.7)

Put
\[ H^q(t, x) \triangleq \left[ \beta(0, x) q(t, x) + \int_0^t \beta_t(t - s, x) q(s, x) \, ds \right]_t \\
- \sum_{i,j} \left\{ a^{ij}(x) \left[ b_{xi}(0, x) q(t, x) + \int_0^t \left( b_{txi}(t - s, x) + b_{xi}(t - s, x) \beta(0, x) \right. \right. \right. \\
\left. \left. \left. + \int_s^t b_{xi}(t - \tau, x) \beta_t(\tau - s, x) \, d\tau \right) \right] \right\}_{x_j}. \] (4.8)

By (4.2), (4.5) and (4.7)–(4.8), we see that \( q \) solves
\[ \begin{cases} 
q_{tt} - \sum_{i,j} (a^{ij} q_{x_i})_{x_j} = H^q(t, x) & \text{in } Q, \\
q = 0 & \text{on } \Sigma, \\
q(0, x) = 0, \quad q_t(0, x) = z_0(x) & \text{in } \Omega.
\end{cases} \] (4.9)

**Step 2.** Recall (2.12) for the definitions of \( R_1 \) and \( T_0 \). Since \( T > T_0 \), we may assume in what follows that
\[ T > R_1. \] (4.10)

Consequently, one can choose a constant \( \delta \in (0, 2) \) so that
\[ \frac{R_2^2}{T^2} < \frac{\delta}{2} < \frac{R_1}{T}. \] (4.11)

Henceforth, we choose
\[ \phi(t, x) \triangleq d(x) - \frac{\delta}{2} t^2, \quad (t, x) \in Q, \] (4.12)
where \( d(\cdot) \) is the function given by (2.11). By (4.11), it follows that
\[
\phi(T, x) = d(x) - \frac{\delta}{2} T^2 \leq R_1^2 - \frac{\delta}{2} T^2 < 0, \quad \forall x \in \Omega.
\]

Therefore, there is a \( T_1 \in (0, T) \), such that
\[
\phi(t, x) < 0, \quad \forall (t, x) \in [T_1, T] \times \Omega. \tag{4.13}
\]

On the other hand, by (2.11) and (2.12), we see that \( R_0 > 0 \). Hence
\[
\phi(0, x) = d(x) \geq R_0^2 > 0, \quad \forall x \in \Omega.
\]

Therefore, one deduces that there exists a sufficiently small \( T_0 \in (0, T_1) \) such that
\[
\phi(t, x) \geq R_0^2/2, \quad \forall (t, x) \in [0, T_0] \times \Omega. \tag{4.14}
\]

By Lemma 3.1, with \( b^{ij} \) and \( w \) replaced by \( a^{ij} \) and \( q \) (defined by (4.4)), \( \delta \) and \( \phi \) given by (4.11) and (4.12), for any \( \varepsilon > 0 \) and \( \lambda > 0 \), we have

\[
\theta^2 \left[ q_{tt} - \sum_{i, j} (a^{ij} q_{x_i})_{x_j} \right] + M_t
+ 2 \sum_j \left\{\sum_{i, i', j'} a^{ij} \ell_{i}^{i'} \ell_{j}^{j'} v_{x_i} v_{x_j} - \sum_{i, i', j'} a^{ij} \ell_{i}^{i'} \ell_{j}^{j'} v_{x_i} v_{x_j} + \Psi v \sum_{i} a^{ij} v_{x_i} \right\}
- 2 \ell_{t} v_t \sum_{i} a^{ij} v_{x_i} + \sum_{i, i'} a^{ij} \ell_{i} v_{x_i} v_{x_i}' - \sum_{i} a^{ij} \left[ (A + \Psi) \ell_{i} + \frac{\Psi v_{x_i}}{2} v_{x_i}' \right] v_{x_i}'
\geq 2 \lambda \varepsilon v_t^2 + 2 \lambda \sum_{i, j} \left\{ \sum_{i', j'} \left[ 2a^{ij} (a^{i'j} d_{x_{i'}})_{x_j} - (a^{ij})_{x_j} a^{i'j} d_{x_{i'}} \right] - a^{ij} (2\delta + \varepsilon) \right\} v_{x_i} v_{x_j} + B v^2, \tag{4.15}
\]

where

\[
\begin{align*}
M & \triangleq 2 \left[ \ell_{t} \left( v_t^2 + \sum_{i, j} a^{ij} v_{x_i} v_{x_j} \right) - 2 \sum_{i, j} a^{ij} \ell_{i} v_{x_i} v_{x_j} v_t - \Psi v v_t + (A + \Psi) \ell_t v_t^2 \right], \\
\nu & = \theta q, \quad \theta = e^\ell, \quad \ell \triangleq \lambda \phi, \quad \Psi \triangleq \lambda \sum_{i, j} (a^{ij} d_{x_i})_{x_j} - \delta - \varepsilon,
\end{align*}
\]

\[
A = \lambda^2 \left( \delta^2 t^2 - \sum_{i, j} a^{ij} d_{x_i} d_{x_j} \right) + \lambda (2\delta + \varepsilon),
\]

\[
B = 2\lambda^3 \left[ (2\delta + \varepsilon) \sum_{i, j} a^{ij} d_{x_i} d_{x_j} + \sum_{i, j} a^{ij} d_{x_i} \left( \sum_{i', j'} a^{i'j'} d_{x_{i'}} d_{x_{j'}} \right)_{x_j} - (3\delta + \varepsilon) \delta^2 t^2 \right] + O(\lambda^5).
\]

We now choose (note \( \delta \in (0, 2) \))
\[
\varepsilon \in (0, 4 - 2\delta).
\]
Then, by (2.7) and the first condition of (2.11), recalling the definition of $a^{ij}$ in (2.5), and noting (2.2), we conclude that for some positive constant $c_0 > 0$, it holds

$$
\varepsilon \xi_0^2 + \sum_{i,j} \left[ \sum_{\ell',j'} (2a^{ij'j'}(a^{ij'}d_{x_i})_{x_j} - (a^{ij})_{x_j}a^{ij'}d_{x_i}) - a^{ij}(2\delta + \varepsilon) \right] \xi_i \xi_j \\
\geq \varepsilon \xi_0^2 + (4 - 2\delta - \varepsilon) \sum_{i,j} d^{ij}\xi_i \xi_j \geq c_0(\xi_0^2 + \cdots + \xi_n^2), \quad \forall (x, \xi_0, \xi_1, \ldots, \xi_n) \in \Omega \times \mathbb{R}^{1+n}. \quad (4.17)
$$

On the other hand, by [7, (11.6) in Appendix B, p. 1604], we have

$$
\mu_0 \sum_{i,j} a^{ij}d_{x_i}d_{x_j} \leq \sum_{i,j} a^{ij}d_{x_i} \left( \sum_{\ell',j'} a^{ij'}d_{x_i}d_{x_j} \right)_{x_j}. \quad (4.18)
$$

Therefore, by the definition of $B$ in (4.16), noting (2.11) and (2.12),

$$
B \geq 2\lambda^3 \left[ (2\delta + \varepsilon + \mu_0) \sum_{i,j} a^{ij}d_{x_i}d_{x_j} - (3\delta + \varepsilon)\delta^2 t^2 \right] + O(\lambda^3) \\
\geq 2(3\delta + \varepsilon) \lambda^3 \left[ \sum_{i,j} a^{ij}d_{x_i}d_{x_j} - \delta^2 t^2 \right] + O(\lambda^3) \\
\geq 2(3\delta + \varepsilon)(4R_1^2 - \delta^2 T^2)\lambda^3 + O(\lambda^3), \quad \forall (t, x) \in Q. \quad (4.19)
$$

By (4.11), we see that $4R_1^2 - \delta^2 T^2 > 0$. Hence, there are two constants $c_1 > 0$ and $\lambda_0 > 1$ such that for any $\lambda \geq \lambda_0$, it holds

$$
B \geq c_1 \lambda^3, \quad \forall (t, x) \in Q. \quad (4.20)
$$

**Step 3.** Integrating (4.15) on $Q_1 = (0, T_1) \times \Omega$, using integration by parts, by (4.17), (4.20) and recalling $\ell = \lambda \phi$ with $\phi$ given by (4.12), and noting the definition of $\Gamma_0$ in (2.8), $\sum_{i,j} a^{ij}v_i v_j = 1$ and $v_{x_i} = \partial v / \partial x_i$ on $\Sigma$ (which follows from (2.9) and $v|_{\Sigma} = 0$, respectively), we find that

$$
2c_0\lambda \int_{Q_1} \left( v_t^2 + |\nabla v|^2 \right) dt dx + c_1 \lambda^3 \int_{Q_1} v^2 dt dx \\
\leq \int_{Q_1} \theta^2 \left| q_{tt} - \sum_{i,j} \left( a^{ij}q_{x_i} \right)_{x_j} \right|^2 dt dx + \int_{\Omega} M(T_1, x) dx - \int_{\Omega} M(0, x) dx \\
+ 2\lambda \int_0^{T_1} \int_{\Gamma} \left( \sum_{i} a^{ij}v_i v_j \right) \left( \sum_{j'} a^{ij'}d_{x_j} v_{j'} \right) \left| \frac{\partial v}{\partial x} \right|^2 dt d\Gamma \\
\leq \int_{Q_1} \theta^2 \left| q_{tt} - \sum_{i,j} \left( a^{ij}q_{x_i} \right)_{x_j} \right|^2 dt dx + \int_{\Omega} M(T_1, x) dx - \int_{\Omega} M(0, x) dx \\
+ C\lambda \int_0^{T_1} \int_{\Gamma_0} \left| \frac{\partial v}{\partial x} \right|^2 dt d\Gamma_0, \quad \forall \lambda > \lambda_0. \quad (4.21)
$$
Denote the energy of system (4.9) by

\[
E(t) = \frac{1}{2} \int_\Omega \left[ |q_t(t, x)|^2 + |\nabla q(t, x)|^2 \right] \, dx. \tag{4.22}
\]

By (4.16) and recalling (4.12), noting \( \ell_t(0, x) \equiv 0 \) and \( q(0, x) \equiv 0 \), using (4.13) and Poincaré’s inequality, we have

\[
M(0, x) = 0, \quad \left| \int_\Omega M(T_1, x) \, dx \right| \leq C\lambda^3 E(T_1). \tag{4.23}
\]

However, by (4.9) and (4.8), noting \( b \in C^3(\overline{Q}) \) and hence \( \beta \in C^3(\overline{Q}) \), we have

\[
\int_{Q_1} \theta^2 \left| q_{tt} - \sum_{i,j} (a_{ij} q_{x_i})_{x_j} \right|^2 \, dt \, dx
\]

\[
\leq \int_{Q_1} \theta^2(t, x) \left[ \beta(0, x)q(t, x) + \int_0^t \beta_t(t - s, x)q(s, x) \, ds \right] \left[ b_{x_i} (0, x)q(t, x) + \int_0^t b_{tx_i} (t - s, x) + b_{x_i} (t - s, x) \beta(0, x) \right.
\]

\[
+ \left. \int_s^t b_{x_i} (t - \tau, x) \beta_t (\tau - s, x) \, d\tau \right] q(s, x) \, ds \right] \, dx \, dt
\]

\[
\leq C \int_{Q_1} \theta^2(t, x) \left\{ q^2(t, x) + q_t^2(t, x) + |\nabla q(t, x)|^2 + \int_0^t [q^2(s, x) + |\nabla q(s, x)|^2] \, ds \right\} \, dt \, dx. \tag{4.24}
\]

On the other hand, we note

\[
\int_{Q_1} \theta^2(t, x) \left\{ \int_0^t [q^2(s, x) + |\nabla q(s, x)|^2] \, ds \right\} \, dt \, dx
\]

\[
\leq \int_{Q_1} \theta^2(t, x) \left[ q^2(s, x) + |\nabla q(s, x)|^2 \right] \, ds \, dt \, dx
\]

\[
\leq C \int_{Q_1} \theta^2(t, x) \left[ q^2(t, x) + |\nabla q(t, x)|^2 \right] \, dt \, dx.
\]

Therefore,

\[
\int_{Q_1} \theta^2 \left| q_{tt} - \sum_{i,j} (a_{ij} q_{x_i})_{x_j} \right|^2 \, dt \, dx \leq C \int_{Q_1} \theta^2 \left( q^2 + q_t^2 + |\nabla q|^2 \right) \, dt \, dx. \tag{4.24}
\]
Further, by \( q = e^{-\lambda \phi} v \), \( \theta = e^{\lambda \phi} \) and noting (4.12), using \( q|_{\Sigma} = 0 \), we get

\[
\int_{Q_1} \theta^2 (q_t^2 + |\nabla q|^2) \, dt \, dx + \lambda^2 \int_{Q_1} \theta^2 q^2 \, dt \, dx \leq C \left[ \int_{Q_1} (v_t^2 + |\nabla v|^2) \, dt \, dx + \lambda^2 \int_{Q_1} v^2 \, dt \, dx \right],
\]

(4.25)

and

\[
\int_{0}^{T_1} \int_{R_0} |\partial v/\partial n| \, dt \, d\Gamma_0 \leq C e^{C_{1} \lambda} \int_{0}^{T_1} \int_{R_0} |\partial q/\partial v| \, dt \, d\Gamma_0.
\]

(4.26)

Then, by (4.21), (4.23), (4.24), (4.25) and (4.26), we conclude that

\[
\lambda \int_{Q_1} \theta^2 (q_t^2 + |\nabla q|^2) \, dt \, dx + \lambda^3 \int_{Q_1} \theta^2 q^2 \, dt \, dx \leq C_1 \left[ \int_{Q_1} \theta^2 (q_t^2 + q_t^2 + |\nabla q|^2) \, dt \, dx + \lambda^3 E(T_1) + \lambda e^{C_{1} \lambda} \int_{0}^{T_1} \int_{R_0} |\partial q/\partial v| \, dt \, d\Gamma_0 \right], \quad \forall \lambda \geq \lambda_0.
\]

(4.27)

where \( C_1 > 0 \) is a constant, independent of \( \lambda \). By choosing \( \lambda \) sufficiently large, the term \( "C_1 \int_{Q_1} \theta^2 (q_t^2 + q_t^2 + |\nabla q|^2) \, dt \, dx " \) in the right hand side of (4.27) can be absorbed by its left hand side. Therefore, there is a constant \( \lambda_1 \geq \lambda_0 \) such that

\[
\int_{Q_1} \theta^2 (q_t^2 + |\nabla q|^2) \, dt \, dx \leq C \left[ \lambda^2 E(T_1) + e^{C_{1} \lambda} \int_{0}^{T_1} \int_{R_0} |\partial q/\partial v| \, dt \, d\Gamma_0 \right], \quad \forall \lambda \geq \lambda_1.
\]

(4.28)

However, by (4.14), we have

\[
\int_{Q_1} \theta^2 (q_t^2 + |\nabla q|^2) \, dt \, dx \geq e^{R_0^2 \lambda} \int_{0}^{T_0} \int_{\Omega} (q_t^2 + |\nabla q|^2) \, dt \, dx = 2e^{R_0^2 \lambda} \int_{0}^{T_0} E(t) \, dt.
\]

(4.29)

Thus, by (4.28)--(4.29), we conclude that

\[
e^{R_0^2 \lambda} \int_{0}^{T_0} E(t) \, dt \leq C \left[ \lambda^2 E(T_1) + e^{C_{1} \lambda} \int_{0}^{T_1} \int_{R_0} |\partial q/\partial v| \, dt \, d\Gamma_0 \right], \quad \forall \lambda \geq \lambda_1.
\]

(4.30)

Now, recalling that \( \omega = \mathcal{O}_{b_0}(R_0) \cap \Omega \), similar to [7] and [26], using Lemma 3.3 and noting (4.9) and (4.4), one can show that

\[
\int_{0}^{T_1} \int_{R_0} |\partial q/\partial v| \, dt \, d\Gamma_0 \leq C \int_{\omega}^{T} \int_{\omega} (q_t^2 + q_t^2) \, dt \, dx \leq C \int_{\omega}^{T} \int_{\omega} z^2 \, dt \, dx.
\]

(4.31)
Combining (4.30) and (4.31), it follows

\[
e^{R_0^2 \lambda} \int_0^T E(t) \, dt \leq C \left[ \lambda^2 E(T_1) + e^{C\lambda} \int_0^T \int_\omega z^2 \, dt \, dx \right], \quad \forall \lambda \geq \lambda_1.
\] (4.32)

**Step 4.** Let us complete the proof of Theorem 2.1. First of all, multiplying the first equation of (4.9) by \(q_t\), integrating it on \((0, t) \times \Omega\), noting that \(a_{ij}^{ij} = a_{ji}^{ij}\) and the boundary condition in (4.9), and using (4.22), (4.8) and Poincaré’s inequality, we obtain that

\[
E(t) - E(0) = \int_0^t \int_\Omega q_t H^q \, dt \, dx
\]

\[
\leq C \int_0^t \int_\Omega \left( |H^q(t, x)|^2 + q_t^2(t, x) \right) \, dt \, dx
\]

\[
\leq C \int_0^t \int_\Omega \left\{ q^2(t, x) + q_t^2(t, x) + |\nabla q(t, x)|^2 + \int_0^t \left[ q^2(s, x) + |\nabla q(s, x)|^2 \right] ds \right\} \, dt \, dx
\]

\[
\leq C \int_0^t \int_\Omega \left( q^2(t, x) + q_t^2(t, x) + |\nabla q(t, x)|^2 \right) \, dt \, dx
\]

\[
\leq C \int_0^t E(s) \, ds, \quad \forall t \in [0, T].
\] (4.33)

Therefore, by Gronwall’s inequality, we conclude that

\[
E(t) \leq CE(0), \quad \forall t \in [0, T].
\] (4.34)

Due to the time-reversibility of system (4.9), similar to (4.33), one gets

\[
E(0) \leq E(t) + C \int_0^t \int_\Omega |H^q(t, x)|^2 \, dx \, dt \leq E(t) + C \int_0^t E(s) \, ds, \quad \forall t \in [0, T].
\] (4.35)

Note that, by (4.34), one has

\[
\int_0^t E(s) \, ds \leq CtE(0), \quad \forall t \in [0, T].
\] (4.36)

Thus, by (4.35) and (4.36), there exists a sufficiently small \(t_0 \in (0, T_0)\) such that

\[
E(0) \leq CE(t), \quad \forall t \in [0, t_0].
\] (4.37)
Finally, in view of (4.32), and noting (4.34) and (4.37), we see that
\[
e^{R_{\Omega}^2} E(0) \leq C_2 \left[ \lambda^2 E(0) + e^{C_2 \lambda} \int_0^T \int_\omega z^2 \, dt \, dx \right], \quad \forall \lambda \geq \lambda_1.
\]

(4.38)

Therefore, if we take \( \lambda \geq \lambda_1 \) so that \( e^{R_{\Omega}^2} \geq C_2 \lambda^2 + 1 \), from (4.38), we deduce that
\[
E(0) \leq C \int_0^T \int_\omega z^2 \, dt \, dx.
\]

(4.39)

Recalling (4.22) and the third equation in (4.9), we see that (4.39) implies (4.3) immediately, which completes the proof of Theorem 2.1.

5. Highly concentrated approximate solutions for hyperbolic equations with memory in \( \mathbb{R}^n \)

As a preliminary to prove Theorem 2.2, we construct in this section highly concentrated approximate solutions for hyperbolic equations with memory in \( \mathbb{R}^n \).

Fix \( T > 0 \), and put
\[
\Lambda_T^+ = \{ (t, s) \in (0, T) \times (0, T) \mid 0 < s < t < T \}, \\
\Lambda_T^- = \{ (t, s) \in (-T, 0) \times (-T, 0) \mid -T < t < s < 0 \}, \\
\Lambda_T = \Lambda_T^+ \cup \Lambda_T^-, \quad \Upsilon_T = (-T, 0) \cup (0, T).
\]

(5.1)

One can check that \( W^{1,\infty}(-T, T) \hookrightarrow W^{1,\infty}(\Upsilon_T) \), but \( W^{1,\infty}(\Upsilon_T) \neq W^{1,\infty}(-T, T) \) although the only difference between \( \Upsilon_T \) and \( (-T, T) \) is the point \( \{0\} \). Indeed, the function \( f \) defined by
\[
f(x) = \begin{cases} 1, & x \in (0, T), \\ 0, & x \in (-T, 0) \end{cases}
\]

belongs to \( W^{1,\infty}(\Upsilon_T) \), but not in \( W^{1,\infty}(-T, T) \).

We consider the following hyperbolic equation with memory in \( \mathbb{R}^n \):
\[
W u = F(u), \quad \text{in} \ \Upsilon_T \times \mathbb{R}^n,
\]

(5.2)

where \( W \) is defined by (2.16),
\[
F(u) \triangleq B_1(t, x)u(t, x) + B_2(t, x)u_t(t, x) + B_3(t, x) \cdot \nabla u(t, x) \\
+ \int_0^t \left[ B_4(t, s, x)u(s, x) + B_5(t, s, x)u_s(s, x) + B_6(t, s, x) \cdot \nabla u(s, x) \right] \, ds,
\]

(5.3)

and \( B_1, B_2, B_3, B_4, B_5 \) and \( B_6 \) are some given functions defined on \( \Upsilon_T \times \mathbb{R}^n \) and/or \( \Lambda_T \times \mathbb{R}^n \). Here, \( a \cdot b \) stands for the usual scalar product of \( a \) and \( b \) in \( \mathbb{R}^n \).
Remark 5.1. From Step 1 in the proof of Theorem 2.1, and noting the transformations (4.1), (4.4) and (4.6), it is easy to see that the original heat equation with hyperbolic memory kernel, i.e. system (1.4), is equivalent to system (4.9). Note that, by (4.8), the first equation of (4.9) is a hyperbolic equation with memory.

The main purpose of this section is to construct approximate solutions for system (5.2) so that their energies are highly concentrated in a small neighborhood of some ray of operator \( W \). Given a ray \((t, x(t))\) of operator \( W \), we construct first a family of highly localized approximate solutions to Eq. (5.2) in the following form

\[
 w_\varepsilon(t, x) = \varepsilon^{1-\frac{n}{2}} \left[ a(t) e^{i\phi(t,x)/\varepsilon} + \varepsilon \int_0^t A(t,s) e^{i\phi(s,x)/\varepsilon} \, ds \right], \quad \varepsilon \in (0, 1). \tag{5.4}
\]

In (5.4), we take the phase function \( \phi \) to be of the form

\[
 \phi(t, x) = \xi(t)^T (x-x(t)) + \frac{1}{2} (x-x(t))^T M(t)(x-x(t)), \tag{5.5}
\]

where \( M(t) \) is an \((n \times n)\) complex symmetric matrix (i.e., \( M(t)^T = M(t) \)) with positive definite imaginary part (we shall denote this by \( \text{Im} \, M(t) > 0 \)). Note that \( M(t) \) is not self-adjoint. The construction of approximate solutions (5.4) requires an appropriate choice of \( a(t), \, A(t,s), \) and \( M(t) \).

Remark 5.2. Compared with the system without memory, say \( W u = 0 \) in \((0, T) \times \mathbb{R}^n\) considered in [23], the main change on the ansatz of approximate solutions \( u = w_\varepsilon \) to system (5.2) is that we introduce a new memory term \( \varepsilon \int_0^t A(t,s) e^{i\phi(s,x)/\varepsilon} \, ds \), which will play a crucial role in treating the present memory situation. As we shall see later also that this will lead to some technical complexity for the analysis.

We need the following result (recall (5.1) for \( \Upsilon_T \) and \( \Lambda_T \)):

**Theorem 5.1.** Let \( a^{ij} \in W^{3,\infty}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n) \) \((i, j = 1, 2, \ldots, n)\), and \((t, x(t))\) be a ray of operator \( W \). For given \( T > 0 \), let \( B_1 \in L^\infty(\Upsilon_T \times \mathbb{R}^n), \, B_2 \in W^{1,\infty}(\Upsilon_T \times \mathbb{R}^n), \, B_3 \in W^{1,\infty}(\Upsilon_T \times \mathbb{R}^n), \, B_4 \in L^\infty(\Lambda_T \times \mathbb{R}^n), \, B_5 \in W^{2,\infty}(\Lambda_T \times \mathbb{R}^n) \) and \( B_6 \in W^{2,\infty}(\Lambda_T \times \mathbb{R}^n) \cap W^{2,\infty}(\Upsilon_T) \) and a complex-valued function \( a(\cdot) \in C([-T, T], \mathbb{C}) \cap \{0\} \) given by (5.7) for \( a(\cdot) \) is a ray of operator \( W \) with

\[
 M(0) = M_0, \quad \text{Im} \, M(t) > 0 \quad \text{for all} \quad t \in [-T, T], \quad a(t_0) = a_0, \tag{5.6}
\]

such that

1. The family \( \{w_\varepsilon\}_{\varepsilon > 0} \) given by (5.4) are approximate solutions of (5.2) in the sense that

\[
 \text{ess sup}_{t \in \Upsilon_T} |W w_\varepsilon(t, \cdot) - F(w_\varepsilon(t, \cdot))|_{L^2(\mathbb{R}^n)} = O(\varepsilon^{\frac{1}{2}}), \quad \text{as} \ \varepsilon \rightarrow 0; \tag{5.7}
\]

2. The initial energy of \( w_\varepsilon \) is bounded below as \( \varepsilon \to 0 \), i.e.,

\[
 \left| \left| w_\varepsilon(0, \cdot), \partial_t w_\varepsilon(0, \cdot) \right| \right|_{H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} \geq c_0 \tag{5.8}
\]

for some \( c_0 > 0 \), independent of \( \varepsilon \);
(3) The energy of \( w_\varepsilon \) is polynomially small off the ray \((t, x(t))\) (recall (2.1) for the definition of \( \mathcal{O}_{x^{1/4}}(x(t)) \)):

\[
\text{ess sup}_{t \in \mathcal{T}} \int_{\mathbb{R}^n \setminus \mathcal{O}_{x^{1/4}}(x(t))} \left[ |\partial_t w_\varepsilon(t, x)|^2 + |w_\varepsilon(t, x)|^2 + |\nabla w_\varepsilon(t, x)|^2 \right] \, dx = O(\varepsilon^2), \quad \text{as} \ \varepsilon \to 0. \tag{5.9}
\]

**Proof.** The proof is divided into two steps.

**Step 1.** Assignment of \( M(t), a(t) \) and \( A(t, s) \). Let \( w_\varepsilon \) be of the form (5.4). By a direct computation and noting that \( M(t), a(t) \) and \( A(t, s) \) will be chosen to be independent of \( x \), we have

\[
W \cdot w_\varepsilon - F\left(w_\varepsilon\right) = \varepsilon^{2-\frac{n}{2}} r_1 + \varepsilon^{1-\frac{n}{2}} r_2 + \varepsilon^{-\frac{n}{2}} r_3 + \varepsilon^{-1-\frac{n}{2}} r_4, \tag{5.10}
\]

where

\[
\begin{align*}
 r_1 & \triangleq \left[A(t(t), t) + A_t(t, \tau)\right]_{\tau=t} - B_2(t, x)A(t, t)\right]e^{i\phi(t, x)/\varepsilon} \\
 & + \int_0^t \left[A_{tt}(t, s) - B_1(t, x)A(t, s) - B_2(t, x)A_t(t, s) - B_5(t, s, x)A(s, s)\right]e^{i\phi(s, x)/\varepsilon} \, ds \\
 & - \int_0^t \int_0^s \left(B_4(t, s, x)A(s, \tau) + B_5(t, s, x)A_3(s, \tau)\right)e^{i\phi(t, x)/\varepsilon} \, d\tau \, ds, \tag{5.11}
\end{align*}
\]

\[
\begin{align*}
 r_2 & \triangleq [a_{tt}(t) + iA(t, t)\phi_t(t, x) - B_1(t, x)a(t) - B_2(t, x)a_t(t)]e^{i\phi(t, x)/\varepsilon} \\
 & - \int_0^t \left[i \sum_{i,j} a^{ij}(x)A(s, s)\phi_{x_i x_j}(s, x) + iA(t, s)B_3(t, x) \cdot \nabla \phi(s, x) \\
 & + B_4(t, s, x)a(s)\right]e^{i\phi(s, x)/\varepsilon} + iB_6(t, s, x) \cdot \int_0^s A(s, \tau)e^{i\phi(t, x)/\varepsilon} \, d\tau \, ds, \tag{5.12}
\end{align*}
\]

\[
\begin{align*}
 r_3 & \triangleq i \left[2a_t(t)\phi_t(t, x) + a(t)W\phi(t, x) - a(t)\left(B_2(t, x)\phi_t(t, x) + B_3(t, x) \cdot \nabla \phi(t, x)\right)\right]e^{i\phi(t, x)/\varepsilon} \\
 & + \int_0^t A_t(t, s) \sum_{i,j} a^{ij}(x)\phi_{x_i}(s, x)\phi_{x_j}(s, x) - i\varepsilon\phi_2^2(t, x) \cdot \nabla \phi(s, x) \\
 & - i\varepsilon\phi_2(t, x) \cdot \nabla \phi(s, x)\right]e^{i\phi(s, x)/\varepsilon} \, ds, \tag{5.13}
\end{align*}
\]

and

\[
\begin{align*}
 r_4 & \triangleq a(t) \left[\sum_{i,j} a^{ij}(x)\phi_{x_i}(t, x)\phi_{x_j}(t, x) - \phi_t^2(t, x)\right]e^{i\phi(t, x)/\varepsilon}. \tag{5.14}
\end{align*}
\]

First, by \([17,22]\), we choose \( M(t) \in C^2([-T, T]; \mathbb{C}^{n \times n}) \) appearing in (5.5) (defining function \( \phi \)), with \( M(t_0) = M_0 \) and \( \text{Im} M(t) > 0 \), so that for each fixed \( t \in [-T, T] \), it holds

\[
\sum_{i,j} a^{ij}(x)\phi_{x_i}(t, x)\phi_{x_j}(t, x) - \phi_t^2(t, x) = O\left(|x - x(t)|^3\right), \quad \text{as} \ x \to x(t). \tag{5.15}
\]
Indeed, $M(t)$ can be uniquely determined as the solution of the following Riccati equation:

$$\begin{cases}
\dot{M}(t) + M(t)G_1(t)M(t) + G_2(t)M(t) + M(t)G_2(t)^\top + G_3(t) = 0, \\
M(t_0) = M_0,
\end{cases}$$

(5.16)

where $G_1(t)$, $G_2(t)$ and $G_3(t)$ are suitable $(n \times n)$ matrices whose coefficients are determined by the first and second derivatives of the function $g(x, \xi)$ evaluated along the ray $(t, x(t), \xi(t))$ (recall (2.13) for $g(x, \xi)$). We refer to [22] for the global existence of solutions to this nonlinear ordinal differential equations with any initial data $M_0$ satisfying $\text{Im} M_0 > 0$. By Lemma 3.4 and noting (5.15), we find

$$|r_4(t, \cdot)|_{L^2(\mathbb{R}^n)} = O\left(\varepsilon^{\frac{n+1}{2}}\right), \text{ uniformly for a.e. } t \in \mathcal{T}_\tau. \quad (5.17)$$

Next, we choose $a(\cdot) \in C([-T, T] ; \mathbb{C} \setminus \{0\}) \cap W^{2,\infty}(\mathcal{T}_\tau)$ with $a(t_0) = a_0$ so that

$$2\dot{a}(t)\phi_i(t, x) + a(t)W\phi(t, x) - a(t)\left(B_2(t, x)\phi_i(t, x) + B_3(t, x) \cdot \nabla \phi(t, x)\right)$$

$$= O\left(\varepsilon |x - x(t)|\right), \text{ as } x \to x(t), \text{ uniformly for a.e. } t \in \mathcal{T}_\tau. \quad (5.18)$$

For this purpose, we note that, by (5.5), (2.17) and (2.18), it follows

$$\phi_i(t, x(t)) = -\frac{1}{2}, \quad \nabla \phi(t, x(t)) = \xi(t), \quad \forall t \in \mathbb{R}. \quad (5.19)$$

Hence, $a(t)$ is determined by the following linear ordinal differential equation:

$$\begin{cases}
\dot{a}(t) = a(t)\left[\frac{1}{2}B_2(t, x(t)) - B_3(t, x(t)) \cdot \xi(t)\right], \\
a(t_0) = a_0.
\end{cases}$$

(5.20)

By Lemma 3.4 and noting (5.18), we get

$$\int_{\mathbb{R}^n} \left[2\dot{a}(t)\phi_i(t, x) + a(t)W\phi(t, x) - a(t)\left(B_2(t, x)\phi_i(t, x) + B_3(t, x) \cdot \nabla \phi(t, x)\right)\right] e^{i\phi(t, x)/\varepsilon} dx = O\left(\varepsilon^{\frac{n+1}{2}}\right). \quad (5.21)$$

Finally, by (5.15) and (5.19), we have

$$\sum_{i, j} d_{ij}^j(x(s))\phi_{x_i}(s, x(s))\phi_{x_j}(s, x(s)) = \phi_1^2(s, x(s)) = \frac{1}{4}. \quad (5.22)$$

Hence, by choosing

$$A(t, s) = -2ia(s)B_5(t, s, x(s)) + 4ia(s)B_6(t, s, x(s)) \cdot \xi(s), \quad (5.23)$$

and noting again (5.19), we find that

$$A(t, s) \sum_{i, j} d_{ij}^j(x)\phi_{x_i}(s, x)\phi_{x_j}(s, x) - ia(s)B_5(t, s, x)\phi_5(s, x) - ia(s)B_6(t, s, x) \cdot \nabla \phi(s, x)$$

$$= O\left(|x - x(s)|\right), \text{ as } x \to x(s), \text{ uniformly for a.e. } t, s \in \mathcal{A}_T. \quad (5.23)$$
Therefore, by Lemma 3.4, noting (5.23) and $\text{Im } M(s) > 0$, and recalling that

$$\text{Im } \phi(s, x) = \frac{1}{2} (x - x(s))^T \text{Im } M(s)(x - x(s)),$$

we have

$$\int_{\mathbb{R}^n} \left| \int_0^t \left[ A(t, s) \sum_{i,j} a^{ij}(x) \phi_{x_i}(s, x) \phi_{x_j}(s, x) - i a(s) B_2(t, s, x) \phi_2(s, x) \right] e^{i \phi(s, x)/\varepsilon} \, ds \right|^2 \, dx$$

$$\leq C \int \int_{\mathbb{R}^n} \left| A(t, s) \sum_{i,j} a^{ij}(x) \phi_{x_i}(s, x) \phi_{x_j}(s, x) - i a(s) B_2(t, s, x) \cdot \nabla \phi(s, x) \right| e^{i \phi(s, x)/\varepsilon} \, ds \, dx$$

$$= C \int \int_{\mathbb{R}^n} \left[ A(t, s) \sum_{i,j} a^{ij}(x) \phi_{x_i}(s, x) \phi_{x_j}(s, x) - i a(s) B_2(t, s, x) \cdot \nabla \phi(s, x) \right] e^{-\frac{\text{Im } \phi(s, x)}{\varepsilon}} \, ds \, ds$$

$$= C \int_0^T \int_{\mathbb{R}^n} O \left( \varepsilon^{\frac{n}{2} + 1} \right) \, ds = O \left( \varepsilon^{\frac{n}{2} + 1} \right).$$

(5.24)

Combining (5.21) and (5.24), we conclude that

$$|r_3(t, \cdot)|_{L^2(\mathbb{R}^n)} = O \left( \varepsilon^{\frac{n}{2} + 1} \right), \quad \text{uniformly for a.e. } t \in \mathcal{T}_T.$$  

(5.25)

Also, using Lemma 3.4 again, one has

$$|r_1(t, \cdot)|_{L^2(\mathbb{R}^n)} + |r_2(t, \cdot)|_{L^2(\mathbb{R}^n)} = O \left( \varepsilon^{\frac{n}{2}} \right), \quad \text{uniformly for a.e. } t \in \mathcal{T}_T.$$  

(5.26)

**Step 2. Verification of (5.7)–(5.9).** First, the first conclusion in Theorem 5.1, i.e., (5.7), follows from (5.10), (5.17), (5.25) and (5.26).

Next, we note that, compared to its leading term $\varepsilon^{1 - \frac{n}{4}} a(t) e^{i \phi(t, x)/\varepsilon}$, the memory term $\varepsilon^{2 - \frac{n}{4}} \int_0^t A(t, s) e^{i \phi(t, x)/\varepsilon} \, ds$ in $w_\varepsilon$ (given by (5.4)) is a higher order one with respect to $\varepsilon$. Therefore, proceeding as in the proof of [17, Theorem 1], one arrives at the second conclusion, (5.8), in Theorem 5.1. Finally, by the definition of $w_\varepsilon(t, x)$ in (5.4), we have

$$\left\{ \begin{array}{l}
\partial_t w_\varepsilon(t, x) = \varepsilon^{2 - \frac{n}{4}} \left[ A(t, t) e^{i \phi(t, x)/\varepsilon} + \int_0^t A(t, s) e^{i \phi(s, x)/\varepsilon} \, ds \right] \\
\quad + \varepsilon^{1 - \frac{n}{4}} a(t) (t) e^{i \phi(t, x)/\varepsilon} + \varepsilon \int_0^t e^{i \phi(s, x)/\varepsilon} A(t, s) \nabla \phi(s, x) \, ds,
\end{array} \right.$$  

(5.27)

Hence,
\[
\text{ess sup}_{t \in T_T} \int_{\mathbb{R}^n \setminus O_x,4(x(t))} \left[ \left| \partial_t w_{\epsilon}(t, x) \right|^2 + \left| w_{\epsilon}(t, x) \right|^2 + \left| \nabla w_{\epsilon}(t, x) \right|^2 \right] dx \\
\leq C \text{ess sup}_{t \in T_T} \int_{\mathbb{R}^n \setminus O_x,4(x(t))} e^{-\frac{2}{\epsilon} \Im(\phi(s,x))} dx + \epsilon^{2 - \frac{2}{\epsilon}} \int_{\mathbb{R}^n \setminus O_x,4(x(t))} \left| \int_0^t e^{-2 \Im(\phi(s,x))} ds \right| dx.
\]
\[
(5.28)
\]

As shown in the proof of [17, Theorem 1], there is a constant \(\beta > 0\), independent of \(\epsilon\), such that
\[
\epsilon^{-\frac{2}{\epsilon}} \text{ess sup}_{t \in T_T} \int_{\mathbb{R}^n \setminus O_x,4(x(t))} e^{-2 \Im(\phi(s,x))} dx \leq Ce^{-\beta/\sqrt{\epsilon}}.
\]
\[
(5.29)
\]

On the other hand, by Lemma 3.4, one deduces that
\[
\text{ess sup}_{t \in T_T} \int_{\mathbb{R}^n \setminus O_x,4(x(t))} \left| \int_0^t e^{-2 \Im(\phi(s,x))} ds \right| dx = \text{ess sup}_{t \in T_T} \left| \int_0^t \int_{\mathbb{R}^n} e^{-2 \Im(\phi(s,x))} dx ds \right|
\leq \text{ess sup}_{t \in T_T} \left| \int_0^t \int_{\mathbb{R}^n} e^{-2 \Im(\phi(s,x))} dx ds \right| \leq Ce^{n/2}.
\]
\[
(5.30)
\]

Now, combining (5.28)-(5.30), we conclude the desired estimate (5.9). This completes the proof of Theorem 5.1. \(\square\)

Now, several remarks are in order.

**Remark 5.3.** From (5.16), we see that \(M(t)\) is uniquely determined by its initial data \(M_0\) and the ray \((t, x(t), \xi(t))\). On the other hand, by (5.20) and (5.22), we see that \(a(t)\) and \(A(t, s)\) are uniquely determined by the initial data \(a_0\) of \(a(t)\) and the ray \((t, x(t), \xi(t))\). Therefore, whenever the ray \((t, x(t), \xi(t))\) is given, \(M(t)\) and \(a(t)\) (hence also \(A(t, s)\)) can be uniquely determined respectively by their values \(M_0\) (with \(\text{Im} M_0 > 0\)) and \(a_0\) at any instant \(t_0 \in \mathbb{R}\).

**Remark 5.4.** From (5.22), we see that \(A(t, s)\) only depends on \(B_5(t, s, x)\) and \(B_6(t, s, x)\). Hence, if \(B_5(t, s, x) \equiv 0\) and \(B_6(t, s, x) \equiv 0\) in \(\Lambda_T \times \mathbb{R}^n\), then it is not necessary to introduce the memory term \(\epsilon \int_0^t A(t, s) e\phi(s,x) ds\) in (5.4).

**Remark 5.5.** From the proof of [17, Theorem 1], one sees that (5.8) in Theorem 5.1 can be improved as follows:
\[
\left| \partial_t w_{\epsilon}(0, \cdot) \right|_{L^2(\mathbb{R}^n)} \geq c_0.
\]
\[
(5.31)
\]

We shall use this fact later in an essential way.

**Remark 5.6.** Due to the appearance of the memory term in Eq. (5.2), the estimate (5.9) for \(w_{\epsilon}\) is much weaker than its counterpart in [17, Theorem 1]. Indeed, for the case without memory in [17],
the energy of the similar highly concentrated approximate solutions is exponentially small off the ray \((t, x(t))\), i.e.,

\[
\text{ess sup}_{t \in (0, T)} \int_{\mathbb{R}^n \setminus O_{1/4}(x(t))} \left[ |\partial_t w_{\varepsilon} (t, x)|^2 + |w_{\varepsilon} (t, x)|^2 + |\nabla w_{\varepsilon} (t, x)|^2 \right] dx = O \left( e^{-\beta/\sqrt{\varepsilon}} \right)
\]

for some \(\beta > 0\) as \(\varepsilon \to 0\). But, from (5.30), it seems that (5.9) is sharp. That is, in the present case, the energy of \(w_{\varepsilon}\) is only polynomially small off the ray \((t, x(t))\). Note however that, by denoting the first term of \(w_{\varepsilon} (t, x)\) in (5.4) by

\[
z_{\varepsilon} (t, x) = \varepsilon^{1-\frac{3}{4}} a(t) e^{i \phi(t, x)/\varepsilon}, \tag{5.32}
\]

similar to the proof of [17, Theorem 1], one deduces that the energy of \(z_{\varepsilon}\) is exponentially small off the ray \((t, x(t))\). More precisely, for some \(\beta > 0\), it holds

\[
\text{ess sup}_{t \in T} \int_{\mathbb{R}^n \setminus O_{1/4}(x(t))} \left[ |\partial_t z_{\varepsilon} (t, x)|^2 + |z_{\varepsilon} (t, x)|^2 + |\nabla z_{\varepsilon} (t, x)|^2 \right] dx = O \left( e^{-\beta/\sqrt{\varepsilon}} \right), \tag{5.33}
\]

Recall that, as shown in [17], for any given cut-off function \(\rho_0 \in C_0^\infty (\mathbb{R}^{1+n})\) which is identically equal to 1 in a neighborhood of the ray \(\{(t, x(t)) \mid t \in [-T, T]\}\), one may check that the function \(\rho_0 w_{\varepsilon}\) also satisfies the corresponding estimates (5.7)–(5.9) for the case without memory and lower order terms in (5.2) (i.e., the \(B_i \equiv 0\) in (5.3) for \(i = 1, \ldots, 6\)). Hence, in this case, one can easily choose \(w_{\varepsilon}\) such that they are supported in any given small neighborhood of the ray. Note however that, as shown in the proof of the next theorem, in the present case with memory, in order to construct highly concentrated approximate solutions for Eq. (5.2), we have to correct some terms in \(w_{\varepsilon}\) (given by (5.4)) rather than simply multiplying it by a cut-off function. Indeed, for any given \(S \in (-T, T)\) and any given cut-off function \(\varrho = \varrho(t, x) \in C_0^\infty (\mathbb{R}^{1+n})\) which is identically equal to 1 in a neighborhood of the ray \(\{(t, x(t)) \mid t \in [S, T]\}\), put

\[
u_{\varepsilon} \triangleq \varepsilon^{1-\frac{3}{4}} \left[ \varrho(t, x) a(t) e^{i \phi(t, x)/\varepsilon} + \varepsilon \int_0^t \varrho(s, x) A(t, s) e^{i \phi(s, x)/\varepsilon} ds \right], \tag{5.34}
\]

where \(\phi\) is given by (5.5). Clearly, \(\nu_{\varepsilon} \neq \varrho w_{\varepsilon}\). We have he following result:

**Theorem 5.2.** Let the assumptions in Theorem 5.1 hold, and the complex-valued symmetric matrix \(M(\cdot) \in C^2([-T, T]; \mathbb{C}^{n \times n})\), the complex-valued function \(a(\cdot) \in C([-T, T]; \mathbb{C} \setminus \{0\}) \cap W^{2,\infty}(\mathcal{Y}_T)\) and the complex-valued function \(A(\cdot, \cdot) \in W^{2,\infty}(\Lambda_T)\) be constructed as in Theorem 5.1. Then, for sufficiently small \(\varepsilon > 0\), \(u_{\varepsilon}\) given by (5.34) satisfies (5.7) and (5.9) (with \(w_{\varepsilon}\) replaced by \(u_{\varepsilon}\)), and

1. If \(S \leq 0\), then the initial energy of \(u_{\varepsilon}\) satisfies (5.31) (with \(w_{\varepsilon}\) replaced by \(u_{\varepsilon}\));
2. If \(S > 0\) and \(\text{supp} \varrho \subset O_{5/2} (\{(t, x(t)) \mid t \in [S, T]\})\), then the initial energy of \(u_{\varepsilon}\) vanishes.

**Proof.** By (5.32) and (5.34), it holds

\[
u_{\varepsilon} = \varrho(t, x) z_{\varepsilon} (t, x) + \varepsilon^{2-\frac{3}{4}} \int_0^t \varrho(s, x) A(t, s) e^{i \phi(s, x)/\varepsilon} ds. \tag{5.35}
\]
Hence, similar to (5.10), by (5.34) and by a direct computation, noting that \( M(t) \), \( a(t) \) and \( A(t,s) \) are independent of \( x \), we have

\[
W u_\varepsilon - F(u_\varepsilon) = \varepsilon^{2-\frac{n}{2}} r_1^\prime + \varepsilon^{1-\frac{n}{2}} r_2^\prime + \varepsilon^{-\frac{n}{4}} r_3^\prime + \varepsilon^{-1-\frac{n}{4}} r_4^\prime + 2 \left( Q_t \partial_t z_\varepsilon - \sum_{ij} a^{ij} \partial_{x_i} Q \partial_{x_j} z_\varepsilon \right),
\]

(5.36)

where \( r_1^\prime \) and \( r_2^\prime \) are respectively similar terms as \( r_1 \) and \( r_2 \) in (5.11) and (5.12) such that the counterpart of (5.26) holds\(^1\):

\[
|r_1^\prime(t, \cdot)|_{L^2(\mathbb{R}^n)} + |r_2^\prime(t, \cdot)|_{L^2(\mathbb{R}^n)} = O(\varepsilon^{\frac{3}{2}}), \quad \text{uniformly for a.e. } t \in \mathcal{Y}_T;
\]

(5.37)

while

\[
r_3^\prime \triangleq iQ(t, x) \left[ 2\dot{a}(t) \phi_t(t, x) + a(t) W \phi(t, x)
- a(t) \left( B_2(t, x) \phi_t(t, x) + B_3(t, x) \cdot \nabla \phi(t, x) \right) \right] e^{i\phi(t, x)/\varepsilon}
+ \int_0^t Q(s, x) \left[ A(t, s) \sum_{i,j} a^{ij}(x) \phi_{x_i}(s, x) \phi_{x_j}(s, x) - ia(s) B_5(t, s, x) \phi_5(s, x) \right.
- \left. ia(s) B_6(t, s, x) \cdot \nabla \phi(s, x) \right] e^{i\phi(s, x)/\varepsilon} \, ds,
\]

(5.38)

and

\[
r_4^\prime \triangleq a(t) Q(t, x) \left[ \sum_{i,j} a^{ij}(x) \phi_{x_i}(t, x) \phi_{x_j}(t, x) - \phi_1^2(t, x) \right] e^{i\phi(t, x)/\varepsilon}.
\]

(5.39)

By (5.33) in Remark 5.6, we have

\[
\left| Q_t(t, \cdot) \partial_t z_\varepsilon(t, \cdot) - \sum_{ij} a^{ij}(\cdot) \partial_{x_i} Q(t, \cdot) \partial_{x_j} z_\varepsilon(t, \cdot) \right|_{L^2(\mathbb{R}^n)} = O(\varepsilon^{\frac{1}{2}}), \quad \text{uniformly for a.e. } t \in \mathcal{Y}_T.
\]

(5.40)

Now, for \( M(\cdot) \in C^2([-T, T]; C^{n \times n}) \), \( a(\cdot) \in C([-T, T]; \mathbb{C} \setminus \{0\}) \cap W^{2,\infty}(\mathcal{Y}_T) \) and \( A(\cdot, \cdot) \in W^{2,\infty}(\Lambda_T) \) constructed in Theorem 5.1, similar to the proof of Theorem 5.1, we can show that

\[
|r_3^\prime(t, \cdot)|_{L^2(\mathbb{R}^n)} = O(\varepsilon^{\frac{n+1}{2}}), \quad |r_4^\prime(t, \cdot)|_{L^2(\mathbb{R}^n)} = O(\varepsilon^{\frac{n+3}{2}}), \quad \text{uniformly for a.e. } t \in \mathcal{Y}_T.
\]

(5.41)

Combining (5.36), (5.37), (5.40) and (5.41), we conclude that \( u_\varepsilon \) satisfies (5.7). Noting Remark 5.5, the proof of the rest assertions in Theorem 5.2 is either similar to that of Theorem 5.1 or obvious. \( \square \)

---

\(^1\) As shown in the proof of Theorem 5.1, we do not need the exact expression of \( r_1^\prime \) and \( r_2^\prime \).

---
6. Highly concentrated approximate solutions for hyperbolic equations with memory in bounded domains

As a further preliminary to prove Theorem 2.2, we construct in this section highly concentrated approximate solutions for hyperbolic equations with memory in bounded domains.

We now consider the following hyperbolic equations with memory in the bounded domain $\Omega$ with boundary $\Gamma \in C^3$:

$$
\begin{align*}
W u &= F(u) \quad \text{in } Q, \\
u &= 0 \quad \text{on } \Sigma.
\end{align*}
$$

(6.1)

The operators $W$ and $F$ in (6.1) are respectively defined similar to (2.16) and (5.3), but with coefficients (recall (5.1) for the definition of $A_T^+$)

$$
d^{ij} \in C^3(\overline{\Omega}), \quad i, j = 1, 2, \ldots, n,
B_1 \in L^\infty(Q), \quad B_2 \in W^{1,\infty}(0, T; C^1(\overline{\Omega})), \quad B_3 \in W^{1,\infty}(0, T; C^1(\overline{\Omega}; \mathbb{R}^n)),
B_4 \in L^\infty(A_T^+ \times \Omega), \quad B_5 \in W^{2,\infty}(A_T^+; C^2(\overline{\Omega})), \quad B_6 \in W^{2,\infty}(A_T^+; C^2(\overline{\Omega}; \mathbb{R}^n)),
$$

(6.2)

for given $T > 0$. Also, $d^{ij}$ satisfies the elliptic condition in (2.2).

The main purpose of this section is to adapt the construction of approximate solutions for $Wu = F(u)$ in $\mathcal{Y}_T \times \mathbb{R}^n$ in the last section (see Theorem 5.2 and recall (5.1) for the definition of $\mathcal{Y}_T$) to obtain highly concentrated approximate solutions to system (6.1). Recall that, in the last section, all coefficients of $W$ and $F$ are defined in $\mathcal{Y}_T$ for $t$, in $A_T$ for $(t, s)$, and in $\mathbb{R}^n$ for $x$. Therefore, we need first extend the domain of these coefficients as follows.

We extend first $d^{ij}, B_2, B_3, B_5$ and $B_6$ in $\mathbb{R}^n \setminus \overline{\Omega}$ for $x$-variable by continuity up to their original order of derivatives (say, one can use the classical Lions’s extension [16]); while for $B_1$ and $B_4$, we extend them in $\mathbb{R}^n \setminus \overline{\Omega}$ for $x$-variable as 0. Then, for $t$-variable and $(t, s)$-variable, we choose respectively odd extension for $B_{k_1}$ $(k_1 = 1, 2, 3)$ and $B_{k_2}$ $(k_2 = 4, 5, 6)$ on $(-T, 0) \times \mathbb{R}^n$ and $A_T^- \times \mathbb{R}^n$ (recall (5.1) for the definition of $A_T^-$), i.e.,

$$
B_{k_1}(t, \cdot) = -B_{k_1}(-t, \cdot) \quad \text{for } t \in (-T, 0), \quad B_{k_2}(t, s, \cdot) = -B_{k_2}(-t, -s, \cdot) \quad \text{for } (t, s) \in A_T^-.
$$

(6.3)

To simplify the notation, we still denote these extensions by their original notations. Hence, we may assume that (recall (5.1) for the definition of $\mathcal{Y}_T$ and $A_T$)

$$
A(x) \triangleq (d^{ij}(x))_{1 \leq i, j \leq n} \in C^3(\mathbb{R}^n; \mathbb{R}^{n \times n}),
B_1 \in L^\infty(\mathcal{Y}_T \times \mathbb{R}^n), \quad B_2 \in W^{1,\infty}(\mathcal{Y}_T; C^1(\mathbb{R}^n)), \quad B_3 \in W^{1,\infty}(\mathcal{Y}_T; C^1(\mathbb{R}^n; \mathbb{R}^n)),
B_4 \in L^\infty(A_T \times \mathbb{R}^n), \quad B_5 \in W^{2,\infty}(A_T; C^2(\mathbb{R}^n)), \quad B_6 \in W^{2,\infty}(A_T; C^2(\mathbb{R}^n; \mathbb{R}^n)).
$$

(6.4)

Similar to Section 2, replacing $A(x)$ by $\rho A(x) + (1 - \rho)I$ (recall (2.14) for the cut-off function $\rho$), and $B_k$ by $\rho B_k$ for $k = 2, 3, 6$ if necessary, we may assume that $A(x)$ satisfies the elliptic condition (2.15) and

$$
A(x) \in W^{3,\infty}(\mathbb{R}^n; \mathbb{R}^{n \times n}) \cap C^3(\mathbb{R}^n; \mathbb{R}^{n \times n}), \quad B_2 \in W^{1,\infty}(\mathcal{Y}_T \times \mathbb{R}^n), \quad B_5 \in W^{2,\infty}(A_T \times \mathbb{R}^n), \quad B_6 \in W^{2,\infty}(A_T \times \mathbb{R}^n; \mathbb{R}^n).
$$

(6.5)
The main task in the sequel is to construct suitable approximate solutions to the following system:

\[
\begin{align*}
Wu &= F(u) \quad \text{in } \mathcal{T}_T \times \Omega, \\
\varepsilon - u &= 0 \quad \text{on } \mathcal{T}_T \times \Gamma.
\end{align*}
\]  
(6.6)

Noting (2.15) and Proposition 3.2, any ray \((t, x(t), \xi(t))\) of operator \(W\) is defined globally for \(t \in \mathbb{R}\). Hence, by Theorems 5.1–5.2, one may construct approximate solutions \(u_{\varepsilon} = u_{\varepsilon}(t, x)\) as (5.34) for equation \(Wu = F(u)\) in \(\mathcal{T}_T \times \mathbb{R}^n\).

In some special situation, there may exist a ray \((t, x(t), \xi(t))\) of \(W\) starting from \(\Omega\) but never arrives at its boundary \(\Gamma\), i.e., \(x(t) \in \Omega\) for each \(t \geq 0\) (e.g. [2]). This (rarely happened) case is quite easy to treat since the highly concentrated approximate solutions for equation \(Wu = F(u)\) in \(\mathcal{T}_T \times \mathbb{R}^n\) constructed in the last section (see Theorem 5.2) are also approximated solutions for equation \(\varepsilon - u = 0\) in \(\mathcal{T}_T \times \Omega\). Therefore, in the sequel, we shall not consider this special but easy case.

In what follows, we always assume that any ray of operator \(W\) starting from \(\Omega\) will exit \(\Omega\) in finite time. In this case, the approximate solutions for \(Wu = F(u)\) in \(\mathcal{T}_T \times \mathbb{R}^n\) constructed in the last section will not satisfy in general the homogeneous Dirichlet boundary condition. In order to overcome this difficulty, one has to superpose two approximate solutions concentrated respectively in a small neighborhood of two different rays of operator \(W\), one reflects of the other at the boundary. This is indeed the motivation to introduce the notion of generalized ray of operator \(W\) (see Definition 2.2).

Similar to [23, Lemma 2.2], by means of the perturbation technique, one can show the following geometric lemma.

**Lemma 6.1.** Suppose that the triple \((\Omega, \omega, T)\) satisfies Assumption 2.1. Then there is a generalized ray \((t, x^i(t), \xi^i(t)) | t \in [s_i, s_{i+1}]\}_{i=0}^{m-1}\) of operator \(W\) in \(\overline{\Omega}\) such that

1. \(x^i(0) \in \Omega\) and \(x^i(T) \in \Omega\);
2. it does not meet \(\partial \Omega\), i.e., \(x^{i+1}(t) \notin \partial \Omega\) for all \(i \in \{0, 1, \ldots, m - 1\}\) and \(t \in [s_i, s_{i+1}]\);
3. it always meets \(\Gamma\) transversally, i.e.,

\[
\xi^i(s_{i+1})^\top A(x^i(s_{i+1})) \nu(x^i(s_{i+1})) \neq 0, \quad \forall i \in \{0, 1, 2, \ldots, m - 2\},
\]  
(6.7)

where \(s_{i+1}\) is the \((i+1)\)-th reflected instant of this generalized ray.

**Remark 6.1.** When \(A(x) \equiv I\), (6.7) means that the ray \((t, x^i(t))\) is nonperpendicular to the direction \(\nu(x^i(s_{i+1}))\). Generally, (6.7) means the direction of the ray \((t, x^i(t))\) at \(t = s_{i+1}\) is nonperpendicular to \(\nu(x^i(s_{i+1}))\) under the metric induced by \(g(x, \cdot)\). Therefore, the condition (6.7) guarantees that the ray \((t, x^i(t), \xi^i(t))\) exits \(\Omega\) at time \(t = s_{i+1}\), and hence at least this ray exits \(\Omega\) locally.

### 6.1. Ansatz of the incoming and reflected waves

Assume \((t, x^-(t), \xi^-(t))\) is a ray of operator \(W\) starting from \(\Omega\) at time \(t = 0\), i.e., \(x^-(0) \in \Omega\), and arriving at the boundary \(\Gamma\) at time \(t = t_0\), i.e., \(x_0 = x^-(t_0) \in \Gamma\).

As mentioned before, by Theorem 5.2, we can construct a family of approximate solutions \(u_\varepsilon^- = u_\varepsilon^-(t, x)\) to the first equation in (6.6). However, \(u_\varepsilon^-\) may not satisfy the homogeneous Dirichlet boundary condition \(u_\varepsilon^- = 0\) on \(\mathcal{T}_T \times \Gamma\), i.e., the second equation in (6.6). One has to superpose \(u_\varepsilon^-\) with another approximate solution \(u_\varepsilon^+\). The later is constructed from the ray \((t, x^+(t), \xi^+(t))\), which reflects the original one, \((t, x^-(t), \xi^-(t))\), at the boundary (see Fig. 3). The point is to select approximate solutions \(u_\varepsilon^\pm\) to the first equation in (6.6), concentrated in a small neighborhood of the reflected ray \((t, x^\pm(t), \xi^\pm(t))\), such that \(u_\varepsilon^- + u_\varepsilon^+\) satisfies approximately the homogeneous Dirichlet boundary condition.
According to (2.17), \((x^-(t), \xi^-(t))\) satisfies

\[
\begin{aligned}
\dot{x}^-(t) &= \nabla_\xi g(x^-(t), \xi^-(t)), \\
\dot{\xi}^-(t) &= -\nabla_x g(x^-(t), \xi^-(t)), \\
x^-(t_0) &= x_0, \quad \xi^-(t_0) = \xi^-(t_0).
\end{aligned}
\]

We choose \((x^+(t), \xi^+(t))\) to satisfy

\[
\begin{aligned}
\dot{x}^+(t) &= \nabla_\xi g(x^+(t), \xi^+(t)), \\
\dot{\xi}^+(t) &= -\nabla_x g(x^+(t), \xi^+(t)), \\
x^+(t_0) &= x_0, \\
\xi^+(t_0) &= \xi^-(t_0) - 2\left[\nu(x_0)^\top A(x_0)\xi^-(t_0)\right]\nu(x_0).
\end{aligned}
\]

Here, similar to (2.9), \(\nu(x_0) = (\nu_1(x_0), \ldots, \nu_n(x_0))^\top\) is the outward normal vector of \(\Omega\) at \(x_0 \in \Gamma\) so that

\[\nu(x_0)^\top A(x_0)\nu(x_0) = 1.\] (6.10)

On the other hand, from (2.18), one has \(g(x_0, \xi^-(t_0)) = \frac{1}{4}\). Hence, noting (6.10), one can check that \(g(x_0, \xi^+(t_0)) = \frac{1}{4}\). Therefore,

\[g(x^\pm(t), \xi^\pm(t)) = \xi^\pm(t)^\top A(x^\pm(t))\xi^\pm(t) = \frac{1}{4}, \quad \forall t \in \mathbb{R}.\] (6.11)

We assume that \(\xi^-(t)\) is transversal to the boundary \(\Gamma\) at time \(t = t_0\) (with respect to the metric \(g\)), i.e.,

\[\xi^-(t_0)^\top A(x_0)\nu(x_0) \neq 0.\] (6.12)

Denote by \(T_1 > 0\) the instant when the reflected ray arrives at \(\Gamma\), i.e., \(x^+(T_1) \in \Gamma\) (note that \(0 < t_0 < T_1\)). Fix any

\[T^* \in (t_0, T_1).\] (6.13)
Fix any cut-off function \( \varrho^- = \varrho^-(t, x) \in C_0^\infty(\mathbb{R}^{1+n}) \) which is identically equal to 1 in a neighborhood of the ray \( \{(t, x^-(t)) \mid t \in [0, t_0]\} \), with

\[
\text{supp} \varrho^- \subseteq O(t^* - t_0)/4 \{(t, x^-(t)) \mid t \in [0, t_0]\}.
\]  

(6.14)

According to (5.34) and Theorem 5.2, we may construct approximate solutions to equation \( Wu = F(u) \) in \( \mathcal{T}_T \times \mathbb{R}^n \) as follows:

\[
u^-_\varepsilon(t, x) = \varepsilon^{1-n/4} \left[ Q^- (t, x) a^- (t) e^{i\phi^- (t, x)/\varepsilon} + \varepsilon \int_0^t Q^- (s, x) A^- (t, s) e^{i\phi^- (s, x)/\varepsilon} ds \right],
\]

(6.15)

where

\[
\phi^- (t, x) = \xi^- (t)^\top (x - x^- (t)) + \frac{1}{2} (x - x^- (t))^\top M^- (t) (x - x^- (t)).
\]

(6.16)

In (6.16), \( M^- (t) \) is some given \( n \times n \) complex symmetric matrix with positive definite imaginary part.

Fix any cut-off function \( \varrho^+ = \varrho^+ (t, x) \in C_0^\infty (\mathbb{R}^{1+n}) \) which is identically equal to 1 in a neighborhood of the ray \( \{(t, x^+(t)) \mid t \in [t_0, T_1]\} \) with

\[
\text{supp} \varrho^+ \subseteq O_{\text{min}(t_0, T_1 - T^*)}/4 \{(t, x^+(t)) \mid t \in [t_0, T_1]\}.
\]

(6.17)

Our aim is to find another approximate solution

\[
u^+_\varepsilon(t, x) = \varepsilon^{1-n/4} \left[ Q^+ (t, x) a^+ (t) e^{i\phi^+ (t, x)/\varepsilon} + \varepsilon \int_0^t Q^+ (s, x) A^+ (t, s) e^{i\phi^+ (s, x)/\varepsilon} ds \right]
\]

(6.18)

of equation \( Wu = F(u) \) in \( \mathcal{T}_T \times \mathbb{R}^n \), which is concentrated in a small neighborhood of the reflected ray \( (t, x^+(t), \xi^+(t)) \) such that the following approximate Dirichlet boundary condition holds

\[
|u^-_\varepsilon + u^+_\varepsilon|_{H^1((0, T^*) \times \Gamma')} = O(\varepsilon^{1/2}).
\]

(6.19)

Here, similar to (6.16), we take \( \phi^+ \) to be of the form:

\[
\phi^+ (t, x) = \xi^+ (t)^\top (x - x^+ (t)) + \frac{1}{2} (x - x^+ (t))^\top M^+ (t) (x - x^+ (t)),
\]

(6.20)

where \( M^+ (t) \) is a suitable \( (n \times n) \) complex symmetric matrix with positive definite imaginary part, which will be determined later.

From (6.18) and (6.16), it is easy to see that it remains to construct \( M^+ (t) \), \( a^+ (t) \) and \( A^+ (t, s) \). For this purpose, we borrow some idea in [23]. First, by Remark 5.3, \( a^+ (t) \) and \( A^+ (t, s) \) are uniquely determined by the ray \( (t, x^+(t), \xi^+(t)) \) and the initial value of \( a^+ (t) \) at \( t = t_0 \), which is assigned to be

\[
a^+ (t_0) = -a^- (t_0).
\]

(6.21)

Next, \( M^+ (t) \) is determined by its initial \( M^+ (t_0) \) and the reflected ray \( (t, x^+(t), \xi^+(t)) \). Note that \( (x^+(t), \xi^+(t)) \) is given by (6.9). Hence, it suffices to assign \( M^+ (t_0) \). This will be done below.
6.2. Assignment of the reflected phase function and its properties

In order to determine \( M^+(t_0) \), we introduce local coordinates near the reflected point \( x_0 \in \Gamma \), called henceforth \( \hat{x} \equiv \hat{x}(x) = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)^\top \) (in the sequel, we denote \((\hat{x}_2, \ldots, \hat{x}_n)^\top \) by \( \hat{x}' \)), centered at the reflected point \( \hat{x}_0 \equiv (0, \hat{x}_0') \), the new coordinate of \( x_0 \), such that \( \Omega \) is locally given by \( \hat{x}_1 \geq 0 \), and \( \Gamma \) is flat near \( \hat{x}_0 \). Denote the inverse Jacobian matrix of \( \hat{x} = \hat{x}(x) \) by \( J(\hat{x}) \), i.e.,

\[
J(\hat{x}) \equiv (g_{ij}(\hat{x}))_{1 \leq i, j \leq n} \triangleq \frac{\partial (x_1, x_2, \ldots, x_n)}{\partial (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)}.
\]  

(6.22)

By \( \Gamma \in C^3 \), we see that \( J(\hat{x}) \in C^2 \). In the new coordinates the outward normal vector \( \nu(x_0) \) at the reflected point becomes \((-1, 0, \ldots, 0)^\top \). Hence

\[
\left( J(\hat{x}_0) \right)^{-1} \nu(x_0) = \left| \left( J(\hat{x}_0) \right)^{-1} \nu(x_0) \right| (-1, 0, \ldots, 0)^\top.
\]

(6.23)

Write the expression of \( \varphi^\pm(t, x) \) in the \( \hat{x}' \)-coordinates as

\[
\hat{\varphi}^\pm(t, \hat{x}) = \hat{\xi}^\pm(t) (x(\hat{x}) - x^\pm(t)) + \frac{1}{2} (x(\hat{x}) - x^\pm(t))^\top M^\pm(t)(x(\hat{x}) - x^\pm(t)).
\]

(6.24)

Put

\[
\sigma^\pm \equiv \begin{pmatrix} \sigma_1^\pm \\ \sigma_2^\pm \end{pmatrix} \triangleq \left( J(\hat{x}_0) \right)^\top \hat{\xi}^\pm(t_0),
\]

\[
\eta^\pm \equiv \begin{pmatrix} \eta_1^\pm \\ \eta_2^\pm \\ \vdots \\ \eta_n^\pm \end{pmatrix} \triangleq \left( J(\hat{x}_0) \right)^{-1} A(x_0) \hat{\xi}^\pm(t_0).
\]

(6.25)

where \( \sigma_\pm, \eta_\pm \in \mathbb{R}^{n-1} \). Both \( \sigma^\pm \) and \( \eta^\pm \) will be needed to compute the derivatives of \( \hat{\varphi}^\pm(t, 0, \hat{x}') \) at \((t_0, \hat{x}_0)\) up to second order.

We have the following result.

**Proposition 6.1.** Under the assumption (6.12), it holds:

\[
\eta_1^+ = -\eta_1^- \neq 0, \quad \sigma'_+ = \sigma'_-.
\]

(6.26)

We refer to Appendix A for the proof of Proposition 6.1.

Denote

\[
\hat{M}^\pm(t_0) \triangleq (J(\hat{x}_0))^{-1} M^\pm(t_0) J(\hat{x}_0).
\]

(6.27)

Obviously, determining \( M^+(t_0) \) is equivalent to choosing \( \hat{M}^+(t_0) \). Therefore, in the rest of this subsection, we shall devote to choosing \( \hat{M}^+(t_0) \). For this purpose, put

\[
\hat{x}^\pm(t) = \hat{x}(x^\pm(t)).
\]

(6.28)

Then, one has

\[
x^\pm(t) = x(\hat{x}^\pm(t)).
\]

(6.29)

We need to compute the derivatives of \( \hat{\varphi}^\pm(t, 0, \hat{x}') \) at \((t_0, \hat{x}_0')\), up to second order.
Proposition 6.2. As \((t, \hat{x}')\) tends to \((t_0, \hat{x}_0')\), the following estimates hold

\[
\left( \begin{array}{c}
0 \\
\hat{x}'
\end{array} \right) - \hat{x}'(t) = \left( \begin{array}{c}
-2\eta_t^\pm(t - t_0) \\
\hat{x}' - \hat{x}_0'
\end{array} \right) - O(|t - t_0|^2),
\]

\[
\hat{\phi}^\pm(t, 0, \hat{x}') = 0(|t - t_0| + |\hat{x}' - \hat{x}_0'|),
\]

\[
\partial_t \hat{\phi}^\pm(t, 0, \hat{x}') = -\frac{1}{2} + O(|t - t_0| + |\hat{x}' - \hat{x}_0'|),
\]

\[
\nabla_{\hat{x}} \hat{\phi}^\pm(t, 0, \hat{x}') = \sigma^\pm + O(|t - t_0| + |\hat{x}' - \hat{x}_0'|),
\]

\[
\partial_{tt} \hat{\phi}^\pm(t, 0, \hat{x}') = -2(\nabla_x(\xi^\pm(t_0) \top A(x_0) \xi^\pm(t_0))) \top A(x_0) \xi^\pm(t_0)
\]

\[
+ 4(\eta^\pm) \top \tilde{M}^\pm(t_0) \eta^\pm + O(|t - t_0| + |\hat{x}' - \hat{x}_0'|),
\]

\[
\partial_t \nabla_{\hat{x}} \hat{\phi}^\pm(t, 0, \hat{x}') = -\left( \nabla_x(\xi^\pm(t_0) \top A(x_0) \xi^\pm(t_0)) \right) \top J(\hat{x}_0)
\]

\[
- 2\tilde{M}^\pm(t_0) \eta^\pm + O(|t - t_0| + |\hat{x}' - \hat{x}_0'|),
\]

\[
\nabla^2_{\hat{x}} \hat{\phi}^\pm(t, 0, \hat{x}') = \nabla_{\hat{x}}(J(\hat{x}_0)) \top \xi^\pm(t_0)) + \tilde{M}^\pm(t_0) + O(|t - t_0| + |\hat{x}' - \hat{x}_0'|).
\]

We refer to Appendix A for the proof of Proposition 6.2.

Now, we write

\[
\nabla_{\hat{x}}(\left( J(\hat{x}_0) \right) \top \xi^\pm(t_0)) = (h^\pm)_{1 \leq i, j \leq n},
\]

\[
\left( \nabla_x(\xi^\pm(t_0) \top A(x_0) \xi^\pm(t_0)) \right) \top J(\hat{x}_0) = \left( \kappa^\pm \right),
\]

\[
\tilde{M}^\pm(t_0) = (m^\pm_{ij})_{1 \leq i, j \leq n} = \left( \begin{array}{c}
m^\pm_{11} \\
\vdots \\
m^\pm_{nn}
\end{array} \right),
\]

where \(\kappa^\pm = (\kappa^\pm_1, \ldots, \kappa^\pm_n)\top\), \(\vartheta^\pm = (m^\pm_{21}, \ldots, m^\pm_{n1})\top\) and \(\tilde{M}^\pm = (m^\pm_{ij})_{2 \leq i, j \leq n}\). Note that all \(h^\pm_{ij}, \kappa^\pm_1, \kappa^\pm_2\) and \(m^\pm_{ij}\) are known. We now assign all \(m^\pm_{ij}\) to then obtain \(\tilde{M}^\pm(t_0)\) in (6.27).

The main idea to determine \(\tilde{M}^\pm(t_0)\) is to choose all \(m^\pm_{ij}\) such that the second derivatives of \(\hat{\phi}^\pm(t, 0, \hat{x}')\) with respect to \(t\) and \(\hat{x}'\) coincide at the reflection point. This can be done by choosing (recall (6.25) for \(\eta'_\pm\))

\[
m^\pm_{ij} = h^\pm_{ij} + m^\pm_{ij} - h^\pm_{ij}, \quad 2 \leq i, j \leq n,
\]

\[
\vartheta^\pm = (m^\pm_{21}, \ldots, m^\pm_{n1}) \top = \frac{\kappa^\pm - \vartheta^\pm + 2(\eta^-_1 \vartheta^- + \tilde{M}^- \vartheta^-_1 - \tilde{M}^+ \vartheta^+_1)}{2\eta^+_1},
\]

and

\[
m^\pm_{11} = \frac{1}{2|\eta^+_1|^2} \left[ (\nabla_x(\xi^+(t_0) \top A(x_0) \xi^+(t_0))) \top A(x_0) \xi^+(t_0) \right.
\]

\[
- (\nabla_x(\xi^-(t_0) \top A(x_0) \xi^-(t_0))) \top A(x_0) \xi^-(t_0)
\]

\[
+ 2(m^\pm_{11} |\eta^-_1|^2) + 2m^\pm_{11} \vartheta^- \eta^- + (\eta^-) \top \tilde{M}^- \vartheta^- - 2\eta^+_1 \vartheta^+_1 \eta^+_1 - (\eta^+_1) \top \tilde{M}^+ \vartheta^+_1 \right].
\]
which completes the assignment of $\hat{M}^+(t_0)$, and hence $M^+(t_0)$. Indeed, we have the following result:

**Proposition 6.3.** If $m^+_{ij}$ (1 ≤ $i$, $j$ ≤ $n$) are chosen as in (6.38)–(6.40), then, as $(t, \dot{x})$ tends to $(t_0, \dot{x}_0)$, it holds

\[
\begin{align*}
\partial_{tt} \hat{\phi}^+(t, 0, \dot{x}) &- \partial_{tt} \hat{\phi}^-(t, 0, \dot{x}) = O\left(|t - t_0| + |\dot{x} - \dot{x}_0|\right), \\
\partial_t \nabla_x \hat{\phi}^+(t, 0, \dot{x}) &- \partial_t \nabla_x \hat{\phi}^-(t, 0, \dot{x}) = O\left(|t - t_0| + |\dot{x} - \dot{x}_0|\right), \\
\nabla^2_x \hat{\phi}^+(t, 0, \dot{x}) &- \nabla^2_x \hat{\phi}^-(t, 0, \dot{x}) = O\left(|t - t_0| + |\dot{x} - \dot{x}_0|\right).
\end{align*}
\]

(6.41)

(6.42)

(6.43)

**Proof.** First of all, from (6.36) in Proposition 6.2 and noting (6.37)–(6.38), one sees that (6.43) holds by choosing $m^+_{ij} = h^+_{ij} + m^-_{ij} - h^+_{ij}$ as in (6.38), 2 ≤ $i$, $j$ ≤ $n$. This determines $\hat{M}^+$.

Next, by (6.25) and (6.37), we see that

\[
\hat{M}^\pm(t_0) \eta^\pm = \begin{pmatrix} m^\pm_{11} \eta^\pm_1 + \vartheta^\pm_1 \eta'_\pm \\
\eta^\pm_1 \vartheta^\pm_2 + \tilde{M}^\pm \eta'\pm \end{pmatrix}.
\]

(6.44)

Hence, we choose $m^+_{1j} = m^+_{1j}$ for $j = 2, \ldots, n$ as in (6.39). Then, by (6.35) in Proposition 6.2, and noting (6.37), (6.39) and (6.44), we get (6.42).

Finally, from (6.25) and (6.44), we have

\[
(\eta^\pm)^T \hat{M}^\pm(t_0) \eta^\pm = m^\pm_{11} |\eta^\pm_1|^2 + 2 \eta^\pm_1 \vartheta^\pm_1 \eta'_\pm + (\eta'_\pm)^T \tilde{M}^\pm \eta'_\pm.
\]

(6.45)

Then, by choosing $m^+_{11}$ as in (6.40) and noting (6.45), we get

\[
-2(\nabla_x (\xi^+ (t_0)^T A(x_0) \xi^+ (t_0)))^T A(x_0) \xi^+ (t_0) + 4(\eta^+)^T \hat{M}^+ (t_0) \eta^+
= -2(\nabla_x (\xi^- (t_0)^T A(x_0) \xi^- (t_0)))^T A(x_0) \xi^- (t_0) + 4(\eta^-)^T \hat{M}^- (t_0) \eta^-.
\]

(6.46)

Combining (6.34) in Proposition 6.2 and (6.46), we arrive at (6.41). This completes the proof of Proposition 6.3.

Thanks to Taylor’s formula, it follows from Propositions 6.1–6.3 that

**Proposition 6.4.** As $(t, \dot{x})$ tends to $(t_0, \dot{x}_0)$, it holds

\[
\hat{\phi}^+(t, 0, \dot{x}) - \hat{\phi}^-(t, 0, \dot{x}) = O\left(|t - t_0|^3 + |\dot{x} - \dot{x}_0|^3\right).
\]

(6.47)

As mentioned before, it is crucial to show the following result:

**Proposition 6.5.** Both $\hat{M}^+(t_0)$ constructed above and the desired $M^+(t_0)$, and hence $M^+(t)$, are $(n \times n)$ complex symmetric matrices with positive definite imaginary part.

**Proof.** First, from (6.38) and noting that $h^+_{ij} \in \mathbb{R}$, one finds

\[
\text{Im} \hat{M}^+ = \text{Im} \hat{M}^-.
\]

(6.48)

Next, by (6.39) and (6.48), and noting that $\eta'_\pm \in \mathbb{R}^{n-1}$ and $\kappa_\pm \in \mathbb{R}^{n-1}$, we get

\[
\text{Im} \vartheta_+ = \frac{\eta^+_1 \text{Im} \vartheta_- + \text{Im} \hat{M}^- (\eta'_- - \eta'_+)}{\eta^+_1}.
\]

(6.49)
Finally, by (6.40) and (6.48)–(6.49), noting that both \((\nabla_x(\xi^+(t_0))^\top A(x_0)\xi^+(t_0))\) and \((\nabla_x(\xi^-(t_0))^\top A(x_0)\xi^-(t_0))\) are real numbers, we see that

\[
\Im m_{11}^- = \frac{1}{|\eta_1^-|^2} \left[ |\eta_1^-|^2 \Im m_{11}^- + 2\eta_1^- (\Im \vartheta_-)\Im m_{11}^- + \Im M^- \eta_1^- \right]
\]

Now, combining (6.48)–(6.50), we arrive at

\[
\Im \hat{M}^+(t_0) = \left( \frac{\eta_1^- / \eta_1^+}{(\eta_1^- - \eta_1^+)} \right) \Im \hat{M}^-(t_0) \left( \frac{\eta_1^- / \eta_1^+}{(\eta_1^- - \eta_1^+)} \right),
\]

where \(l_{n-1}\) stands for the \((n-1) \times (n-1)\) identity matrix. Recalling that \(\Im \hat{M}^-(t_0) > 0\) and \(\eta_1^\pm \neq 0\), we conclude the desired result. This completes the proof of Proposition 6.5. \(\square\)

Similar to [23, Proposition 4.6], by combining Proposition 6.5, (6.30) in Proposition 6.2 and the conclusion \(\eta_1^\pm \neq 0\) in Proposition 6.1, we conclude the following useful result.

**Proposition 6.6.** As \((t, \hat{x}')\) tends to \((t_0, \hat{x}_0')\), the following estimate

\[
\Im \hat{\phi}^\pm (t, 0, \hat{x}') \geq c (|t - t_0|^2 + |\hat{x}' - \hat{x}_0'|^2)
\]

holds for some constant \(c > 0\).

### 6.3. Verification of the approximate Dirichlet boundary condition

Now, we are in the position to show that

**Lemma 6.2.** Let (6.4)–(6.5) hold. Then the approximate solutions \(u^\pm\) of the first equation in (6.6), constructed by (6.15) and (6.18), with \(a^+(t_0)\) and \(M^+(t_0)\) given by (6.21), (6.27) and (6.38)–(6.40), satisfy (6.19) for sufficiently small \(\varepsilon > 0\).

**Proof.** Let \(\hat{u}^\pm(t, \hat{x})\) be the new coordinate expressions of \(u^\pm(t, x)\). According to Theorem 5.2, noting assumptions (6.14) and (6.17) for the cut-off functions \(\phi^\pm\), without loss of generality, we may assume \(\text{supp} u^\pm_{\varepsilon} \subset (0, T^+)^{-}\) for some small \(\varepsilon > 0\) (recall (2.1) for the definition of \(\text{O}_{\varepsilon_0}(t_0)\) and \(O_0(x_0)\)). Denote by \(\Omega(x_0)\) the image of \(\text{O}_{\varepsilon_0}(t_0) \cap \Gamma\) under the map \(x \mapsto \hat{x}\). We now use the change of variable \(x \mapsto \hat{x}\) to get

\[
|u^\pm_{\varepsilon}(t_0, \hat{x}')|_{H^1((0, T^+) \times \Gamma)} \leq C \left| \hat{u}^\pm_{\varepsilon}(t, 0, \hat{x}') \right|_{H^1(\Omega_{\varepsilon_0}(t_0) \times (\text{O}_{\varepsilon_0}(x_0) \cap \Gamma))},
\]

where \(C\) depends only on \(x_0, t_0, T, \Gamma, \varepsilon_0\).
Noting that $\varrho^{\pm} \equiv 1$ in a neighborhood of $(t_0, x_0)$, we deduce that, for any $(t, \hat{x}') \in O_{\epsilon_0}(t_0) \times O(\hat{x}_0)$, it holds

$$
\hat{u}_{\epsilon}^-(t_0, \hat{x}') + \hat{u}_{\epsilon}^+(t_0, \hat{x}')
$$

$$
= \varepsilon^{1-n/4} \left[ a^{-}(t) e^{i\phi^{-}(t, \hat{x}')/\varepsilon} + a^{+}(t) e^{i\phi^{+}(t, \hat{x}')/\varepsilon} \right]
$$

$$
+ \varepsilon^{2-n/4} \int_{t_0}^{t} \left[ \hat{A}^{-}(s, 0, \hat{x}') A^{-}(t, s) e^{i\phi^{-}(s, 0, \hat{x}')/\varepsilon} \hat{\varrho}^{-}(s, 0, \hat{x}') + \hat{A}^{+}(s, 0, \hat{x}') A^{+}(t, s) e^{i\phi^{+}(s, 0, \hat{x}')/\varepsilon} \right] ds,
$$

(6.53)

where $\hat{\varrho}^{\pm}(t, \hat{x})$ are the new coordinate expressions of $\varrho^{\pm}(t, x)$. By (6.53) and (6.21), noting that Proposition 6.4 yields $\nabla_{\hat{x}'}\hat{\varrho}^{-}(t_0, x_0) = \nabla_{\hat{x}'}\hat{\varrho}^{+}(t_0, \hat{x}_0)$, we conclude that, when $(t, \hat{x}')$ closes $(t_0, \hat{x}_0')$, it holds

$$
\nabla_{\hat{x}'}(\hat{u}_{\epsilon}^-(t_0, \hat{x}') + \hat{u}_{\epsilon}^+(t_0, \hat{x}'))
$$

$$
= i \varepsilon^{-n/4} \left[ a^{-}(t_0) e^{i\phi^{-}(t_0, \hat{x}')/\varepsilon} \nabla_{\hat{x}'}\hat{\varrho}^{-}(t_0, \hat{x}') + a^{+}(t_0) e^{i\phi^{+}(t_0, \hat{x}')/\varepsilon} \nabla_{\hat{x}'}\hat{\varrho}^{+}(t_0, \hat{x}') \right]
$$

$$
+ i \varepsilon^{1-n/4} \int_{t_0}^{t} \left[ \hat{A}^{-}(s, 0, \hat{x}') A^{-}(t, s) e^{i\phi^{-}(s, 0, \hat{x}')/\varepsilon} \nabla_{\hat{x}'}\hat{\varrho}^{-}(s, 0, \hat{x}') + \hat{A}^{+}(s, 0, \hat{x}') A^{+}(t, s) e^{i\phi^{+}(s, 0, \hat{x}')/\varepsilon} \nabla_{\hat{x}'}\hat{\varrho}^{+}(s, 0, \hat{x}') \right] ds
$$

$$
+ A^{+}(t, s) e^{i\phi^{+}(s, 0, \hat{x}')/\varepsilon} \nabla_{\hat{x}'}\hat{\varrho}^{+}(s, 0, \hat{x}') \right] ds
$$

$$
+ A^{-}(t, s) e^{i\phi^{-}(s, 0, \hat{x}')/\varepsilon} \nabla_{\hat{x}'}\hat{\varrho}^{-}(s, 0, \hat{x}') \right] ds.
$$

(6.54)

Also, by Proposition 6.4, we see that as $(t, \hat{x}') \rightarrow (t_0, \hat{x}_0')$, it holds

$$
e^{i\phi^{+}(t, 0, \hat{x}')/\varepsilon} - e^{i\phi^{-}(t, 0, \hat{x}')/\varepsilon}
$$

$$
= \frac{i}{\varepsilon} (\hat{\phi}^{-}(t, \hat{x}') - \hat{\phi}^{+}(t, \hat{x}')) \int_{t_0}^{t} e^{i\eta\hat{\phi}^{+}(t_0, \hat{x}') + \tau(\hat{\phi}^{-}(t_0, \hat{x}') - \hat{\phi}^{+}(t_0, \hat{x}'))/\varepsilon} d\tau
$$

$$
= \frac{i}{\varepsilon} \int_{t_0}^{t} e^{i\hat{\phi}^{+}(t_0, \hat{x}') + \tau(\hat{\phi}^{-}(t_0, \hat{x}') - \hat{\phi}^{+}(t_0, \hat{x}'))/\varepsilon} d\tau O(|t - t_0|^3 + |\hat{x}' - \hat{x}_0'|^3).
$$

(6.55)
By Proposition 6.6, for sufficiently small $\varepsilon > 0$, we see that the factors $e^{-2\text{Im} \hat{\phi}^+(t,0,\hat{x})/\varepsilon}$ and $e^{-2\text{Im} \hat{\phi}^-(t,0,\hat{x})+\tau(\hat{\phi}^-(t,0,\hat{x})-\hat{\phi}^+(t,0,\hat{x}))/\varepsilon}$ localize the integrand in the region

$$|t - t_0|^2 + |\hat{x} - \hat{x}_0|^2 = O(\varepsilon).$$

Therefore, by (6.54)–(6.55), we conclude that, for some positive constant $C$, it holds

$$|\nabla_{\hat{x}} \left( \hat{u}_e^{-}(t,0,\hat{x}) + \hat{u}_e^{+}(t,0,\hat{x}) \right)|^2_{L^2(\mathcal{O}_{t_0}(t_0) \times \mathcal{O}(\hat{x}_0))} \leq C \varepsilon^{-2} \int_{\mathcal{O}_{t_0}(t_0) \times \mathcal{O}(\hat{x}_0)} e^{-2} \int_{0}^{t} \int_{0}^{t} \left[ e^{-2 \text{Im} \hat{\phi}^+(s,0,\hat{x})/\varepsilon} + e^{-2 \text{Im} \hat{\phi}^+(t,0,\hat{x})/\varepsilon} \right] ds \, dt \, d\hat{x}'$$

$$+ e^{-2 \text{Im} \hat{\phi}^-(t,0,\hat{x})/\varepsilon} O \left( |t - t_0|^2 + |\hat{x} - \hat{x}_0|^2 \right)$$

$$+ e^{-2 \text{Im} \hat{\phi}^+(t,0,\hat{x})/\varepsilon} O \left( |t - t_0|^2 + |\hat{x} - \hat{x}_0|^2 \right)$$

Finally, combining (6.52) and (6.56)–(6.57), we arrive at the desired result (6.19). This completes the proof of Lemma 6.2. \qed

Similarly, one shows that

$$|\partial_t \left( \hat{u}_e^{-}(t,0,\hat{x}) + \hat{u}_e^{+}(t,0,\hat{x}) \right)|_{L^2(\mathcal{O}_{t_0}(t_0) \times \mathcal{O}(\hat{x}_0))} = O(\varepsilon^{1/2}).$$

(6.57)
Remark 6.2. Note that the rays \((t, x^\pm(t), \xi^\pm(t))\), \(a^\pm(t)\) and \(A^\pm(t, s)\) are also defined for each \(t \in (-T, 0)\) and \((t, s) \in \Lambda^-\). Hence, \(u_\varepsilon^\pm(t, \cdot)\) in (6.15) and (6.18) are also well-defined for each \(t \in (-T, 0)\). From (6.14), (6.17), and the proof of Lemma 6.2, one sees that, if necessary, by choosing the supports of \(\varphi^\pm\) to be smaller, the approximate solutions \(u_\varepsilon^\pm\) constructed in Lemma 6.2 satisfy a stronger version of (6.19):

\[
|u_\varepsilon^- + u_\varepsilon^+|_{H^1((0, T) \times I')} + |u_\varepsilon^- + u_\varepsilon^+|_{H^1((-T, 0) \times I')} = O(\varepsilon^{1/2}).
\]

(6.58)

7. Highly concentrated solutions for hyperbolic equations with memory in bounded domains and proof of the negative observability/controllability result

In this section, we consider the following initial–boundary problem for hyperbolic equations with memory in a bounded domain \(\Omega\) with boundary \(\Gamma^\varepsilon \subset C^3\):

\[
\begin{aligned}
W u &= F(u) \quad \text{in } Q, \\
u &= 0 \quad \text{on } \Sigma, \\
u(0, x) &= 0 \quad \text{in } \Omega,
\end{aligned}
\]

(7.1)

with coefficients given by (6.2).

The main purpose of this section is to construct exact solutions for system (7.1) whose energy are localized in \(\Omega \setminus \overline{\omega}\), and via which we shall give a proof of Theorem 2.2. We have the following result:

**Theorem 7.1.** Let (6.2) hold and \(a^{ij}\) satisfy the elliptic condition in (2.2). Let \(\Omega\) be a bounded domain with boundary \(\Gamma^\varepsilon \subset C^3\) and \(\omega\) be a nonempty open subset of \(\Omega\). Assume that there is a generalized ray \(\{(t, x^i(t), \xi^i(t)) | t \in [s_i, s_{i+1}]\}_{i=0}^{m-1}\) of operator \(W\) in \(\overline{\Omega}\) such that

1. \(x^1(0) \in \Omega\) and \(x^m(T) \in \Omega\);
2. it does not meet \(\overline{\omega}\), i.e., \(x^i(t) \notin \overline{\omega}\) for all \(i \in \{0, 1, \ldots, m-1\}\) and \(t \in [s_i, s_{i+1}]\);
3. it always meets \(\Gamma\) transversally, i.e.,

\[
\xi^i(s_{i+1})^T A(x^i(s_{i+1})) \nu(x^i(s_{i+1})) \neq 0, \quad \forall i \in \{0, 1, 2, \ldots, m-2\},
\]

where \(s_{i+1}\) is the \((i+1)\)-th reflected instant of this generalized ray.

Then there is a family of solutions \(\{u_\varepsilon\}_{\varepsilon > 0}\) to system (7.1) in \((0, T)\) such that, for any small \(\varepsilon > 0\) and some constant \(c_0 > 0\) (independent of \(\varepsilon\)), it holds

\[
|\partial_t u_\varepsilon(0, \cdot)|_{L^2(\Omega)} \geq c_0, \quad |u_\varepsilon|_{H^1((0, T); L^2(\omega))} = O(\varepsilon^{1/2}).
\]

(7.2)

**Proof.** We divide the proof into three steps.

**Step 1.** In this step, let us construct a suitable approximate solutions \(\{U_\varepsilon\}_{\varepsilon > 0}\) to system (6.6) whose energies are concentrated in a neighborhood of the generalized ray \(\{(t, x^i(t)) | t \in [s_i, s_{i+1}]\}_{i=0}^{m-1}\). For this purpose, first, choose any cut-off function \(\varphi^1 = \varphi^1(t, x) \in C_0^\infty(\mathbb{R}^{1+n})\) which is identically equal to 1 in a neighborhood of the ray \(\{(t, x^1(t)) | t \in [0, s_1]\}\) with

\[
\text{supp } \varphi^1 \subset \mathcal{O}(s_2-s_1)/4 \{ (t, x^1(t)) | t \in [0, s_1] \}.
\]

(7.3)

Choose any \((n \times n)\) complex symmetric matrix \(M_0^1\) with \(\text{Im } M_0^1 > 0\) and any \(a_0^i \in \mathbb{C} \setminus \{0\}\). According to Theorem 5.2 and Proposition 3.1, and noting our assumptions (1) and (2), we conclude that there exist
a complex-valued symmetric matrix $M^1(\cdot) \in \mathcal{C}^2([-T, T]; \mathbb{C}^{n \times n})$, a complex-valued function $a^1(\cdot) \in \mathcal{C}([-T, T]; \mathbb{C} \setminus \{0\}) \cap \mathcal{W}^{2, \infty}(\gamma_T)$ and a complex-valued function $A^1(\cdot, \cdot) \in \mathcal{W}^{2, \infty}(A_T)$ with

$$M^1(0) = M^1_0, \quad \text{Im} M^1(t) > 0 \quad \text{for all } t \in [-T, T], \quad a^1(0) = a^1_0,$$

such that

$$u^1_\varepsilon(t, x) = \varepsilon^{1-n/4} \left[ Q^1(t, x) a^1(t) e^{i \phi^1(t, x)/\varepsilon} + \varepsilon \int_0^t Q^1(s, x) A^1(t, s) e^{i \phi^1(s, x)/\varepsilon} \, ds \right],$$

with $\phi^1(t, x) = \xi^1(t)^\top (x - x^1(t)) + \frac{1}{2} (x - x^1(t))^\top M^1(t)(x - x^1(t))$, satisfies

$$\text{ess sup}_{t \in \gamma_T} |W u^1_\varepsilon(t, \cdot) - F(u^1_\varepsilon(t, \cdot))|_{L^2(\Omega)} = O(\varepsilon^{1/2}),$$

$$|\partial_t u^1_\varepsilon(0, \cdot)|_{L^2(\Omega)} \geq c_0,$$

$$|u^1_\varepsilon|_{H^1(0, T; L^2(\omega))} + |u^1_\varepsilon|_{H^1(-T, 0; L^2(\omega))} = O(\varepsilon),$$

(7.4)

for any small $\varepsilon > 0$ and some constant $c_0 > 0$.

Next, choose any cut-off function $\rho^2 = \rho^2(t, x) \in \mathcal{C}^\infty(\mathbb{R}^{1+n})$ which is identically equal to 1 in a neighborhood of the ray $\{(t, x^2(t)) \mid t \in [s_1, s_2]\}$ with

$$\text{supp} \rho^2 \subset \mathcal{C}_{\min(s_1, s_2 - s_2)} \setminus \{t, x^2(t) \mid t \in [s_1, s_2]\}.$$

(7.5)

According to Theorem 5.2, Lemma 6.2 and Remark 6.2, and noting our assumptions, we conclude that there exist a complex-valued symmetric matrix $M^2(\cdot) \in \mathcal{C}^2([-T, T]; \mathbb{C}^{n \times n})$, a complex-valued function $a^2(\cdot) \in \mathcal{C}([-T, T]; \mathbb{C} \setminus \{0\}) \cap \mathcal{W}^{2, \infty}(\gamma_T)$ and a complex-valued function $A^2(\cdot, \cdot) \in \mathcal{W}^{2, \infty}(A_T)$ with

$$\text{Im} M^2(t) > 0 \quad \text{for all } t \in [-T, T],$$

such that

$$u^2_\varepsilon(t, x) = \varepsilon^{1-n/4} \left[ Q^2(t, x) a^2(t) e^{i \phi^2(t, x)/\varepsilon} + \varepsilon \int_0^t Q^2(s, x) A^2(t, s) e^{i \phi^2(s, x)/\varepsilon} \, ds \right],$$

with $\phi^2(t, x) = \xi^2(t)^\top (x - x^2(t)) + \frac{1}{2} (x - x^2(t))^\top M^2(t)(x - x^2(t))$, satisfies

$$\text{ess sup}_{t \in \gamma_T} |W u^2_\varepsilon(t, \cdot) - F(u^2_\varepsilon(t, \cdot))|_{L^2(\Omega)} = O(\varepsilon^{1/2}),$$

$$|u^1_\varepsilon + u^2_\varepsilon|_{H^1((0, S) \times \Gamma)} + |u^1_\varepsilon + u^2_\varepsilon|_{H^1((-T, 0) \times \Gamma)} = O(\varepsilon^{1/2}),$$

$$|u^2_\varepsilon(0, \cdot)|_{H^1(0, T; L^2(\omega))} + |u^2_\varepsilon|_{H^1(-T, 0; L^2(\omega))} = O(\varepsilon^{1/2}),$$

(7.6)

for any small $\varepsilon > 0$ and $S \in (0, s_2)$. 
Further, inductively, for any $i = 3, \ldots, m$, choose any cut-off function $\varrho^i = \varrho^i(t, x) \in C_0^\infty(\mathbb{R}^{1+n})$ which is identically equal to 1 in a neighborhood of the ray $\{(t, x^t(t)) \mid t \in [s_{i-1}, s_i]\}$ with

$$\text{supp } \varrho^i \subset C_{\min(s_{i-1}, \ldots, s_i+2, s_i-1)}(\mathbb{R}^{1+n}) \setminus \{t \in [s_{i-1}, s_i]\}. \quad (7.7)$$

Here $s_m = T$ and $s_{m+1} = 2T$. According to Theorem 5.2, Lemma 6.2 and Remark 6.2, and noting our assumptions, we conclude that there exist a complex-valued symmetric matrix function $M^t(\cdot) \in C^2([-T, T]; \mathbb{C}^{\infty \times \infty})$, complex-valued functions $d^i(\cdot) \in C([-T, T]; C \setminus \{0\}) \cap W^{2, \infty}(T_\tau)$ and $A^i(\cdot, \cdot) \in W^{2, \infty}(A_T)$ with

$$\text{Im } M^t(t) > 0 \quad \text{for all } t \in [-T, T],$$

such that

$$u^i_\epsilon(t, x) = \epsilon^{1-n/4} \left[ \varrho^i(t, x) d^i(t) e^{i\phi(t, x)/\epsilon} + \epsilon \int_0^t \varrho^i(s, x) A^i(t, s) e^{i\phi(s, x)/\epsilon} \, ds \right],$$

with $\phi^i(t, x) = \xi^i(t) (x - x^t(t)) + \frac{1}{2} (x - x^t(t))^T M^t(t) (x - x^t(t))$, satisfies

$$\begin{align*}
\text{ess sup}_{t \in T_\tau} |W u^i_\epsilon(t, \cdot) - F(u^i_\epsilon(t, \cdot))|_{L^2(\Omega)} &= O(\epsilon^{1/2}), \\
|u^i_\epsilon|_{H^1([0, T]; L^2(\Omega))} + |u^i_{\epsilon - 1}|_{H^1([0, T]; L^2(\Omega))} &= O(\epsilon^{1/2}), \\
\|u^i_\epsilon(0, \cdot), \partial_t u^i_\epsilon(0, \cdot)\|_{H^1(\Omega)} &= 0, \\
\|u^i_\epsilon|_{H^1([0, T], L^2(\Omega))} &= O(\epsilon^{1/2}), \quad (7.8)
\end{align*}$$

for any small $\epsilon > 0$ and

$$S \in \begin{cases} 
(0, s_i), & \text{if } i = 3, \ldots, m - 1, \\
(0, T), & \text{if } i = m.
\end{cases}$$

Now, put

$$U_\epsilon = \sum_{i=1}^m u^i_\epsilon.$$

By (7.4), (7.6) and (7.8), and noting the support of $\varrho^k (k = 1, 2, \ldots, m)$ satisfying (7.3), (7.5) and (7.7), we conclude that

$$\begin{align*}
\text{ess sup}_{t \in T_\tau} |W U_\epsilon(t, \cdot) - F(U_\epsilon(t, \cdot))|_{L^2(\Omega)} &= O(\epsilon^{1/2}), \\
|U_\epsilon|_{H^1(\Sigma)} + |U_\epsilon|_{H^1([-T, 0]; L^2(\Omega))} &= O(\epsilon^{1/2}), \\
\|\partial_t U_\epsilon(0, \cdot)|_{L^2(\Omega)} &\geq c_0, \\
\|U_\epsilon|_{H^1([0, T], L^2(\omega))} + |U_\epsilon|_{H^1([-T, 0], L^2(\omega))} &= O(\epsilon^{1/2}), \quad (7.9)
\end{align*}$$

for any small $\epsilon > 0$. The above $\{U_\epsilon\}_{\epsilon > 0}$ is the desired approximate solutions to system (6.6).
Step 2. Put

\[ V_\varepsilon(t, \cdot) = U_\varepsilon(t, \cdot) - U_\varepsilon(-t, \cdot), \quad \forall t \in [0, T]. \tag{7.10} \]

Noting that \( B_k \) \((k = 1, \ldots, 6)\) satisfy (6.3), by (7.9)–(7.10), we see that \( \{V_\varepsilon\}_{\varepsilon > 0} \) are approximate solutions to system (7.1). More precisely, we have

\[
\begin{align*}
\text{ess sup}_{t \in (0, T)} |W V_\varepsilon(t, \cdot) - F(V_\varepsilon(t, \cdot))|_{L^2(\Omega)} &= O(\varepsilon^{1/2}), \\
|V_\varepsilon|_{H^1(\Sigma)} &= O(\varepsilon^{1/2}), \\
V_\varepsilon(0, x) &= 0 \quad \text{in } \Omega, \\
|\partial_t V_\varepsilon(0, \cdot)|_{L^2(\Omega)} &\geq c_0, \\
|V_\varepsilon|_{H^1(0, T; L^2(\omega))} &= O(\varepsilon^{1/2}).
\end{align*}
\tag{7.11}
\]

for any small \( \varepsilon > 0 \).

Step 3. We now correct \( \{V_\varepsilon\}_{\varepsilon > 0} \) to become a family of exact solutions to system (7.1). For this, let

\[ u_\varepsilon = V_\varepsilon + v_\varepsilon, \tag{7.12} \]

where \( v_\varepsilon \) solves

\[
\begin{align*}
W v_\varepsilon - F(v_\varepsilon) &= -W V_\varepsilon + F(V_\varepsilon) \quad \text{in } Q, \\
v_\varepsilon &= -V_\varepsilon \quad \text{on } \Sigma, \\
v_\varepsilon(0, x) &= 0, \quad \partial_t v_\varepsilon(0, x) = 0 \quad \text{in } \Omega. \tag{7.13}
\end{align*}
\]

However, similar to the proof of [17, Corollary 11, p. 10], it holds

\[
\max_{t \in [0, T]} \left| \left( v_\varepsilon(t, \cdot), \partial_t v_\varepsilon(t, \cdot) \right) \right|_{H^1_0(\Omega) \times L^2(\Omega)} \leq C \left[ -W V_\varepsilon + F(V_\varepsilon) \right]_{L^1(0, T; L^2(\Omega))} + |V_\varepsilon|_{H^1(\Sigma)}. \tag{7.14}
\]

By the first two conclusions in (7.11), the above yields

\[
\max_{t \in [0, T]} \left| \left( v_\varepsilon(t, \cdot), \partial_t v_\varepsilon(t, \cdot) \right) \right|_{H^1_0(\Omega) \times L^2(\Omega)} \leq C \varepsilon^{1/2}. \tag{7.14}
\]

Finally, in view of (7.12) and (7.14), and noting the last three assertions in (7.11), we conclude that \( u_\varepsilon \) are the desired solutions satisfying (7.2). \( \square \)

Now, we are ready to prove our negative observability/controllability result for heat equations with hyperbolic memory kernel, i.e., Theorem 2.2.

Proof of Theorem 2.2. Again, by the standard duality argument (see [14, p. 282, Lemma 2.4], for examples), it suffices to show the first assertion in Theorem 2.2. From Step 1 in the proof of Theorem 2.1, one sees that it suffices to show that there is no constant \( C > 0 \) such that

\[
\int_\Omega |q_t(0, x)|^2 \, dx \leq C \int_0^T \int_\omega \left( |q_t(t, x)|^2 + |q(t, x)|^2 \right) \, dt \, dx.
\]
for any solution $q$ to system (4.9). From (4.8), and noting our assumptions, using Lemmas 3.2 and 6.1, one can apply Theorem 7.1 to system (4.9) to conclude the desired result. □

Appendix A. Proofs of some technical results

This appendix is addressed to the proofs of Propositions 6.1 and 6.2, which are respectively similar to [23, Propositions 4.1 and 4.2]. But we shall give the details here for the reader’s convenience.

Proof of Proposition 6.1. First, we claim that

$$\left((J(\hat{x}_0))^{-1}\right)^\top (J(\hat{x}_0))^{-1} v(x_0) \parallel v(x_0),$$

(A.1)

where $\xi \parallel \eta$ stands for that vectors $\xi$ and $\eta$ are parallel. Indeed, denote by $\mathcal{T}$ the tangent space to $\Gamma$ at the reflected point $x_0$. Then, it is obvious that $(J(\hat{x}_0))^{-1}\mathcal{T} = \{ \xi_1 = 0 \}$. Noting (6.23), this means that $(J(\hat{x}_0))^{-1} v(x_0) \perp (J(\hat{x}_0))^{-1} \mathcal{T}$. Hence, $(J(\hat{x}_0))^{-1} (J(\hat{x}_0))^{-1} v(x_0) \perp \mathcal{T}$. This yields (A.1).

Next, by the last equation in (6.9), and noting (6.10) and (6.12), we deduce that

$$\xi^+(t_0)^\top A(x_0) v(x_0) = -\xi^-(t_0)^\top A(x_0) v(x_0) \neq 0.$$  

(A.2)

Hence, by the definition of $\eta^+_1$ in (6.25), and noting (6.23), (A.1) and (A.2), we obtain

$$\eta^+ = -\eta^- \neq 0.$$ 

(A.3)

Combining (A.2) and (A.3), we conclude that $\eta^+_1 = -\eta^-_1 \neq 0$.

On the other hand, by the last equation in (6.9), we have

$$\left((J(\hat{x}_0))^{-1}\right)^\top \xi^+(t_0) = \left((J(\hat{x}_0))^{-1}\right)^\top \xi^+(t_0) - 2 \xi^-(t_0)^\top A(x_0) v(x_0) (J(\hat{x}_0))^{-1} v(x_0).$$

(A.4)

Clearly, (A.1) yields

$$\left((J(\hat{x}_0))^{-1}\right)^\top v(x_0) \parallel v(x_0).$$

(A.5)

Note that (A.5) and (6.23) imply that $(J(\hat{x}_0))^{-1} v(x_0) \parallel (1, 0, \ldots, 0)^\top$. Hence, in view of (A.4), we see that the $j$-th component of $(J(\hat{x}_0))^{-1} \xi^+(t_0)$ is equal to that of $(J(\hat{x}_0))^{-1} \xi^-(t_0)$ for $j = 2, \ldots, n$. This means $\sigma^+_1 = \sigma^-$. This completes the proof of Proposition 6.1. □

Proof of Proposition 6.2. The computations are as follows.

Verification of (6.30)–(6.31): Recalling that $\hat{x}(t) = \hat{x}(x_0) = \hat{x}_0$, from (6.8)–(6.9), (6.25) and (6.28), one finds (recall (6.22) for $J(\hat{x})$)

$$\hat{x}^\pm(t) = \left((J(\hat{x}_0))^{-1}\right)^\top \chi^\pm(t) = 2\left((J(\hat{x}_0))^{-1}\right)^\top A(\chi^\pm(t)) \xi^\pm(t)$$

$$= 2\left((J(\hat{x}_0))^{-1}\right)^\top A(x_0) \xi^\pm(t_0) + O(|t - t_0|) = 2(\eta^+_1, \eta^-_1)^\top + O(|t - t_0|)$$

(A.6)

as $t \to t_0$. Hence, recalling that $\hat{x}_0 = (0, \hat{x}_0)$, we see that when $t$ tends to $t_0$, it holds...
\[
\hat{x} - \hat{x}^\pm(t) = \hat{x} - \hat{x}_0 - \hat{x}_1(t-t_0) + O\left(|t-t_0|^2\right)
\]
\[
= \hat{x} - \hat{x}_0 - 2\left(\eta_1^\pm\right)\eta_1^\pm(t-t_0) + O\left(|t-t_0|^2\right)
\]
\[
= \left(\hat{x}_1 - 2\eta_1^\pm(t-t_0)\right) + O\left(|t-t_0|^2\right),
\]
which gives (6.30).

On the other hand, by (6.24) and (6.29), we get \(\hat{\phi}^\pm(t, \hat{x}) = O(|\hat{x} - \hat{x}^\pm(t)|)\), which, combined with (6.30), yields (6.31) immediately.

**Verification of (6.32)–(6.33):** From (6.24) and (6.29), we see that
\[
\hat{\partial_t \phi}^\pm(t, \hat{x}) = -\xi^\pm(t)\hat{x}_1^\pm(t) + \xi_t^\pm(t)^T(x(\hat{x}) - x^\pm(t)) - (x(\hat{x}) - x^\pm(t))^T M^\pm(t)x_1^\pm(t)
\]
\[
= -\xi^\pm(t)^T x_1^\pm(t) + O\left(|\hat{x} - \hat{x}^\pm(t)|\right).
\]
\[\tag{A.8}\]
However, by (6.8) and (6.9), and noting (6.11), we have
\[
\xi^\pm(t)^T x_1^\pm(t) = 2\xi^\pm(t)^T A(x^\pm(t))\xi^\pm(t) = \frac{1}{2}.
\]
\[\tag{A.9}\]
Combining (A.8) and (A.9), we arrive at (6.32).

On the other hand, from (6.24)–(6.25) and (6.29), we see that
\[
\nabla_x \hat{\phi}^\pm(t, \hat{x}) = \left(\begin{array}{c}
J(\hat{x})
\end{array}\right)^T \xi^\pm(t) + (J(\hat{x}))^T M^\pm(t)(x(\hat{x}) - x^\pm(t))
\]
\[
= (J(\hat{x}))^T \xi^\pm(t_0) + O\left(|t-t_0| + |\hat{x} - \hat{x}_0| + |\hat{x} - \hat{x}^\pm(t)|\right)
\]
\[
= \sigma^\pm + O\left(|t-t_0| + |\hat{x} - \hat{x}_0| + |\hat{x} - \hat{x}^\pm(t)|\right).
\]
\[\tag{A.10}\]
From (6.8)–(6.9) and noting (2.13), we have
\[
\xi_t^\pm(t) = -\nabla_x (\xi^\pm(t)^T A(x)\xi^\pm(t))|_{x=x^\pm(t)} = -\nabla_x (\xi^\pm(t_0)^T A(x_0)\xi^\pm(t_0)) + O\left(|t-t_0|\right).
\]
\[\tag{A.11}\]
Combining (A.11) and (A.12), and noting (6.30) and (6.8)–(6.9), we arrive at
\[
\partial_{tt} \hat{\phi}^\pm(t, 0, \hat{x}') = -\xi_t^\pm(t_0)^T x_1^\pm(t_0) + x_1^\pm(t_0)^T M^\pm(t_0)x_1^\pm(t_0) + O\left(|t-t_0| + |\hat{x}' - \hat{x}_0'|\right)
\]
\[
= -2\left(V_x(\xi^\pm(t_0)^T A(x_0)\xi^\pm(t_0))\right)^T A(x_0)\xi^\pm(t_0)
\]
\[
+ 4(A(x_0)\xi^\pm(t_0))^T M^\pm(t_0)A(x_0)\xi^\pm(t_0) + O\left(|t-t_0| + |\hat{x}' - \hat{x}_0'|\right).
\]
\[\tag{A.13}\]
Noting (6.25) and (6.27), this fact yields (6.34).
Verification of (6.35): From the first equality in (A.8), and noting (A.12), (6.8) and (6.9), we see that
\[
\partial_t \nabla_x \phi^\pm(t, \tilde{x}) = \xi^\pm(t)^\top J(\tilde{x}) - (J(\tilde{x}))^\top M^\pm (t) x^\pm(t) + O \left( |\tilde{x} - \tilde{x}^\pm(t)| \right)
\]
\[
= -\left( \nabla_x (\xi^\pm(t))^\top A(x) \xi^\pm(t) \right) |_{x=x^\pm(t)}^\top J(\tilde{x}) - 2(J(\tilde{x}))^\top M^\pm(t) A(x^\pm(t)) \xi^\pm(t) + O \left( |\tilde{x} - \tilde{x}^\pm(t)| \right).
\]
Hence, combining (A.16) and (6.30), and noting (6.27), we end up with (6.36). This completes the proof of Proposition 6.2. \(\square\)

References


