The global self-similarity of a chemical reaction system with critical nonlinearity

Y. W. Qi
Department of Mathematics, University of Central Florida, Orlando, FL 32816-1364, USA (yqi@pegasus.cc.ucf.edu)

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In this paper, we study the Cauchy problem of a cubic autocatalytic chemical reaction system

\[ \begin{align*}
    u_{1,t} &= u_{1,xx} - u_1 u_2^2, \\
    u_{2,t} &= d u_{2,xx} + u_1 u_2^2
\end{align*} \]

with non-negative initial data, where the constant \( d > 0 \) is the Lewis number. Our purpose is to study the global dynamics of solutions under mild decay of initial data as \( |x| \to \infty \). In particular, we show that, for a substantial class of \( L^1 \) initial data, the exact large-time behaviour of solutions is characterized by a universal, non-Gaussian spatio-temporal profile, subject to the apparent conservation of total mass.

1. Introduction

In this paper, we study the initial-value problem of the reaction–diffusion system

\[ \begin{align*}
    u_{1,t} &= u_{1,xx} - u_1 u_2^2, \\
    u_{2,t} &= d u_{2,xx} + u_1 u_2^2
\end{align*} \]  

(1.1)

in \( \mathbb{R}^1 \), where \( d > 0 \) is the Lewis number. We assume that

\[ u_1(x,0) = a_1(x) \geq 0, \quad u_2(x,0) = a_2(x) \geq 0, \quad a_1(x), a_2(x) \in L^1(\mathbb{R}^1) \cap L^\infty(\mathbb{R}^1). \]  

(1.2)

We are concerned with deriving sharp large-time dynamics when the apparent scaling law of the system fails to describe the large-time behaviour of solutions. In particular, we are interested in the case where initial data have modest decay as \( |x| \to \infty \).

The system (1.1) is a model of pre-mixed cubic autocatalytic chemical reaction of the type

\[ A + 2B \to 3B, \]

under the usual assumption that the isothermal reaction rate is proportional to \( u_1 u_2^2 \). Here \( u_1 \) is the concentration density of reactant A and \( u_2 \) is the concentration density of autocatalyst B. The system is also used to model the thermal-diffusion–combustion problem [5].

As is well known in chemical reaction theory, autocatalytic chemical reactions play a very important role in complex chain reactions which fulfill many important functions in a living cell and they are the focus of intensive research in cell biology.
For recent works on similar systems with applications in mathematical biology, see [6] and references therein.

The study of the system in a bounded domain has been carried out by many authors (in particular, see [1, 8, 12, 13]). Among other things, these authors established boundedness, global existence and large-time behaviour of solutions. For homogeneous Dirichlet or Neumann boundary conditions, the large-time behaviour is that \((u_1, u_2)\) converges to a constant vector \((c_1, c_2)\) such that \(c_1 \cdot c_2 = 0\) [13].

The existence of the travelling front solution was established by Billingham and Needham in [3, 4]. These authors also study the large-time behaviour of solutions using formal methods and numerical computation. For the most recent progress on related travelling front solutions see [9, 10, 14].

The present paper is more closely related to the recent works of Berlyand and Xin [2] and Bricmont et al. [5], where the initial-value problem is studied. In particular, the large-time spatio-temporal profile is derived in [5] when initial values have very steep decay as \(|x| \to \infty\). The result of [5], though very powerful, is very restrictive as to what kind of initial values meet the conditions there.

The main purpose of this paper is to show that the large-time dynamics, as prescribed in [5], hold for a substantially larger class of initial values with more modest decay as \(|x| \to \infty\). The difference between our result and that of [5] is greater when the initial values are small in \(L^1\)-norm.

Our approach is a combination of partial differential equation analysis and the renormalization group (RG) method, as opposed to the purely RG approach in [5]. In particular, deriving a priori estimates plays a crucial role in our study. It allows us not only to get better results, but also to make the analysis in general (and RG iteration in particular) much simpler and more transparent. Another significant aspect of this paper is a detailed analysis of the limiting linear eigenvalue problem.

We consider the system with initial values \(a_1, a_2 \in \mathcal{B}\), where \(\mathcal{B}\) is the Banach space of continuous functions in \(R^1\) with norm

\[
\|f\| = \sup_{x \in \mathbb{R}} |f(x)|(1 + |x|)^q, \quad q > 1. \tag{1.3}
\]

Let \(\phi\) be the Gaussian

\[
\phi(x) = \frac{1}{\sqrt{4\pi d}} e^{-x^2/4d}.
\]

For \(A > 0\), let \(\psi_A\) be the normalized \((\int \psi_A^2(x) \, d\mu(x) = 1\), see below) principal eigenfunction and let \(\lambda_A\) be the principal eigenvalue of differential operator

\[
\mathcal{L}_A = -\frac{d^2}{dx^2} - \frac{x}{2} \frac{d}{dx} - \frac{1}{2} + A^2 \phi_A^2(x)
\]

on \(L^2(R^1, d\mu)\), with \(d\mu(x) = e^{x^2/4} \, dx\).

Our main result is the following theorem.

**Theorem 1.1.** Suppose the initial values \(a_1, a_2 \in \mathcal{B}\) and \(a_i \geq 0 \neq 0, i = 1, 2\). Let

\[
A = \int_R a_1(x) + a_2(x) \, dx > 0
\]
be the total mass, which is conserved in time. Then the system (1.1) has a unique global classical solution \((u_1(x,t), u_2(x,t)) \in B \times B\) for all \(t \geq 0\). Furthermore, if
\begin{equation}
q > q(A) \equiv 1 + 2\lambda_A, \tag{1.4}
\end{equation}
there exists a positive constant \(B\) depending continuously on \((a_1, a_2)\) such that
\[
\| t^{1/2+\lambda_A} u_1(\sqrt{t} \cdot, t) - B\psi_A(\cdot) \| \to 0,
\]
\[
\| t^{1/2} u_2(\sqrt{t} \cdot, t) - A\phi(\cdot) \| \to 0
\]
as \(t \to \infty\).

**Remark 1.2.** As pointed out in [5], the extra decay power \(\lambda_A\) in time is due to the critical cubic nonlinearity of system (1.1). In other words, the scaling law which works for non-cubic nonlinearity no longer works for cubic nonlinearity, and thus the appearance of the anomalous exponent \(\lambda_A\) (for more details, see [5,11]).

**Remark 1.3.** Our result is clearly an improvement of the main result in [5] since we can quantify how large \(q\) should be in terms of \(A\). In particular, it is a much more accurate condition for the decay of initial data to have the large-time dynamics when \(A\) is small. Furthermore, as we shall show below, the theorem is false in general if \(q < q(A)\).

To understand heuristically the result of theorem 1.1, let us suppose that the nonlinear term in the \(u_1\) equation causes some extra decay in time on \(u_1\) of the order of \(t^{-\delta}\), where \(\delta > 0\), on top of the pure diffusion decay of \(t^{-1/2}\). This, in turn, results in \(u_2\) having pure diffusion behaviour, assuming \(u_2\) has a \(t^{-1/2}\) decay, since the nonlinear term \(u_1 u_2^2\) is of the order of \(t^{-(1+\delta)} u_2\). In RG terminology, the nonlinear term is irrelevant in the \(u_2\) equation. Then, it is a relatively simple matter to show that
\[
t^{1/2} u_2(\sqrt{t} x, t) \to A\phi(x) \quad \text{as} \quad t \to \infty.
\]
That is,
\[
w_2(y,s) = t^{1/2} u_2(\sqrt{t} x, t), \quad y = \frac{x}{\sqrt{t}}, \quad s = \log t,
\]
converges to the steady solution of \(u_s = \mathcal{L} u\), where
\[
\mathcal{L} = -d^2 \frac{d}{dx^2} - \frac{x}{2} \frac{d}{dx} - \frac{1}{2}.
\]
If we substitute the limiting profile of \(u_2\) into the \(u_1\) equation, we have
\[
u_{1,t} = u_{1,xx} - t^{-1} A^2 \phi^2 \left( \frac{x}{\sqrt{t}} \right) u_1,
\]
which, after a self-similar change of variables
\[
u_1(y,s) = t^{1/2} u_1(x,t), \quad y = \frac{x}{\sqrt{t}}, \quad s = \log t,
\]
turns into
\[
u_{1,s} = -\mathcal{L}_A u_1.
\]
Therefore, it is reasonable to imagine that the large-time behaviour of $u_1$ is determined by the first eigenvalue $\lambda_A$ and the corresponding eigenfunction $\psi_A$ of $L_A$, so that

$$u_1(y, s) e^{\lambda_A s} \to B \psi_A(y) \quad \text{as } s \to \infty$$

with some constant $B > 0$, provided that $u_1$ has sufficiently steep decay, such as exponential decay in $x$ as $|x| \to \infty$.

However, since we are dealing with an initial-value problem with strong nonlinear terms, it is a non-trivial matter to prove the result rigorously. In particular, this is true if we assume that $u_1$ has only modest decay in $x$ as $|x| \to \infty$.

The organization of the paper is as follows. In §2, we deduce a priori estimates on the solution of the system and consequently the large-time behaviour of $u_2$. A lower bound for $u_1$ will also be shown. In §3, we consider the limiting linear eigenvalue problem and provide detailed analysis. In §4, we use a refined RG method to prove the convergence of $u_1$ to a first eigenfunction of $L_A$, completing the proof of the main theorem.

Throughout the paper, $\| \cdot \|_2$ denotes the $L^2(R^1, d\mu)$-norm of a function, unless otherwise stated and $\| \cdot \|$ denotes the norm defined in (1.3). Also, for simplicity of notation, we shall not distinguish the generic constant $C$ from line to line.

2. A priori estimates

First, we summarize some basic facts about $u_1$ and $u_2$.

**Proposition 2.1.**

(i) If $0 \leq a_1, a_2 \in B$, then $u_1(\cdot, t), u_2(\cdot, t) \in B$ for any $t > 0$, and there exists $C = C(q, \|a_1\|, \|a_2\|) > 0$ such that

$$u_1(x, t) + u_2(x, t) \leq e^{Ct}(1 + |x|)^{-q}.$$  

(ii) For any $t_0 > 0$, there exists $\delta > 0$ such that

$$u_2(x, t) \geq \delta (1 + t)^{-1/2} \phi \left( \frac{x}{\sqrt{1 + t}} \right) \text{ for } t \geq t_0.$$  

(iii) $u_1(\cdot, t)$ and $u_2(\cdot, t)$ are positive, classical solutions to (1.1) with uniformly bounded $L^\infty$ norms.

**Proof.** It is an easy exercise, and we omit the proof (see [11] for more details). \qed

Next, we observe that, by making the change of variables

$$w_i(y, s) = (1 + t)^{1/2} u_i(x, t), \quad i = 1, 2, \quad y = \frac{x}{\sqrt{1 + t}}, \quad s = \log(1 + t),$$

the system is changed to

$$\begin{align*}
  w_{1,s} &= w_{1,yy} + \frac{1}{2} y w_{1,y} + \frac{1}{2} w_1 - w_1 w_2^2, \\
  w_{2,s} &= dw_{2,yy} + \frac{1}{2} y w_{2,y} + \frac{1}{2} w_2 + w_1 w_2. 
\end{align*} \quad (2.1)$$

This is a more convenient formulation to work with.
Lemma 2.2. Suppose that
\[ a_1 \in B, \quad u_2(x, t) \geq c_0 \phi(y)(1 + t)^{-1/2}. \]

Then there exist \( c_1 = c_1(c_0, q, d) > 0 \) and \( E = E(c_0, q, ||a_1||) \) such that
\[ u_1(x, t) \leq E(1 + t)^{-(1+c_1)/2}(1 + |y|)^{-q}. \]

Proof. We use the formulation (2.1). The conclusion is the same as showing that
\[ w_1(y, s) \leq e^{-sc_1/2}(1 + |y|)^{-q}. \]

Let \( D > 0 \) be such that
\[ D^2 > \frac{4q(q + 1)}{q - 1} \]
and, for all \( |y| \geq D, \)
\[ e^{-|y|^2/4} \leq |y|^{-q} \quad \text{and} \quad -|y|e^{-|y|^2/4} \geq -2q|y|^{-q-1}. \]

Let \( \bar{c} = \min_{|y| \leq D} c_0^2 \phi^2(y) \). Without loss of generality, we assume that \( \bar{c} \leq \frac{1}{4}(q - 1) \).

Define
\[ I(w) = w_s - w_{yy} - \frac{1}{2}yw_y - \frac{1}{2}w + c_0^2 \phi^2(y)w. \]

Let
\[ \bar{w} = \begin{cases} \\
E_1 e^{-|y|^2/4} e^{-c_1 s/2} + M e^{-c_1 s/2}, & |y| \leq D, \\
E_1 e^{-|y|^2/4} e^{-c_1 s/2} + M e^{-c_1 s/2} D^q |y|^{-q}, & |y| \geq D,
\end{cases} \tag{2.2} \]

where \( c_1 = \delta \bar{c} \), with \( \delta > 0 \) being a small number to be determined later, and \( E_1 \) and \( M \) are positive numbers satisfying
\[ c_1 E_1 < \frac{1}{4} M(q - 1 - 2c_1)e^{D^2/4}. \]

It is clear that with suitable choice of \( M \), and consequently \( E_1, \bar{w}(y, 0) \geq w_1(y, 0) \).

We now demonstrate that \( I(\bar{w}) \geq 0 \), which, together with \( I(w_1) \leq 0 \), yields the conclusion of the lemma.

If \( |y| \leq D \), it is easy to compute
\[ I(\bar{w}) \geq e^{-c_1 s/2}(1 - \delta)\bar{c} (E_1 e^{-|y|^2/4} + M) > 0. \]

If \( |y| \geq D \), a more detailed calculation gives
\[ I(\bar{w}) \geq e^{-c_1 s/2}(-\frac{1}{2}c_1 E_1 e^{-|y|^2/4} + \frac{1}{2} M(q - 1 - c_1) D^q |y|^{-q} - M(q+1)|y|^{-(q+2)} D^q) > 0 \]
by our careful choice of \( c_1, D \) and \( E_1 \). This completes the proof of lemma 2.2. \( \Box \)

Lemma 2.3. Suppose \( a_1, a_2 \in \mathcal{B} \). There exists \( \delta > 0 \) such that
\[ \lim_{t \to \infty} \left| t^{1/2} u_2(x, t) - A \phi \left( \frac{x}{\sqrt{t}} \right) \right|^{1/2} = 0 \quad \text{uniformly in} \quad \{ x : |x| \leq C \sqrt{t} \}, \tag{2.3} \]
\[ \lim_{t \to \infty} \int_{R^1} u_2(x, t) \, dx - A t^{\delta} \to 0. \tag{2.4} \]
Proof. It is easy to see that $\|u_1 + u_2\|_1 = \|a_1 + a_2\|_1$ and that $\|u_1\|_1$ is decreasing in $t$ and $\|u_2\|_1$ is increasing. By lemma 2.2,

$$\|u_1\|_1 \to 0 \quad \text{as} \quad t \to \infty$$

and, therefore,

$$\|u_2\|_1 \to A \quad \text{as} \quad t \to \infty.$$ 

Moreover, using the $u_1$ bound in lemma 2.2 and [5, proposition 2], which shows that

$$\|u_2(t)\|_\infty \leq C\|u_2(\sqrt{T}, t)\| \leq C(1 + t)^{-1/2},$$

we deduce that, for some $\delta_0 > 0$, $u_2$ satisfies

$$u_{2,t} = du_{2,xx} + O(t^{-1-\delta_0})u_2.$$ 

Clearly, the nonlinear term is irrelevant and the conclusion follows from the classical result for the heat equation. \qed

With the limiting profile of $u_2$ settled, we can now derive a better estimate for $u_1$ than the one given in lemma 2.2.

**Lemma 2.4.** Suppose that $a_1, a_2 \in \mathcal{B}$ and $q > q(A)$. For any $\varepsilon > 0$ there exists $t_0 > 0$ such that if $t \geq t_0$,

$$u_1(x, t) \leq M(1 + |y|)^{-q}t^{-\lambda - 1/2},$$

where $\lambda = \lambda_{A-\varepsilon}$ is the first eigenvalue of $\mathcal{L}_{A-\varepsilon}$ in $L^2(R^1, d\mu)$ and $M$ is a positive constant.

**Proof.** Let $t_1 > 1$ be sufficiently large that

$$w_2(y, s) \geq A\phi(y) - e^{-\delta s} \geq \frac{1}{2} A\phi(y)$$

for $|y| \leq D$ and $s \geq s_1 \equiv \log t_1$, where $D$ is a large positive number to be determined later. Set

$$\bar{w} = e^{-\lambda s}(M_1 \psi(y) + (1 + |y|^2)^{-q/2})$$

Define

$$J(w) = w_s - w_{yy} - \frac{1}{2}yw_y - \frac{1}{2}w + w_2^2w.$$ 

Let $\psi(y)$ be the eigenfunction of $\mathcal{L}_{A-\varepsilon}$ corresponding to $\lambda$ with $\max_{y \in R^1} \psi(y) = 1$. It is easy to see that

$$J(e^{-\lambda s}\psi(y)) \geq \begin{cases} e^{-\lambda s}\psi(y)[A\varepsilon\phi^2(y) - e^{-\delta s}], & |y| \leq D, \\ -e^{-\lambda s}A^2\psi(y)\phi^2(y), & |y| \geq D. \end{cases}$$

Similarly,

$$J(e^{-\lambda s}(1 + |y|^2)^{-q/2})$$

$$\geq e^{-\lambda s}\left( \left( \frac{q - 1}{2} - \lambda \right)(1 + |y|^2) - \frac{q(q + 2)}{2} + \frac{q(q + 2)|y|^2}{1 + |y|^2} \right)(1 + |y|^2)^{-q/2-1}$$

$$> e^{-\lambda s}(1 + |y|^2)^{-q/2}(\frac{q - 1 - 2\lambda}{4}) > 0 \quad \text{if} \quad |y| \geq D_1(\lambda, q),$$

$$\to 0 \quad \text{as} \quad t \to \infty.$$
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where $D_1$ is the first positive number such that
\[
\frac{1}{4}(q - 1 - 2\lambda)(1 + |y|^2)^2 \geq \frac{3}{2}q(1 + |y|^2) + q(q + 2)|y|^2 \quad \text{for all } |y| \geq D_1.
\]

Here we assume that $D \gg D_1$. Clearly,
\[
J(e^{-\lambda s}\psi(y)) \geq \frac{1}{2}e^{-\lambda s}A\varepsilon \psi^2 > 0
\]
if $s \geq s_2(A,D)$ and $|y| \leq D$. Furthermore,
\[
J(M_1 e^{-\lambda s}\psi(y)) \geq \frac{1}{2}e^{-\lambda s}M_1 A\varepsilon \psi^2 > 2q(q + 2)e^{-\lambda s}(1 + |y|^2)^{-q/2 - 1}
\]
if $M_1 > M(A,q)$ and $|y| \leq D_1$. This shows that $J(\bar{w}) \geq 0$ if $|y| \leq D$ and $s \geq s_2(A,D)$.

Finally, if $|y| \geq D$ and $D$ is sufficiently large,
\[
J(e^{-\lambda s}(M_1 \psi(y) + (1 + |y|^2)^{-q/2})) \\
\geq ((\frac{1}{4}(q - 1 - 2\lambda))(1 + |y|^2)^{-q/2} - M_1 M_2^2 \psi(y)\phi^2(y)e^{-\lambda s} > 0.
\]

This shows that $\bar{w}$ is a supersolution if $s \geq s_0 = \max(s_1, s_2)$. It is a simple matter to see that $w_1(y, s_0)$ can be bounded above by $M_2\bar{w}(y, s_0)$, where $M_2$ is a big positive number. By the maximum principle $w_1(y, s) \leq M_2\bar{w}(y, s), s > s_0$. This completes the proof of the lemma. \hfill \Box

The success of the RG method depends, crucially, on estimates of solutions to the linear operators $w_d = \mathcal{L}_d w$ and $w_{in} = \mathcal{L}w$. We collect some relevant estimates in the next lemma. In what follows, we use $e^{-s\mathcal{L}_d} f$ and $e^{-s\mathcal{L}} f$ respectively to denote their solutions with initial value $f$.

**Lemma 2.5.** Suppose that $f \in B$.

(i) **There exist** $\delta = \delta(A, q)$ and $s_0 < \infty$ such that, for $s \geq s_0$,
\[
\|e^{-s\mathcal{L}_d} f\| \leq e^{-\delta s}\|f\|.
\]

Moreover, if $q > q(A)$ and $(\psi_A, f) = 0$,
\[
\|e^{-s\mathcal{L}_d} f\| \leq e^{-(\delta + \lambda_A)s}\|f\|,
\]
where $\lambda_A$ is the first eigenvalue of $\mathcal{L}_d$, and $(\cdot, \cdot)$ is the inner product in $L^2(R^1, d\mu)$.

(ii) **If** $g \in L^2(R^1, d\mu)$, **there exists** $c > 0$ such that
\[
\|e^{-\mathcal{L}_d} g\| \leq C\|g\|_2,
\]
where $\|\cdot\|_2$ is the norm in $L^2(R^1, d\mu)$.

(iii) **Let** $\chi_u = \chi(|x| \geq \sigma)$, **the characteristic function**. **There exists** $C > 0$ such that
\[
\|\chi_u e^{-s\mathcal{L}_d} g\| \leq C e^{-(q-1)\sigma/2}\|g\|.
\]
(iv) Suppose that \( \int f \, dx = 0 \). There exist \( \delta = \delta(q) > 0 \) and \( s_0 < \infty \) such that, for \( s \geq s_0 \),
\[
\|e^{-sL}f\| \leq e^{-\delta s}\|f\|.
\]

(v) The quantities \( |\lambda_A - \lambda_{A'}|, |1 - (\psi_A, \psi_{A'})| \) and the operator norm \( \|P_A - P_{A'}\| \) in \( \mathcal{B} \) are all bounded by \( C(M)|A - A'| \), where \( P_A \) is the orthogonal projection in \( L^2(R^4, d\mu) \) on \( \psi_A \) and \( \|P_A\| \leq C(M) \), for \( 0 \leq A, A' \leq M \).

**Proof.** The first part of (i) follows immediately from lemma 2.2. The second part is proved in the proposition below.

To show (ii), observe that
\[
(e^{-Lg})_n(x) = \int_R e^{-Lg} (x, y) e^{-y^2/8} e^{y^2/8} g(y) \, dy.
\]

Then, an application of the Cauchy–Schwarz inequality together with the fact that
\[
\sup_x (1 + |x|)^q \left( \int_R (e^{-Lg}(x, y))^2 e^{-y^2/4} \, dy \right) < \infty
\]
yield the desired result. The validity of (2.7) can be verified by using
\[
e^{-sLg}(x, y) \leq e^{-sL_0}(x, y), \quad \text{by the Feynman–Kac formula},
\]
and the Mehler formula
\[
e^{-sL_0}(x, y) = (4\pi(1 - e^{-s}))^{-1/2} \exp \left( -\frac{|x - e^{-s/4}y|^2}{4(1 - e^{-s})} \right).
\]

Part (iii) is a direct consequence of a property of the ‘transformed’ heat kernel \( e^{-sL_0}(x, y) \) (see [5]).

Part (iv) is essentially the same as the second part of (i), except the first eigenvalue is zero and the corresponding eigenfunction is \( \phi = e^{-x^2/4} \). The condition \( \int f \, dx = 0 \) means that \( (f, \phi) = 0 \), where \( (\cdot, \cdot) \) is the inner product in \( L^2(R^4, \phi \, dx) \).

Part (v) can be proved by using
\[
(\psi_A, \psi'_A) = 0, \quad \lambda'_A \leq (4\pi d)^{-1}
\]
and \( (L_A - \lambda_A)^{-1} \) is a bounded operator on the subspace \( \{f \in \mathcal{B} \mid (f, \psi_A) = 0\} \), which follows from the second part of (i) (for more details, see the appendix of [5]). Here \( \psi'_A \) and \( \lambda'_A \) are derivatives of \( \psi_A \) and \( \lambda_A \) to \( A \), respectively.

**Remark.** Some of the results in lemma 2.5 are proved in [5]. We collect them here for easy reference in later sections.

**Proposition 2.7.** The second part of lemma 2.5(i) is true.

**Proof.** Suppose \( s_1 > 0 \). Let \( \delta > 0 \) be small so that
\[
q > 1 + 2\lambda_A + 4\delta \quad \text{and} \quad \mu_A - \lambda_A > 4\delta,
\]
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where $\mu_A$ is the second eigenvalue of $A$. Take $f \in B$, $(\psi_A, f) = 0$ and $\|f\| = 1$. We proceed to show inductively that, for $s_n = ns_1$, $\chi_b = \chi(|x| \leq s_1)$ and $\chi_u = 1 - \chi_b$, $v(s_n) = e^{-s_nA}f$, we have the inequalities

$$\|\chi_u v(s_n)\| \leq e^{-\beta n},$$  \hspace{1cm} (2.8) \\
$$\|\chi_b v(s_n)\|_2 + \|x_b v(s_n)\| \leq e^{s_1^2/6}e^{-\beta n},$$  \hspace{1cm} (2.9)

where $\beta = (\lambda_A + 2\delta)s_1$. The conclusion of proposition 2.7 follows immediately from (2.8) and (2.9) if $n$ is large ($n\delta > \frac{1}{2}s_1$) and $s = s_n$. For $s \in (s_n, s_{n+1})$, apply the first part of lemma 2.5(i) and use the fact that $(n-1)\delta > \frac{1}{2}s_1 + \lambda_A$ if $n$ is large.

First, we note that if $f \in B$ and $(\psi_A, f) = 0$, then

$$(e^{-sA}f, \psi_A) = 0 \text{ for all } s > 0.$$ 

If $n = 0$, the bounds in (2.8) and (2.9) hold by the obvious inequality

$$\|\chi_b f\|_2 \leq e^{s_1^2/8}\|f\|$$  \hspace{1cm} (2.10)

and our assumption that $\|f\| = 1$.

If $n = 1$, (2.8) is true by lemma 2.5(iii), since $q > 1 + 2\lambda_A + 4\delta$. Write

$$f = \chi_b f + \chi_u f = f_b + f_u.$$ 

Since $(\psi_A, f) = 0$,

$$|(f_b, \psi_A)| = |(f_u, \psi_A)| \leq C(A)s_1^{-(q-1-2\lambda_A)}$$

and

$$\|e^{-s_1A}f_b\|_2 \leq C(A)s_1^{-(q-1-2\lambda_A)}e^{-\lambda_A s_1} + e^{-\mu_A s_1}\|f_b\|_2 \leq \frac{1}{4}e^{-\beta s_1}e^{s_1^2/8}$$  \hspace{1cm} (2.11)

if $s_1$ is reasonably large. Using lemma 2.5(ii), we get

$$\|e^{-sA}f_b\| \leq Ce^{-(s_1-1)A}f_b \|_2 \leq Ce^{-(s_1-1)(\lambda_A + 2\delta)}e^{s_1^2/8} < \frac{1}{4}e^{-\beta e^{s_1^2/8}}$$  \hspace{1cm} (2.12)

if $s_1$ is a reasonably large constant. By combining the first part of lemma 2.5(i) and (2.10), we have

$$\|\chi_b e^{-s_1A}f_u\|_2 \leq e^{s_1^2/8}\|e^{-s_1A}f_u\| \leq Ce^{-\delta s_1}e^{s_1^2/8} \leq \frac{1}{4}e^{-\beta e^{s_1^2/6}}.$$  \hspace{1cm} (2.13)

It is clear that (2.11)–(2.13) give (2.9).

Suppose that (2.8) and (2.9) hold for $n$. We show they hold for $n + 1$. Write $v = v(s_n) = v_b + v_u$. Since $(v, \psi_A) = 0$,

$$|(v_b, \psi_A)| = |(v_u, \psi_A)| \leq C(A)v_u\|s_1^{-(q-1-2\lambda_A)} \leq e^{-\beta n}$$

and

$$\|e^{-s_1A}v_b\|_2 \leq e^{-\beta n}e^{-\lambda_A s_1} + e^{-\mu_A s_1}\|v_b\|_2 \leq \frac{1}{4}e^{-(\beta + \delta s_1)(n+1)}e^{s_1^2/6}.$$  \hspace{1cm} (2.14)

Again, by lemma 2.5(ii), we get

$$\|e^{-sA}v_b\| \leq Ce^{-(s_1-1)A}f_b \|_2 \leq Ce^{-(s_1-1)(\beta + \delta)(s_1+1)}e^{s_1^2/6} < \frac{1}{4}e^{-\beta(n+1)}e^{s_1^2/6}.$$  \hspace{1cm} (2.15)
By combining lemma 2.5(i) and (2.10), we have
\[ \| \chi e^{-s_1 L_A v_u} \|_2 \leq e^{s_1^2/8} \| e^{-s_1 L_A v_u} \| \leq C e^{s_1^2/8} e^{-\delta s_1} \| v_u \| \leq \frac{e^{-\beta(n+1)}}{4} e^{s_1^2/6}. \] (2.16)

It follows immediately that (2.14)–(2.16) imply (2.9) for \( n+1 \).

Finally, (2.8) follows from
\[ \| \chi e^{-s_1 L_A} \chi_b g \| \leq e^{-s_1^2/5} \| \chi_b g \|_2 \]
\[ \] and lemma 2.5(iii) applied to \( g = v_u \) and the bounds (2.8) and (2.9) for \( v_b \) and \( v_u \).

This completes the proof of the proposition.

3. The linear eigenvalue problem

In this section, we study the linear eigenvalue problem
\[ L_A w = \lambda_A w, \]
where \( \lambda_A > 0 \) is the first eigenvalue of \( L_A \). First, we list the following known facts from classical functional analysis and other sources (for details see [5,7]).

(i) The spectrum of \( L_A \), as considered in \( L^2(R^1, d\mu) \), consists of eigenvalues only and the eigenfunctions form a complete orthogonal set in that space.

(ii) The first eigenvalue \( \lambda_A > 0 \) is non-degenerate and the corresponding eigenfunction is positive and even in the whole space with \( w'(0) = 0 \).

(iii) \( \lambda_A \) depends continuously on \( A \), is an increasing function of \( A \) and is strictly less than \( A^2/4\pi d^2 \).

The main purpose of this section is to show that, as \( y \to \infty \), the eigenfunction \( w \) has the asymptotics
\[ w(y) = ce^{-y^2/4}|y|^{2\lambda_A} + \text{higher-order terms} \]
with \( c \) a positive constant.

As a matter of fact, we shall consider the more general case of
\[ \begin{cases} w'' + \frac{1}{2} y w' + \frac{1}{2} w - A^2 \phi^2 w = -\lambda w, \\ w'(0) = 0, \quad w(y) > 0 \text{ for all } y \geq 0 \end{cases} \] (3.1)
and prove that a positive, even solution with \( \lambda > 0 \) has the above asymptotics when \( \lambda_A \) is replaced by \( \lambda \), provided that \( w \in L^2(R^1, d\mu) \). If we integrate equation (3.1) we have
\[ w'(y) + \frac{1}{2} y w = \int_0^y (A^2 \phi^2 - \lambda) w \, dv. \] (3.2)

If there exists \( y_0 > 0 \) such that
\[ w'(y_0) + \frac{1}{2} y_0 w(y_0) < 0 \quad \text{and} \quad A^2 \phi^2(y) - \lambda < 0 \quad \text{for all } y \geq y_0, \]
then \( g(y) = w'(y) + \frac{1}{2} y w \) satisfies \( g(y_0) \leq 0 \) and \( g'(y) < 0 \) for all \( y \geq y_0 \). Hence, there exists \( \eta > 0 \) with the property \( g(y) < -\eta \) for all \( y \geq y_0 + 1 \). An integration then yields
\[
e^{y^2/4} w(y) - e^{y_1^2/4} w(y_1) < -\eta \int_{y_1}^{y} e^{\nu^2/4} \, d\nu,
\]
which in turn implies that
\[
w(y) < -\frac{\eta}{2y} \quad \text{for all } y \gg 1,
\]
a contradiction! Thus, \( g(y) > 0 \) for all \( y \) large. In addition, it is easy to see that \( w' \) cannot change sign for \( y \) large. This, together with \( g(y) > 0 \), shows that \( w'(y) < 0 \) for all \( y \) large. An inspection of (3.2) gives \( g(y) > 0 \) for all \( y > 0 \). Consequently, \( g'(y) \to 0 \) as \( y \to \infty \).

To derive the exact asymptotics, we make a change of variables. Let
\[
z = e^{y^2/4} w.
\]
We know from the above analysis that \( z, z' > 0 \) for all \( y > 0 \). The equation for \( z \) is
\[
z'' - \frac{1}{2} y z' = (A^2 \phi^2 - \lambda) z.
\]
If \( z \to M > 0 \) as \( y \to \infty \), upon an integration we would have
\[
z' > \frac{\lambda z}{y} \quad \text{for all } y \gg 1.
\]
Clearly, this is in contradiction with \( z \to M > 0 \) as \( y \to \infty \). Hence, \( z \to \infty \) as \( y \to \infty \).

Let \( G(y) = \frac{1}{2} y z' - \lambda z \) and
\[
G' = \frac{1}{2} y z'' + (\frac{1}{2} - \lambda) z' = \frac{1}{2} y G + (\frac{1}{2} - \lambda) z' + \frac{1}{2} y A^2 \phi^2 z.
\]
We show that \( G < 0 \) for \( y > 0 \) if \( \lambda \leq \frac{1}{2} \). Otherwise, \( G > 0 \) for all \( y \) large. Furthermore,
\[
Ge^{-y^2/4} > C > 0 \quad \text{for all } y \gg 1.
\]
Consequently,
\[
yz' > Ce^{y^2/4} \iff g(y) > \frac{C}{y} \quad \text{for all } y \gg 1.
\]
Another integration shows \( w(y) \geq C/y^2 \), which is in clear contradiction with \( w \in L^2(R^1, d\mu) \). Hence, \( G < 0 \) for all \( y > 0 \) if \( \lambda \leq \frac{1}{2} \).

Similarly, \( G > 0 \) for all \( y \gg 1 \) if \( \lambda > \frac{1}{2} \). In addition, for any \( \delta > 0 \), \( z' \leq \delta z \) if \( y \gg 1 \).

We now show that
\[
z y^{-2\lambda} \to C \neq 0 \quad \text{as } y \to \infty.
\]
For this purpose, we make another change of variables by letting
\[
\sigma(t) = z y^{-2\lambda}, \quad t = \log y.
\]
The equation for \( \sigma \), using \( \sigma' = d\sigma/dt \), is
\[
\sigma'' + \left(-\frac{1}{2}e^{2t} + 4\lambda - 1\right) \sigma' + 2\lambda(2\lambda - 1) \sigma = e^{2t} A^2 \phi^2 \sigma. \quad (3.3)
\]
Case 1 ($\lambda \leq \frac{1}{2}$). In this case, $\sigma' < 0$ for all $t$ large. A differentiation of equation (3.3) gives

$$
\sigma'''' + (-\frac{1}{2}e^{2t} + 4\lambda - 1)\sigma'' + (-e^{2t} + 2\lambda(2\lambda - 1) - A^2\phi^2)\sigma' = 2e^{2t}A^2\phi\phi'\sigma.
$$

It is clear that $\sigma'' > 0$ for all $t$ large. Hence,

$$
\frac{-e^{2t}}{4} \frac{\sigma'}{\sigma} < 2\lambda(1 - 2\lambda) + e^{2t}A^2\phi^2, \quad t \gg 1.
$$

An integration then yields that $\log(1/\sigma)$ is bounded from above. Therefore, $\sigma \to C > 0$ as $t \to \infty$.

Case 2 ($\lambda > \frac{1}{2}$). First, we observe that, for any $\delta > 0$, $\sigma' < \delta \sigma$ if $t \gg 1$. Otherwise, using (3.3), we would have $\sigma'' > 0$, $(\sigma' - \delta \sigma)' > 0$ for all $t \gg 1$.

Hence,

$$
\frac{\sigma''}{\sigma'} > (\frac{1}{2} - \varepsilon)e^{2t}, \quad \text{for any } \varepsilon > 0 \text{ if } t \gg 1,
$$

an integration of which yields

$$
\log \sigma' > \frac{1}{2}(\frac{1}{2} - \varepsilon)e^{2t}.
$$

Moreover, we see that

$$
\sigma' - \mu \sigma > 0, \quad (\sigma' - \mu \sigma)' > 0
$$

for any $\mu > 0$, if $t \gg 1$. Consequently, we would have $z' \geq \mu z$ if $y \gg 1$, which is a contradiction.

Thus, $\sigma'/\sigma \to 0$ as $t \to \infty$. Consequently, $\sigma'' > \frac{1}{2}\sigma e^{2t}$ if $t \gg 1$.

If $\sigma \to \infty$ as $t \to \infty$, we would have

$$
\frac{\sigma''}{\sigma'} > Me^{2t}, \quad \text{for any } M > 1 \text{ if } t \gg 1.
$$

An integration shows that

$$
\log \sigma' > \frac{1}{2}Me^{2t}, \quad \log z' > \frac{1}{3}My^2 \quad \text{if } y \gg 1.
$$

From here, it is easy to see that

$$
z > e^{\theta^2/4} \quad \text{if } y \gg 1.
$$

But, this is impossible. Hence, again, $\sigma \to C > 0$ as $t \to \infty$.

The asymptotics of $w$ is established.
4. Self-similarity by the renormalization group method

In this section, we use the RG method, in combination with the a priori estimates derived in §2, to prove theorem 1.1.

The RG method is an iterative scheme. Let $L > 1$. We start by defining the RG map $R$, 

$$(a_1, a_2) \xrightarrow{R} (a'_1, a'_2) \text{ in } B \times B$$

as 

$$a'_i(x) = Lu_i(Lx, L^2), \quad i = 1, 2,$$

where $a_i = u_i(x, 1)$, $i = 1, 2$, and $(u_1, u_2)$ solve (1.1). It is clear that 

$$(a_1^n, a_2^n) \equiv R^n(a_1, a_2) = (L^n u_1(L^n x, L^{2n}), L^n u_2(L^n x, L^{2n})) = R(a_1^{n-1}, a_2^{n-1}).$$

Our ultimate goal is to prove that $(u_1, u_2)$ behaves asymptotically as the solution of the limiting linear problem. Accordingly, we decompose the RG map as

$$a'_1 = L^{-2L} a_1 + Ln_1(Lx, L^2), \quad (4.1)$$

$$a'_2 = L^{-2L} a_2 + Ln_2(Lx, L^2), \quad (4.2)$$

where

$$n_1(x, t) = -\int_1^t \int_R G_A(t, \tau, x, y) \left( u_1 u_2^2(y, \tau) - u_1 A^2 \tau^{-1} \phi^2 \left( \frac{y}{\sqrt{\tau}} \right) \right) dy,$$

$$n_2(x, t) = -\int_1^t \int_R G(t - \tau, x - y) u_1 u_2^2(y, \tau) dy$$

and with $A$ as defined in what follows.

To track the evolution, we write

$$a_1(x) = B\psi_A + b_1, \quad a_2(x) = A\phi(x) + b_2(x), \quad (4.3)$$

where $B = (a_1, \psi_A)$, $A = \int a_2 \, dx$. With the normalization

$$\langle \psi_A, \psi_A \rangle = 1 \quad \text{and} \quad \int \phi \, dx = 1,$$

we have

$$\langle \psi_A, b_1 \rangle = 0, \quad \int b_2 \, dx = 0.$$

It is easy to verify that, with the same decomposition as in (4.3) for $a'_1$ and $a'_2$,

$$A' = A + \int Ln_2(Lx, L^2) \, dx, \quad B' = \langle \psi_A, a'_1 \rangle, \quad b'_1 = (1 - P_{A'}{a'_1}), \quad b'_2 = L^{-2L} b_2 + Ln_2(Lx, L^2) + (A - A')\phi.$$

Since the system (1.1) is invariant under the scaling transform

$$u_{ik} = K u_i(Kx, K^2, t), \quad i = 1, 2,$$

and lemma 2.4, we can assume that

$$\|a_1\| \cdot \|a_2\| < \varepsilon \quad \text{and} \quad \|u_1(s)\| \cdot \|u_2(s)\| < \varepsilon, \quad 1 \leq s \leq L^2.$$
We proceed to derive estimates for $A'$, $b'_2$, $B'$ and $b'_1$, in that order. Since,
\[
\int_R G(t - \tau, x - y)(1 + |y|)^{-q} \, dy \leq c_1 e^c(t - \tau)(1 + |x|)^{-q}
\]
for some $c > 0$,
\[
\|L_n^2(Lx, L^2)\| \leq C(L)\|a_2\| \quad \text{and} \quad \|n_2\| = \sup_{t \in [1, L^2]} \|n_2(\cdot, t)\| \leq C(L)\|a_2\|.
\]
It follows that
\[
|A' - A| \leq C(L)\|a_2\|,
\]
and consequently,
\[
\|b'_2\| \leq L^{-2\delta}\|b_2\| + C(L)\|a_2\|
\]
by lemma 2.5(iv).

Now, consider $n_1$, and write
\[
w(y, \tau) = u_2^2(y, \tau) - A^2\tau^{-1}(e^{\frac{y}{\sqrt{\tau}}})
\]
\[
= \left(u_2(y, \tau) + A\tau^{-1/2}(e^{\frac{y}{\sqrt{\tau}}})\right)(e^{d(\tau-1)}b_2(y) + n_2(y, \tau)),
\]
where $\Delta \equiv d^2/dx^2$. By lemma 2.5(iv) and (4.5),
\[
\|w\| \leq C(L)(\|a_2\| + A)(\|b_2\| + \varepsilon\|a_2\|),
\]
and, since $A \leq C\|a_2\|$ and
\[
\int_R G_A(t, \tau, x, y)(1 + |y|)^{-q} \, dy \leq C\left(1 + \frac{|x|}{\sqrt{t - \tau}}\right)^{-q},
\]
which follows from lemma 2.2, we obtain
\[
\|L_n^1(Lx, L^2)\| \leq C(L)\|(b_2\| + \varepsilon\|a_2\|),
\]
using $\|u_1\| \leq C\|a_1\|$ and $\|a_1\| \cdot \|a_2\| \leq \varepsilon$.

Next, we consider $B'$. Since, from (4.1),
\[
B' = (\psi_A', a_1')
\]
\[
= (\psi_A', L^{-2\lambda_A}a_1) + (\psi_A', L_n^1(Lx, L^2))
\]
\[
= (\psi_A', \psi_A)BL^{-2\lambda_A} + (\psi_A', (P_{A'} - P_A)L^{-2\lambda_A}b_1) + (\psi_A', L_n^1(Lx, L^2)),
\]
P_Ab_1 = 0, by lemma 2.5(v) and (4.7), we have
\[
|B' - BL^{-2\lambda_A}| \leq C|A' - A|L^{-2\lambda_A}(B + \|b_1\|) + C(L)\varepsilon(\|b_2\| + \varepsilon\|a_2\|)
\]
\[
\leq C(L)\varepsilon^2 + C(L)\varepsilon^2\|a_2\| + C(L)\varepsilon\|b_2\|.
\]
In the latter inequality, we used (4.6), $\|a_1\| \cdot \|a_2\| \leq \varepsilon$ and $|B| + \|b_1\| \leq C\|a_1\|$.

Finally, we estimate $b'_1$:
\[
b'_1 = (1 - P_{A'})a_1' = BL^{-2\lambda_A}(P_A - P_{A'})\psi_A + L^{-2\lambda_A}b_1 + L^{-2\lambda_A}b_1(P_A - P_{A'}).
again by using \(P_A b_1 = 0\). It follows immediately from (i) and (v) of lemma 2.5, (4.7) and (4.6), that
\[
\|b'_1\| \leq L^{-2(\lambda_A + \delta)}\|b_1\| + C(L)\varepsilon(1 + \|a_2\| + \|b_2\|).
\]

To summarize our estimates, we have the following lemma.

**Lemma 4.1.** Suppose that \(L \geq L_0 = e^{2s_0}\), where \(s_0\) satisfies the conditions in lemmas 2.4 and 2.5. There exist \(\varepsilon_0(L)\) and \(C(L)\) such that if \(\|a_1\| \cdot \|a_2\| \leq \varepsilon \leq \varepsilon_0(L)\), we have

(a) \(|A' - A| \leq C(L)\varepsilon\|a_2\|\),

(b) \(||b'_2|| \leq L^{-2\delta}\|b_2\| + C(L)\varepsilon\|a_2\||,

(c) \(|B' - L^{-2\lambda_A}B| \leq C(L)\varepsilon(1 + \|a_2\| + \|b_2\|),

(d) \(||b'_1|| \leq L^{-2(\lambda_A + \delta)}\|b_1\| + C(L)\varepsilon(1 + \|a_2\| + \|b_2\||).

**Proof of theorem 1.1.** We write \(a^n_i\) as in (4.3) and derive bounds for \(A_n, B_n\) and \(b_n^i\) using lemmas 2.4 and 4.1.

It is clear that \(A_n \to A^* = \int_R (a_1 + a_2) \, dx\).

First, by lemmas 2.4 and 4.1(a),
\[
|A_{n+1} - A_n| \leq C(L)\|a^n_1\| \cdot \|a^n_2\| \cdot \|a^n_2\| \leq C(L)e^{-2n\eta},
\]
with \(0 < \eta < \lambda^*_A\), and hence
\[
|A_n - A^*| \leq C(L)e^{-2n\eta}.
\]

Set
\[
n\lambda_n = \sum_{m=0}^{n-1} \lambda_{A_m}.
\]

Since \(\lambda_A\) is a continuous and increasing function of \(A\) and \(A_n\) is a bounded and increasing sequence,
\[
\lambda_{A_n} \to \lambda^* = \lambda^*_{A^*}, \quad \lambda_n \to \lambda_{A^*}.
\]

If we take \(\eta < \min(\delta, \lambda^*_{A^*})\), by lemmas 2.4 and 4.1(b) we get
\[
\|b^n_2\| \leq e^{-2n\eta}\|a_2\| \quad \text{and} \quad \|b^n_1\| \leq e^{-2n(\lambda^*_{A^*} + \eta)}(1 + \|a_2\|).
\]

This, together with lemmas 2.4 and 4.1(c), gives
\[
|B_{n+1} - B_n L^{-2\lambda_{A_n}}| \leq C(L)L^{-2n(\lambda^*_{A^*} + \eta)}(1 + \|a_2\|),
\]
with a smaller \(\eta\), if necessary.

This shows that there exists \(B^*\) such that
\[
B_n L^{2n\lambda_n} \to B^* \quad \text{as} \ n \to \infty.
\]
Now, since $\lambda_A$ is an increasing and differentiable function of $A$ and $\lambda_A' \leq \frac{1}{4}\pi d$,

$$|n\lambda_n - n\lambda_A'| = \left| \sum_{m=0}^{n-1} [\lambda_{A_m} - \lambda_{A'}] \right| \leq \sum_{m=0}^{n-1} |\lambda_{A_m} - \lambda_{A'}| \leq \sum_{m=0}^{n-1} C(L)e^{-2m\eta},$$

and therefore $n\lambda_n - n\lambda_A'$ converges to a finite limit as $n \to \infty$. Hence,

$$B_n L^{2n\lambda_A'} \to B^{**} \text{ as } n \to \infty.$$  

With $t = L^{2n}$, $L > L_0$, the results are directly translated to

$$\|\sqrt{t}u(t) - A^*\phi\| \leq Ct^{-\eta}\|a_2\|, \quad (4.8)$$

$$\|t^{1/2+\lambda_A'}u_1(t) - B^{**}\psi_A\| \leq Ct^{-\eta}(1 + \|a_2\|). \quad (4.9)$$

This proves theorem 1.1.

Remark 4.2. To see why theorem 1.1 cannot be true if $q \leq q(A)$, we look at the linear problem

$$u_s = -\mathcal{L}_A u. \quad (4.10)$$

Set

$$w_L(y,s) = \begin{cases} e^{-\lambda_A s}\psi_A(y), & \text{if } |y| \leq D, \\ C_0 e^{-\lambda_A s}(1 + |y|^2)^{-q/2}, & \text{if } |y| \geq D, \end{cases}$$

where $C$ is the constant which makes $w_L$ continuous at $|y| = D$. It is easy to verify (see the proof of lemma 2.2) that the function $w_L$ is a subsolution of (4.10) provided that $D$ is sufficiently large that

$$q(q + 2) \frac{|y|^2}{(1 + |y|^2)^2} \geq A^2 \psi^2(y) \quad \text{for all } |y| \geq D$$

and

$$\psi_A'(y) \leq C(1 + |y|^2)^{\eta/2} \quad \text{at } |y| = D.$$  

It is clear that, for a solution to (4.10) with continuous, positive initial value $u_0(y)$ satisfying

$$\lim_{|y| \to \infty} u_0(y)(1 + |y|)^q = M > 0,$$

we can find a $\delta > 0$ such that $u_0(y) > \delta w_L(y,0)$. Consequently,

$$u(y,s) > \delta w_L(y,s), \quad s > 0.$$  

This shows that theorem 1.1 cannot be true for the linear problem (4.10) since $w_L(y,s)$, although it has the desired time decay, but has a much slower decay in the spatial variable $y$ than $\psi_A(y)$. In summary, in spite of the fact that we have not given a direct proof that theorem 1.1 is false when $q \leq q(A)$, the above serves as strong evidence that it should not be true for the nonlinear problem if it is false for the limiting linear problem.
References


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