The critical exponents of parabolic equations and
blow-up in $\mathbb{R}^n$

Yuan-wei Qi

Department of Mathematics, Hong Kong University of Science and
Technology, Hong Kong

(MS received 10 October 1996. Revised MS received 23 January 1997)

In this paper we study the Cauchy problem in $\mathbb{R}^n$ of general parabolic equations which take
the form $u_t - \Delta u^p + |x|^{r} u^q$ with non-negative initial value. Here $s \geq 0$, $m > (n - 2)/n$,
$p > \max(1,m)$ and $\sigma > -1$ if $n = 1$ or $\sigma > -2$ if $n \geq 2$. We prove, among other things, that for
$p \leq p_1$, where $p_1 = m + s(m - 1) + (2 + 2s + \sigma)/n > 1$, every nontrivial solution blows up in
finite time. But for $p > p_1$, a positive global solution exists.

1. Introduction

The study of the blow-up of solutions to nonlinear parabolic equations probably originates from the paper [10] of Fujita in 1966. That paper studied the following Cauchy problem of the semilinear equation:

$$u_t = \Delta u + u^p, \quad t > 0, \quad x \in \mathbb{R}^n,$$

$$u(x, 0) = u_0(x) \geq 0, \quad x \in \mathbb{R}^n,$$

where $p > 1$. Its main result is the following two-part statement:

(a) if $1 < p < 1 + 2/n$, then every nontrivial solution $u(x, t)$ blows up in finite time;
(b) if $p > 1 + 2/n$ and $u_0(x) \leq \delta e^{-1/n}x^2$ ($0 < \delta \ll 1$), then $u(x, t)$ is a global solution.

We shall call the first case in the above statement the blow-up case, or, which is the
same, the global nonexistence case, and the second the global existence case.

This elegant and beautiful work [10] revealed a new phenomenon of nonlinear
PDEs and stimulated the study of similar phenomena for various parabolic, hyperbolic and nonlinear Schrödinger-type equations. Many important results have appeared since then, but we cannot give a detailed account in such a short article.

The interested reader is referred to the survey paper by H. Levine [14] and the
references therein for a good account of related works. A brief review of some related
results on parabolic equations will be given below.

In this paper, we shall consider Cauchy problems of parabolic equations which

$$u_t = \Delta u^m + f(x, t, u),$$

with non-negative initial value. Here $f$ is a positive function and $m > 0$. 
A rather special case is

\[
\begin{align*}
(\text{II}) \\
\begin{cases}
  u_t = \Delta u^m + |x|^\sigma u^p, \\
  u(x, 0) = u_0(x) \geq 0,
\end{cases}
\end{align*}
\]

where \( m > 0, \sigma > -2 \) and \( p > 1 \). For simplicity of presentation, I shall restrict my attention to equations of the form

\[
\begin{align*}
(\text{III}) \\
\begin{cases}
  u_t = \Delta u^m + |x|^\sigma t^s u^p \\
  u(x, 0) = u_0(x) \geq 0,
\end{cases}
\end{align*}
\]

though the method works for more general nonlinear equations. Here \( s \geq 0, m > 0, p > 1 \) and \( \sigma \) (not necessarily positive) are constants. The existence and regularity of solutions to (III) for \( t \) small have been considered in [4–7], but in this paper we are interested in studying the large-time behaviour of solutions. Furthermore, we assume very mild regularity for a solution \( u \) of (II) or (III) such that it belongs to \( C((0, t_{\text{max}}), L^1_{\text{loc}}(\mathbb{R}^n)), C((0, t_{\text{max}}), L^\infty_{\text{loc}}(\mathbb{R}^n)) \) and \( C((0, t_{\text{max}}), L^p_{\text{loc}}(\mathbb{R}^n)) \), where \( t_{\text{max}} \) is their maximum existence time. In this context, '\( u \) blows up in finite time' means that

\[
w(t) = \int_D u(x, t) \, dx \to \infty \quad \text{as} \quad t \to T > 0,
\]

for a finite \( T > 0 \), where \( D \) is a finite domain in \( \mathbb{R}^n \). If we assume further that \( u \in C((0, T) \times \mathbb{R}^n) \), which is shown to be true under very general conditions on initial values in [4–7], then \( u \) blows up in the sense of Friedman and McLeod [9].

The main purpose of the present paper is to give a unified yet simple approach to working out the necessary and sufficient condition that \( n, m, p, s, \sigma \) must satisfy for nonexistence of global solutions, no matter how small the initial value is (provided that it is not identically zero), and to prove the existence of global solutions otherwise. The merit of this paper is that the equation under consideration is more general than in previous studies and a better result is obtained. In addition, the technique used in this paper is very simple and natural, yet powerful, in that it not only groups many known results into a simpler scheme, but also enables us to solve some open problems.

Since various special cases of (II) and (III) have been studied by many people, we shall give a brief review of some known results to clarify progress up to now.

For (1.1) it was proved by Hayakawa [14] and Kabayashi et al. [16] that \( p = 1 + 2/n \) belongs to the blow-up case. See also [2] and [26] for more elegant proofs.

These results on (1.1) have been generalised by Bandle and Levine [3] and collaborators [18–20] to cover the initial-boundary value problems of (1.1) in cone-like domains as well as of equations where \( u^p \) is replaced by \( |x|^\sigma u^p \). In particular, it is proved in [19] that for general cone-like domains there exists a critical exponent \( p^* > 1 \) such that if \( 1 < p \leq p^* \), then every nontrivial solution blows up, and there exist global solutions otherwise. It is also shown in [20] that with nonlinearity \( |x|^\sigma u^p \), the Cauchy problem has no global solution if \( p < 1 + (2 + \sigma)/n \), but global solutions exist when \( p > 1 + (2 + \sigma)/n \). Nonetheless, the critical case \( p = 1 + (2 + \sigma)/n \) is open.
For (II), when \( \sigma = 0 \) and \( m \neq 0 \), a complete resolution is obtained through the works of Galaktionov et al. [11, 12], Mochizuki and Mukai [21] and the present author [22, 24]. We summarise the results as the following theorem:

**Theorem B.** Let \( m > (n - 2)_- + n / n \), \( u \) be a solution of

\[
\begin{align*}
  u_t &= \Delta u^\sigma + u^p, \quad t > 0, \quad x \in \mathbb{R}^n, \\
  u(x, 0) &= u_0(x) \geq 0, \quad x \in \mathbb{R}^n.
\end{align*}
\]

(a) if \( 1 < p \leq m + 2 / n \), then every nontrivial solution \( u(x, t) \) blows up in finite time;
(b) if \( p > m + 2 / n \) and \( u_0(x) \) is sufficiently small, then \( u(x, t) \) is a global solution.

In the past couple of years, new results on weakly coupled systems have appeared [8, 17, 25]. Recently, Aguirre and Escobedo [1] proved a related result on the semilinear heat equation (1.1) with a convection term \( a \cdot \nabla u^\sigma \) added to it.

The main result of this paper is the following theorem:

**Theorem 1.1.** Let \( s \geq 0, m > (n - 2)_- + n / n \), \( \sigma > -1 \) if \( n = 1 \) or \( \sigma > -2 \) if \( n \geq 2 \) and \( p_r \equiv m + (m - 1)s + (2 + \sigma + 2s)/n + 1 \). Then there exists no nontrivial global solution of (III) if

\[
\max \{1, m + s(m - 1)\} < p \leq p_r. \quad (1.4)
\]

For the global existence we can prove the next theorem:

**Theorem 1.2.** Let \( \sigma, s \) and \( m \) be as in Theorem 1.1. Then there exists a positive global solution of (III) if \( p > p_r \).

**Remark 1.3.** When \( s = 0 \), Theorem 1.1 implies that the critical exponent \( p_r = m + (2 + \sigma) / n \) belongs to the blow-up case, provided \( (n - 2)_- / n < m < \infty \), a case which is left open in [20]. In the general case where \( s > 0 \), we believe that there is no result which is comparable to ours.

**Remark 1.4.** It will be clear from the proof of Theorem 1.1 that the same result is true if we replace the Cauchy problem in \( \mathbb{R}^n \) by an initial-boundary-value problem in \( \mathbb{R}^n \setminus D \), where \( D \) is a bounded domain, with the usual Dirichlet or Neumann boundary condition on \( \partial D \) to guarantee the local existence under a mild condition on the initial value.

**Remark 1.5.** Under our assumption \( m > (n - 2)_- + n / n \), any solution of (III) with non-negative and nontrivial initial value cannot become zero at any finite time \( T > 0 \). The case \( m = 1 \) is clear. The case \( m > 1 \) can be demonstrated by using the Barenblatt–Pattle solution and maximum principle, whereas for \( m < 1 \), validity is a direct consequence of a result of Herreo and Pierre [15].

The plan of this paper is that the blow-up result will be proved for the simple case (II) (of \( s = 0 \)) in Section 2. It serves the purpose of demonstrating the basic idea of the method and shows how it can be used effectively for such problems. The techniques involved are more transparent in this case than in the general one. Furthermore, the result for this case is more comparable to the known results. It takes the form of the following theorem:
Theorem 1.6. Let \( m > (n - 2)\gamma/n, p_c = m + (2 + \sigma)/n > 1 \). Then there exists no non-trivial global solution of (II) if

\[
1 < p \leq p_c. \tag{1.5}
\]

The blow-up case for (III) will be established in Section 3, with the proof of Theorem 1.1. In the last section, an ODE will be studied and the existence of ground-state solutions given. It includes (III) with \( p > p_c \) as a special case and hence accomplishes the proof of Theorem 1.2.

Remark 1.7. The method developed in this paper can be used to study similar parabolic equations with a convection term added, as well as weakly coupled systems. We shall study such problems in forthcoming papers.

2. The Simple Case

In this section the Cauchy problem (II) will be studied and Theorem 1.6 proved. Let \( \varphi(x) \) be a smooth, radially symmetric and nonincreasing function which satisfies

\[
0 \leq \varphi \leq 1, \quad \varphi = 1 \text{ for } |x| \leq 1 \quad \text{and} \quad \varphi = 0 \text{ in } |x| \geq 2. \tag{2.1}
\]

It follows that for \( l > 1 \), \( \varphi_l(x) = \varphi(x/l) \) is a smooth, radially symmetric and non-increasing function which satisfies

\[
0 \leq \varphi_l \leq 1, \quad \varphi_l = 1 \text{ for } |x| \leq l \quad \text{and} \quad \varphi_l = 0 \text{ for } |x| \geq 2l. \tag{2.2}
\]

It is easy to see that \( \varphi_l \) satisfies

\[
|\nabla \varphi_l| \leq C/l, \quad |\Delta \varphi_l| \leq C/l^2. \tag{2.3}
\]

Let

\[
w_l(t) = \int_{\Omega} u \varphi_l \, dx. \tag{2.4}
\]

Here \( \Omega = \mathbb{R}^n \setminus B_1 \), with \( B_1 \) being the unit ball with centre at the origin. Then

\[
\frac{d w_l}{dt} = \int_{\Omega} u^n \Delta \varphi_l \, dx + \int_{\Omega} |x|^\sigma u \varphi_l \, dx
\]

\[
\geq - \int_{\Omega} u^n |\Delta \varphi_l| \, dx + \int_{\Omega} |x|^\sigma u \varphi_l \, dx \tag{2.5}
\]

and

\[
\int_{\Omega} u^n |\Delta \varphi_l| \, dx \leq \left( \int_{\Omega} |x|^{-(k-1)\gamma} |\Delta \varphi_l|^{k/(k-1)} \varphi_l^{-1/(k-1)} \, dx \right)^{1/k} \left( \int_{\Omega} |x|^\sigma u^{mk/(k-1)} \varphi_l^{-1/(k-1)} \, dx \right) \tag{2.6}
\]

for any \( k > 1 \). Since \( p > m \), let \( k = p/(p - m) \); then

\[
\int_{\Omega} |x|^\sigma u^{mk/(k-1)} \varphi_l \, dx = \int_{\Omega} |x|^\sigma u \varphi_l \, dx.
\]
Consequently, we derive from (2.5) and (2.6) that
\[
\frac{dw_i}{dt} \geq -C(\phi_i, l) \left( \int_{\Omega} |x|^s u^p \phi_i \, dx \right)^{m/p} + \int_{\Omega} |x|^s u^p \phi_i \, dx.
\]
It is easy to verify that
\[
C(\phi, l) = \left( \int_{\mathbb{R}^n} |x|^{-m(p-\sigma)} |\Delta \phi_i|^p |(p-m)\phi_i^{m/(p-m)} \, dx \right)^{(p-m)/p} = \int_{\mathbb{R}^n} |x|^{-2-(n+\sigma)m/p} C_1,
\]
where \(C_1 = C(\phi, 1)\) is a constant independent of \(l\). Thus,
\[
\frac{dw_i}{dt} \geq \left( \int_{\Omega} |x|^s u^p \phi_i \, dx \right)^{m/p} \left\{ -C_1 l^{n-2-(n+\sigma)m/p} + \left( \int_{\Omega} |x|^s u^p \phi_i \, dx \right)^{(p-m)/p} \right\}.
\]
By Hölder's inequality, we have
\[
\int_{\Omega} |x|^s u^p \phi_i \, dx \geq \left( \int_{\Omega} u \phi_i \, dx \right)^{n} \left( \int_{\Omega} |x|^{-\sigma(p-1)} \phi_i \, dx \right)^{-(p-1)}.
\]
Hence,
\[
\int_{\Omega} |x|^s u^p \phi_i \, dx \geq \begin{cases} 
C_2(n, \sigma, p) w_f l^{n(p-1)+\sigma}, & \text{if } \sigma < n(p-1), \\
C_3(n, \sigma, p) w_f (\log l)^{-\sigma(p-1)}, & \text{if } \sigma = n(p-1), \\
C_4(n, \sigma, p) w_f, & \text{if } \sigma > n(p-1).
\end{cases}
\]
(2.7')

**Lemma 2.1.** Let \(w\) be a positive continuous function which satisfies the following inequality:
\[
\frac{dw}{dt} \geq Cw^q,
\]
in the distributional sense, where \(C > 0\) is a constant and \(q > 1\). Then \(w\) is an increasing function, and there exists a finite \(T > 0\) such that \(w(t) \to \infty\) as \(t \to T\).

**Proof.** If \(w\) is a smooth function, then the result follows immediately. In general, we can construct a sequence of smooth functions \(w_h\) using mollifiers which converge to \(w\) as \(h \to 0\). It can be seen that \(w_h\) satisfies the same inequality as \(w\). Hence, \(w\) must blow up in finite time. \(\square\)

We now prove Theorem 1.6.

**Proof of Theorem 1.6.** First we consider the case \(\sigma < n(p-1)\). It follows from (2.7') that
\[
\frac{dw_i}{dt} \geq \left( \int_{\Omega} |x|^s u^p \phi_i \, dx \right)^{m/p} \left\{ -C_1 l^{n-2-(n+\sigma)m/p} + C_2 w_f^{p-m} l^{(\sigma-n(p-1)m)/(p-m)} \right\}.
\]
(2.8)

(a) \(p < m + (2 + \sigma)/n\). Under this assumption,
\[
[\sigma - n(p-1)](p-m) > p(n-2) - (n+\sigma)m,
\]
and consequently,
\[
\frac{l^{\sigma-n(p-1)}(p-m)}{l^{m-2p(n+\sigma)m}} \to \infty \quad \text{as } l \to \infty.
\] (2.9)

Using the fact that \( w_l \) is an increasing function of \( l \), we find from (2.8) and (2.9) that there exist \( \delta > 0, l \gg 1 \) such that
\[
\frac{dw_l}{dt} \geq \delta \int_{\Omega} |x|^\sigma u^p \phi_l \, dx \geq \delta w_l^p(t)l^{p-p-1}
\] (2.10)

for all \( t > 0 \). Thus, \( w_l \), and consequently \( u \), blows up in finite time by Lemma 2.1.

(b) \( p = m + (2 + \sigma)/n \). In this case,
\[
[p - m - (p-m)](p-m) = p(n-2) - (n+\sigma)m < 0.
\]

If
\[
\lim_{t \to \infty} \int_{\Omega} u^p \phi_l \, dx
\] (2.11)

is an unbounded function of \( t \), then we are done. Otherwise, \( u(\cdot, t) \in L^1(\Omega) \) for all \( t > 0 \) and there exists an \( M > 0 \) such that
\[
\|u(t)\|_{L^1} < M \quad \text{for all } t > 0.
\] (2.12)

We prove that (2.12) is impossible. Suppose the contrary. It is clear from the above that
\[
\int_{\Omega} u(x, t) |\Delta \phi_l| \, dx \to 0 \quad \text{as } l \to \infty,
\]

if \( \int_{\Omega} u^p(x, t) |x|^\sigma \, dx < \infty \). Otherwise, \( w(t) = \int_{\Omega} u(x, t) \, dx \) satisfies \( w(t) \geq 1 \). Hence, upon an integration, we get
\[
w(t) - w(0) \geq \int_0^t \chi(v) \, dv,
\] (2.13)

where \( \chi(t) = \min \{1, \int_{\Omega} u^p(x, t) |x|^\sigma \, dx\} \).

We shall consider different values of \( m \). If \( m > 1 \), we know that
\[
u(x, t) \geq E(x, t; r_0) = (1 + t)^{-1/2} h(y; r_0)
\]
for some \( r_0 > 0 \), where \( E(x, t; r_0) \) is the Barenblatt–Pattle solution of the equation \( \nu_t = \Delta \nu^m \) with
\[
h(y; r_0) = C(n, m)(r_0^2 - y^2)^{(m-1)/2}, \quad y = |x|/(1 + t)^{1/2}.
\]

Here \( \gamma = (m - 1) + 2/n, \nu = n(m - 1) + 2 \). We find easily that
\[
\int_{\Omega} u^p |x|^\sigma \, dx \geq C(1 + t)^{-1}, \quad \text{for } t \text{ sufficiently large}.
\]

Then, \( w(t) \to \infty \) as \( t \to \infty \) by (2.13). But this is a clear contradiction of (2.12). Consequently, (2.12) is not valid, and \( u \) must blow up in finite time.
Critical exponents of parabolic equations

If $m = 1$, then

$$u(x, t) \geq \delta G(x, t),$$  \hfill (2.14)

where $G(x, t)$ is the Gaussian and $\delta$ is a smaller number. Again, we reach a contradiction of (2.12) by employing essentially the same argument as above.

If $(n - 2)_+ / n < m < 1$, thanks to Herrero and Pierre [15], we have an inequality of the form

$$u(x, t) \geq \delta (1 + t)^{-1/\gamma} (1 + C|x|^2/(1 + t)^{2/\gamma})^{1/(1 - m)}$$  \hfill (2.15)

for $|x| \geq 1$, where $C$ and $\delta$ are constants. Here $\gamma = (m - 1)/2 + v$, $v = n(m - 1) + 2$. Again the above inequality implies

$$\lim_{t \to \infty} \int_\Omega u \, dx \to \infty.$$  

Thus, $u$ must blow up in finite time. This completes the proof for the case $\sigma < n(p - 1)$.

We observe that when $\sigma \geq n(p - 1)$, $m > (n - 2)_+ / n$ implies that $n - 2 - (n + \sigma)m/p < 0$.

For the case $\sigma = n(p - 1)$, combining (2.7) and (2.7'), we find that

$$\frac{d w_i}{dt} \geq \left( \int_\Omega |x|^\sigma u^\sigma \rho_i \, dx \right)^{m/p} \left\{ - C_1 |x|^{n - 2 - (n + \sigma)m/p} + C_3 w_i^{p - n} (\log l)^{m - p} (\log l)^{(p - 1)/p} \right\}.$$  \hfill (2.16)

Using the fact that $w_i$ is an increasing function of $l$, we find from (2.16) that there exist $\delta > 0$, $l \gg 1$ such that

$$\frac{d w_i}{dt} \geq \delta \int_\Omega |x|^\sigma u^\sigma \rho_i \, dx \geq \delta w_i^\sigma (\log l)^{-(p - 1)}$$  \hfill (2.17)

for $t > 0$. Thus, $w_i$, and consequently $u$, blows up in finite time.

The case $\sigma > n(p - 1)$ can be handled similarly, using the third inequality in (2.7').

This completes the proof of Theorem 1.6. \qed

**Remark 2.2.** It is clear from the above that if, instead of considering the Cauchy problem in $\mathbb{R}^n$, an initial-boundary value problem is studied in $\mathbb{R}^n \setminus D$ for (II), with $D$ a bounded domain, then the same conclusion is true regardless of the boundary condition on $\partial D$, provided it is so assigned that the local existence can be established under mild conditions on the initial value.

**Remark 2.3.** The reason for using $\Omega = \mathbb{R}^n \setminus B_1$ rather than $\mathbb{R}^n$ itself is that if $\sigma > 0$, then $\int_{B_1} |x|^{-\sigma(p - 1)} \, dx$ may not converge.

### 3. The general case if $p \leq p_c$

In this section, we shall prove Theorem 1.1 for the general case (III). Since the technique for proving Theorem 1.1 is very similar to that used in the last section for proving Theorem 1.6, we shall be brief.
It is easy to see that the function \( w_1(t) \), which is defined in (2.4), satisfies

\[
\frac{d w_1}{dt} = - \int_{\Omega} u^m \Delta \phi_1 \, dx + t^s \int_{\Omega} \phi_1 |x|^\sigma \, dx \\
\geq - \int_{\Omega} u^m |\Delta \phi_1| \, dx + t^s \int_{\Omega} \phi_1 \, dx \\
\geq \left( \int_{\Omega} u^m \phi_1 |x|^\sigma \, dx \right)^{m/p} \left\{ - C_1 t^{n-2-(n+\sigma)/p} + C_2 t^s w^p l^{\sigma-n(p-1)/p} \right\}.
\]  

(3.1)

**Proof of Theorem 1.1.** For simplicity we shall only outline a proof for the case \( m = 1 \). For cases where \( m \neq 1 \), the same line of argument carries through; we shall indicate the necessary modification in the demonstration at the end of the proof.

Again we start with \( \sigma < n(p-1) \). In this case, using (2.7), we get

\[
\frac{d w_1}{dt} \geq \left( \int_{\Omega} u^m \phi_1 |x|^\sigma \, dx \right)^{m/p} \left\{ - C_1 t^{n-2-(n+\sigma)/p} + C_2 t^s w^p l^{\sigma-n(p-1)/p} \right\}.
\]

**I** \( p < p_c \). Let \( \tau = l^2 t \); then

\[
\frac{d w_1}{d \tau} \geq t^2 \frac{d w_1}{dt} \\
\geq t^2 \left( \int_{\Omega} u^m \phi_1 |x|^\sigma \, dx \right)^{1/p} \left\{ - C_1 t^{n-2-(n+\sigma)/p} + C_2 t^s w^p l^{2s+(\sigma-n(p-1)(p-1)/p)} \right\}.
\]  

(3.2)

It is easy to calculate that for \( p \) in the range given, \( n-2-(n+\sigma)/p < 2s + (\sigma-n(p-1)(p-1)/p \). In addition, by the standard result that there exists \( \delta > 0 \) such that \( u(x, t) \geq \delta G(x, t) \) for \( t > 1 \), where \( G(x, t) \) is the Gaussian, we have \( w_1 \geq C(n) \) for all \( t > 1 \). Therefore the second term in the last inequality dominates the first term for all \( l \gg 1 \) if \( \tau \geq 1 \). We can then proceed exactly as in the proof of Theorem 1.6 (this time using \( \tau \) as the time variable) to conclude that \( u \) must blow up in this case. For simplicity, we omit the detail here.

**II** \( p = p_c = 1 + (2+2s+\sigma)/n \). In this case, the two exponents of \( l \) in (3.2) are the same. It is clear that if \( w_1 \) is unbounded as \( l \to \infty \), when \( \tau > 0 \) is fixed, then \( u \) blows up in finite time. We now prove that this is the case. Suppose the contrary: then \( w_1 \) is uniformly bounded for \( l \gg 1 \) for any \( \tau > 0 \). Consequently,

\[
\int_{\Omega} u(x, l^2 \tau) |\Delta \phi| \, dx \leq C \int_{\Omega} u(x, l^2 \tau) \, dx \to 0, \quad \text{as } l \to \infty.
\]

Hence, for all \( l \gg 1 \),

\[
w_1(l^2 \tau) = \int_{\Omega} u(x, l^2 \tau) \phi_1(x) \, dx
\]
Critical exponents of parabolic equations

\[ \frac{dw_i}{dt}(L^2 \tau) \leq \frac{1}{2} l^{2 \tau} \int_\Omega u^p(x, l^2 \tau) |x|^p \, dx. \] (3.3)

Upon an integration, we attain, with the aid of (3.3) and \( u(x, t) \geq \delta G(x, t) \) for all \( t > 1 \), that

\[ (w_i/(L^2 \tau) - w_i((L^2 \tau)) \geq \int_\Omega \int_{\mathbb{R}^n} u^p(x, \sqrt{t} \tau) |x|^p \varphi_i(x) \, dx \, dv \geq C(t') [\log (L^2 \tau) - \log (L^2 \tau)], \] (3.4)

a contradiction! Thus, \( u \) blows up in finite time.

If \( \sigma \geq n(p - 1) \), then \( n - 2 - (n + \sigma)/p < 0 \). Hence,

\[ \frac{dw_i}{dt} \leq \left( \int_\Omega |x|^p u^p \varphi_i \, dx \right)^{1/p} \left\{ -C_1 l^{n - 2 - (n + \sigma)/p} + C_3 a w^p - 1/2 \right\} (\log l)^{(1 - \rho)(p - 1)/p}, \] (3.5)

if \( \sigma = n(p - 1) \);

\[ \frac{dw_i}{dt} \geq \left( \int_\Omega |x|^p u^p \varphi_i \, dx \right)^{1/p} \left\{ -C_1 l^{n - 2 - (n + \sigma)/p} + C_4 a w^p - 1/2 \right\}, \] (3.6)

if \( \sigma > n(p - 1) \). Using the fact that \( w_i \geq C(n) \) and \( s > 0 \), we find that there exist \( \delta > 0 \), \( l \gg 1 \) such that

\[ \frac{dw_i}{dt} \geq \delta \int_\Omega |x|^p u^p \varphi_i \, dx \geq \omega(t) (\log l)^{(p - 1) \rho} \]

for \( t > 0 \). Thus, \( w_i \), and consequently \( u \), blows up in finite time.

For the case \( m \neq 1 \) and \( p < p_c \), the argument for \( m = 1 \) can be carried through in a straightforward way with obvious modifications, such as replacing \( t = l^2 \tau \) by \( t = l^{(m-1)+2\tau} \). Once again, we can use the Barenblatt–Pattle solution \( E(x, t; r_0) \) for \( m > 1 \) and the inequality (2.13) for \( m < 1 \) to bound \( u \) from below, to show that \( w_i(l^{(m-1)+2\tau}) \geq C(n) \) when \( \tau \) is fixed. But for \( p = p_c \), we need some additional work.

(a) \( m < 1 \) and \( p = p_c \). It is clear that if \( \sigma \geq n(p - 1) \), then we simply need to repeat essentially the same argument as for \( m = 1 \) to conclude that \( u \) must blow up because \( n - 2 - (n + \sigma) m/p < 0 \) for \( p > 0 \).

If \( \sigma < n(p - 1) \), the crucial step is to show that

\[ \int_\Omega u^m(x, n(m - 1) + 2\tau) |\Delta \varphi| \, dx \to 0 \quad \text{as} \quad l \to \infty \]

under the condition that \( w_i(n(m - 1) + 2\tau) = \int_\Omega u(x, n(m - 1) + 2\tau) \varphi_i \) is uniformly bounded for all \( l \gg 1 \) and \( \tau > 0 \) fixed. But this is true since

\[ \int_\Omega u(x, n(m - 1) + 2\tau) |\Delta \varphi| \, dx \leq \left( \int_\Omega u(x, n(m - 1) + 2\tau) \, dx \right)^m |\Delta|^m - 2 \]
and \( n(1 - m) - 2 < 0 \) by our assumption that \( m > (n - 2)_+ / n \). Hence, for all \( l \gg 1 \),

\[
\omega_l(l^2 \tau) = \int_{\Omega} u(x, l^2 \tau) \varphi_l(x) \, dx
\]

satisfies

\[
\frac{d\omega_l}{dt} \left( l^{(m-1)+\tau} \right) \geq \frac{1}{2} l^{(m-1)+2\tau} \int_{\Omega} u^p(x, n(m-1)+2\tau) |x|^{\sigma} \, dx.
\]

Upon an integration, we get, with the aid of (2.15), that

\[
\omega_l(L^{n(m-1)+2\tau}) - \omega_l(l^{n(m-1)+2\tau})
\]

\[
\geq C(\tau) \left[ \log (L^{n(m-1)+2\tau}) - \log (l^{n(m-1)+2\tau}) \right],
\]

Consequently, \( \omega_l(l^{n(m-1)+2\tau}) \to \infty \) as \( l \to \infty \) for any \( \tau > 0 \) fixed. Thus, \( u \) blows up in finite time.

\( \mathbf{(b)} \) \( m > 1 \) and \( p = p_c \). Again, we only need to show that when \( \sigma < n(p-1) \),

\[
\int_{\Omega} u^m(x, l^{n(m-1)+2\tau}) |\Delta \varphi_l| \, dx \to 0 \quad \text{as} \quad l \to \infty,
\]

or otherwise \( u \) blows up in finite time. Suppose the contrary: it is clear that

\[
\int_{\Omega} u^m(x, l^{n(m-1)+2\tau}) |\Delta \varphi_l| \, dx \leq \frac{C}{l^2} \int_{B_{2l} \setminus B_l} u^m(x, t) \, dx.
\]

Then

\[
\int_{\Omega} u^m(x, l^{n(m-1)+2\tau}) \, dx \geq Cl^2
\]

for a sequence of \( l \) which goes to \( \infty \).

By Hölder’s inequality, we have

\[
\int_{\Omega} |x|^\sigma u^m \varphi_l \, dx \geq C_2 \left( \int_{\Omega} u^m \varphi_l \, dx \right)^{p/m} \left( \int_{\Omega} |x|^{\sigma - n(p-m)/m} \varphi_l \, dx \right)^{n(p-m)/m}, \quad \text{if} \quad \sigma < n(p-m)/m,
\]

\[
\int_{\Omega} |x|^\sigma u^m \varphi_l \, dx \geq C_3 \left( \int_{\Omega} u^m \varphi_l \, dx \right)^{p/m} (\log l)^{-(p-m)/m}, \quad \text{if} \quad \sigma = n(p-m)/m,
\]

\[
\int_{\Omega} |x|^\sigma u^m \varphi_l \, dx \geq C_4 \left( \int_{\Omega} u^m \varphi_l \, dx \right)^{p/m}, \quad \text{if} \quad \sigma > n(p-m)/m.
\]

Then,

\[
\frac{d\omega_l}{dt} \geq \left( \int_{\Omega} u^p \varphi_l |x|^\sigma \, dx \right)^{m/p} \left\{ -C_4 l^{n-2-(n+\sigma)/m} \right\} + t^4 C_2 \left( \int_{\Omega} u^m \varphi_l \, dx \right)^{(n(p-m)/m)} \left( l^{n(p-m)/m} \right)^{(p-m)/p},
\]

(3.8)
if $\sigma < n(p - m)/m$, or
\[
\frac{dw_l}{dt} \geq \left(\int_{\Omega} u^{p} \varphi_1 |x|^p \, dx\right)^{\frac{m}{p}} \left\{ -C_1 |x|^{-2(\alpha + \sigma)m/p} \right.
\left. + t^\alpha C_3 \left(\int_{\Omega} u^{n} \varphi_1 \, dx \log \frac{l}{t^{(n - m)/p}}\right)^{(p - m)/m} \right\},
\]  
(3.9)

if $\sigma \geq n(p - m)/m$. But, for $m > 1$,
\[\left[\sigma - n(p - m)/m\right] (p - m)/p > \left[\sigma - n(p - 1)(p - m)/p\right] > n - 2 - (n + \sigma)m/p.\]  
(3.10)

This, together with (3.7), (3.8) (or (3.9)) and H"older's inequality, will yield that there exist $l > 1$ and $\tau > 0$ such that at $t_0 = l^{n(m-1)+2\tau}$,
\[
\frac{dw_l}{dt} \geq \delta w_l^{\alpha} |x|^{-n(p-1)}.
\]  
(3.12)

Consequently, $w_l$ is increasing for $t > t_0$, and satisfies the same inequality for all $t \geq t_0$. Hence, $w_l$ and consequently $u$ must blow up in finite time. This completes the proof of Theorem 1.1. □

4. Global existence for $p > p_c$

The global existence of (III) can be studied by using the similarity solution. It takes the form
\[
u(x, t) = t^{-\alpha}v(|x|/t^\beta), \quad \alpha = \frac{2 + \sigma + 2\alpha}{2(p - 1) + (m - 1)\sigma}, \quad \beta = \frac{p - m - s(m - 1)}{2(p - 1) + (m - 1)\sigma}.\]  
(4.1)

The resulting ODE for $v$, after an appropriate scaling, with $r = |x|/t^\beta$, is
\[
(v^m)^" - \frac{n - 1}{r} (v^m)' + \frac{r}{2} v' + kv + r^\sigma v^p = 0,\]  
(4.2)
\[
v(0) = \eta > 0, \quad (v^m)'(r) = -r^\sigma v^p(0)/(n + \sigma), \quad \text{as } r \to 0,
\]
where $k = \alpha/2\beta$. The case $p > p_c$ results in $k < n/2$.

We observe that a function $U(x, t) = U(|x|/t^\beta)$ is a super-solution of (III) if and only if it satisfies the inequality
\[
(U^m)^" - \frac{n - 1}{r} (U^m)' + \frac{r}{2} U' + kU + r^\sigma U^p \leq 0.
\]  
(4.3)

The approach we adopt is to demonstrate that there exists a ground state to (4.2) or a positive function which satisfies the inequality (4.3) with $k < n/2$. The study of (4.2) will follow in the footsteps of previous results [13, 22, 23] on the important special case $\sigma = 0$. The additional complication when $\sigma \neq 0$ is that if $\sigma < 0$, the last term in equation (4.2) dominates the other terms for $r \ll 1$, and may force $v$ to be zero at some finite value of $r$; whereas, for $\sigma > 0$, the last term may get bigger as $r \to \infty$, which has the same effect of forcing $v$ to be zero at some finite value of $r$. Nevertheless, the difficulty can be overcome by a more delicate analysis. Since our
purpose is to use (4.2) and (4.3) to study (III), we shall be brief and derive just the necessary result. Again our presentation will be just for the case \( m = 1 \). Consequently, (4.2) and (4.3) take the form

\[
v'' + \frac{n-1}{r} v' + \frac{r}{2} v' + kv + r^\sigma v^p = 0, \tag{4.4}
\]

\[
v(0) = \eta > 0, \quad v(r) = -r^{\sigma + 1} v^p(0)/(n+\sigma), \quad \text{as } r \to 0,
\]

where \( k = \alpha/2 \beta < n/2 \).

**Lemma 4.1.** Let \( \sigma > 0 \). Then \( U = \eta e^{-A r^\sigma} \) with \( 0 < \eta \ll 1 \) and \( k/2n < A < 1/4 \) satisfies the inequality (4.3) and is therefore a super-solution of (III).

**Proof.** It is elementary to verify the claim and we omit the detail. \( \Box \)

**Lemma 4.2.** Let \( n \geq 2 \) and \( -2 < \sigma < 0 \) or \( n = 1 \) and \( -1 < \sigma < 0 \). Then there exists a ground state solution to (4.4).

**Proof.** Let \( v \) be a solution to (4.4). Then \( v(0) = 0 \). It is easy to see that \( v \) satisfies the following integral equation:

\[
v' r^{\sigma - 1} + \frac{r^\sigma}{2} v = \int_0^r \left( \frac{n}{2} - k \right) v s^{s-1} ds - \int_0^r v^p s^{s-1+\sigma} ds. \tag{4.5}
\]

We shall look at solutions with \( \eta \ll 1 \). First we note that \( v \) is a decreasing function of \( r \) before it reaches zero. But unfortunately the fact that \( \sigma < 0 \) prevents us from concluding that the first term on the right-hand side dominates the second term, which is the essential fact used in [13] to conclude, in the case \( \sigma = 0 \), that \( v \) is a ground state solution for \( \eta \ll 1 \). But the first term on the right-hand side dominates the second term for \( r \geq 1 \), since \( \sigma < 0 \). The task is then to show that \( v \) as well as \( v' r^{\sigma - 1} + r^\sigma v/2 \) stays positive for some \([0, M]\), where \( M > 1 \). In this connection, we consider

\[
w'' + \frac{n-1}{r} w' + \frac{1}{2} w' + kw + \eta^p r^\sigma = 0, \tag{4.6}
\]

\[
w(0) = \eta > 0, \quad w'(0) = 0.
\]

If \( \eta \) is chosen as the initial value of \( v \), then \( v \) and \( v' r^{\sigma - 1} + (r^\sigma/2) v \) will stay positive as long as the corresponding terms in \( w \) do. This can be seen by using the fact that \( y = v - w \) satisfies

\[
v' r^{\sigma - 1} + \frac{r^\sigma}{2} y = \int_0^r \left( \frac{n}{2} - k \right) v s^{s-1} ds + \int_0^r (v^p(0) - v^p(s)) s^{s-1+\sigma} ds > 0 \tag{4.7}
\]

so long as \( w \) and \( v \) are positive in \([0, r]\). But an integration of the above yields that \( y > 0 \) in the same range. Hence, it is impossible for \( v \) to reach zero before \( w \) ever does. Furthermore, an elementary calculation gives that \( w \) and \( r^{\sigma - 1} w + r^\sigma w/2 \) are both positive for \( r \in [0, R] \), where \( R^{\sigma + 1} = (\sigma + 2)(\sigma + n)\eta^{-1} (r - 1) \). One finds by an integration of (4.4) on \([r_0, r]\) that

\[
v' r^{\sigma - 1} + \frac{r^\sigma}{2} v = v(r_0) r_0^{\sigma - 1} + \frac{r^\sigma}{2} v(r_0) + \int_{r_0}^r \left( \frac{n}{2} - k \right) v s^{s-1} ds - \int_{r_0}^r v^p s^{s+\sigma - 1} ds, \tag{4.8}
\]

where \( r_0 = 1/(n/2 - k) \). Thus, \( v \) is a ground state if \( \eta \ll 1 \). \( \Box \)
Critical exponents of parabolic equations

Proof of Theorem 1.2. The case \( m = 1 \) follows directly from the above two lemmas. The case \( m \neq 1 \) follows from the same line of argument and it is omitted. \( \square \)

Acknowledgment

The author wishes to thank Professor B. McLeod and Professor W. M. Ni for stimulating discussion. This research was supported in part by HK government RGC grant HKUST 630/95p.

References


(Issued 24 February 1988)