Critical Exponents of Quasilinear Parabolic Equations

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In this paper we study the critical exponents of the Cauchy problem in $\mathbb{R}^n$ of
the quasilinear singular parabolic equations:

$$u_t = \text{div}(\nabla u^{m+1} \nabla u) + r|x|^\alpha u^p,$$

with non-negative initial data. Here $s \geq 0$, $(n-1)(n+1) < m < 1$, $p > 1$ and $\sigma > n(1-m) - (1 + m + 2s)$. We prove that $p_c = m + (1 + m + 2s + \sigma)/n > 1$ is the critical exponent. That is, if $1 < p \leq p_c$, then every non-trivial solution blows up in finite time, but for $p > p_c$, a small positive global solution exists.

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1. INTRODUCTION

The study of blow-up for nonlinear parabolic equations probably originates from Fujita [7, 8], where he studied the following Cauchy problem of semilinear heat equation

$$u_t = \Delta u + u^p, \quad x \in \mathbb{R}^n, \quad t > 0,$$

$$u(x, 0) = u_0(x) \geq 0, \quad x \in \mathbb{R}^n,$$

where $p > 1$ and obtained the following result:

(a) If $1 < p < 1 + 2/n$, then every non-trivial solution of (1.1) blows up in finite time;
(b) If \( p > 1 + 2/n \) and \( u_0(x) \leq \delta e^{-|x|^2} \) for some \( 0 < \delta \ll 1 \), then (1.1) admits a global solution.

For the critical case \( p = 1 + 2/n \), it was shown by Hayakawa [11] for dimensions \( n = 1, 2 \), Kobayashi et al. [13], and Aronson and Weinberger [2] for all \( n \geq 1 \) that (1.1) possesses no global solution \( u(x, t) \) satisfying

\[
\|u(x, t)\|_q < \infty \quad \text{for} \quad t \geq 0.
\]

Weissler [26] proved that if \( p = 1 + 2/n \), then (1.1) possesses no global solution \( u(x, t) \) satisfying

\[
\|u(x, t)\|_q < \infty \quad \text{for} \quad t \geq 0 \quad \text{and some} \quad q \in [1, +\infty).
\]

The value \( p_c = 1 + 2/n \) is called the critical exponent of (1.1). It plays an important role in studying the behavior of the solution to (1.1).

These elegant works revealed a new phenomenon of nonlinear PDEs and stimulated the study of similar features for various nonlinear evolution equations. Especially, the following Cauchy problems of porous medium equations

\[
\begin{align*}
    u_t &= \Delta u^n + t^s |x|^\sigma u^p, \quad x \in \mathbb{R}^n, \quad t > 0, \\
    u(x, 0) &= u_0(x) \geq 0, \quad x \in \mathbb{R}^n
\end{align*}
\]

(1.2)

were studied by many authors [3, 10, 14–20, 23, 25], where \( m > (n - 2)_+ / n \), \( s \geq 0 \), and \( \sigma > -1 \) if \( n = 1 \) or \( \sigma > -2 \) if \( n \geq 2 \) and \( p > \max\{m, 1\} \).

Recently, Qi [23] proved the following: If \( p_c \triangleq m + (m - 1)s + (2 + \sigma + 2s)/n > 1 \) then \( p_c \) is the critical exponent of (1.2); i.e., when \( 1 < p \leq p_c \), every non-trivial solution of (1.2) blows up in finite time, and when \( p > p_c \), (1.2) admits small positive global solution.

The following Cauchy problems of the quasilinear degenerate parabolic equation

\[
\begin{align*}
    u_t &= \text{div}(|\nabla u|^{m-1} \nabla u) + t^s |x|^\sigma u^p, \quad x \in \mathbb{R}^n, \quad t > 0, \\
    u(x, 0) &= u_0(x) \geq 0, \quad x \in \mathbb{R}^n
\end{align*}
\]

(1.3)

with \( m > 1 \), were studied by the authors of [9, 10, 20, 22]. They obtained that \( p_c \triangleq m + (1 + m)/n \) is the critical exponent of (1.3) and \( p_c \) belongs to the blow-up case ([10, 20]).

In this paper, we shall consider the following general quasilinear “singular” parabolic equations

\[
\begin{align*}
    u_t &= \text{div}(|\nabla u|^{m-1} \nabla u) + t^s |x|^\sigma u^p, \quad x \in \mathbb{R}^n, \quad t > 0, \\
    u(x, 0) &= u_0(x) \geq \neq 0, \quad x \in \mathbb{R}^n
\end{align*}
\]

(1)

where \( (n - 1)/(n + 1) < m < 1 \), \( s \geq 0 \), \( p > 1 \) and \( \sigma > n(1 - m) - (1 + m + 2s) \), \( u_0(x) \) is a continuous function in \( \mathbb{R}^n \).
By a solution of (I) we mean a continuous function \( u : \mathbb{R}^n \times (0, T) \to \mathbb{R}^+ \) with \( Du \in L^1_{\text{loc}}(0, T; L^1_{\text{loc}}(\mathbb{R}^n)) \), and Eq. (I) is satisfied in the sense of distribution in \( \mathbb{R}^n \times (0, T) \), where \( T > 0 \) is the maximal existence time. Since \( m > (n - 1)/(n + 1) \) and \( u_0(x) \) are non-negative and continuous functions in \( \mathbb{R}^n \), the existence, uniqueness, and comparison principle of solutions to (I) had been proved in [5]. Moreover, the following result holds:

**Proposition 1.** If \( u_0(x) \) is a non-trivial and non-negative continuous function, then the solution \( u(x, t) \) of (I) is positive in \( \mathbb{R}^n \) for any \( t > 0 \) if

\[
 w(t) = \int_\Omega u(x, t) \, dx \to +\infty \quad \text{as} \quad t \to T^-
\]

for a finite \( T > 0 \), where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \). It follows easily from Proposition 1 that this definition is the same as that of Friedman and McLeod [6].

Our main result reads as follows:

**Theorem 1.** Assume that \( (n - 1)/(n + 1) < m < 1, \ s \geq 0, \ p > 1 \) and \( \sigma > n(1 - m) - (1 + m + 2s) \). Then \( p_c = m + (1 + m + 2s + \sigma)/n > 1 \) is the critical exponent of (I). That is, if \( 1 < p \leq p_c \), then every non-trivial solution of (I) blows up in finite time; if \( p > p_c \) then (I) has small non-trivial global solutions.

This paper is organized as follows. In Section 2 we discuss the qualitative behaviors and give some estimates of solutions to the homogeneous problem

\[
 u_t = \text{div}(\vert \nabla u \vert^{m-1}\nabla u), \quad x \in \mathbb{R}^n, \quad t > 0,
\]

\[
 u(x, 0) = u_0(x) \geq 0, \quad x \in \mathbb{R}^n. \tag{II}
\]

In Section 3, for convenience, we first discuss the special case of (I): \( s = 0 \), i.e.,

\[
 u_t = \text{div}(\vert \nabla u \vert^{m-1}\nabla u) + \vert x \vert^\sigma u^p, \quad x \in \mathbb{R}^n, \quad t > 0,
\]

\[
 u(x, 0) = u_0(x) \geq 0, \quad x \in \mathbb{R}^n. \tag{III}
\]

and prove that if \( 1 < p \leq p_c \triangleq m + (1 + m + \sigma)/n \) then every non-trivial solution of (III) blows up in finite time. In Section 4 we prove Theorem 1.

Our method is similar in nature to that in [23].
Remark. We end this section with a simple but very useful reduction. Since, by Proposition 1, the solution \( u(x, t) \) of (I) is continuous and positive in \( \mathbb{R}^n \) for any \( t > 0 \), we may assume, without loss of generality, that \( u_0(x) \) is continuous and positive in \( \mathbb{R}^n \). By the comparison principle we need only consider that \( u_0(x) \) is radially symmetric and non-increasing; i.e., \( u_0(x) = u_0(r) \) with \( r = |x| \) and \( u_0(r) \) non-increasing in \( r \). Therefore, the solution \( u(x, t) \) of (I) is also radially symmetric and non-increasing in \( r = |x| \).

2. ESTIMATES OF SOLUTIONS TO (II)

In this section we discuss (II) for the radially symmetric case. The main results are the two propositions.

**Proposition 2.** Assume that \( \frac{n-1}{n+1} < m < 1 \) and \( u_0(x) \) is a non-trivial and non-negative continuous function. If, in addition, \( u_0(x) \) is a radially symmetric and non-increasing function, then the solution \( u(x, t) \) of (II) satisfies

\[
    u_t \geq -\frac{\alpha}{t} u \quad \text{for all} \quad x \in \mathbb{R}^n \quad \text{and} \quad t > 0,
\]

where \( \alpha = n/[1 + m - n(1 - m)] > 0 \).

**Proposition 3.** Under the assumptions of Proposition 2,

\[
    u(x, t) \geq \delta(t - \epsilon)^{-\alpha}(1 + C|x|^{1+m})^{-m(1-m)/(2m)}
\]

for \( |x| \geq 1, t > \epsilon > 0 \), where \( \delta \) and \( C \) are positive constants and \( \beta = (1 + m)/(m(1 + m - n(1 - m))) > 0 \).

**Proof of Proposition 2.** By the uniqueness we know that \( u(x, t) = u(r, t) \) is radially symmetric and non-increasing in \( r, r = |x| \). Let

\[
    v = \frac{m}{m-1} u^{(m-1)/m},
\]

then we have \( v < 0, v' < 0 \) (where \( v' = \frac{dv}{dr} \)) and

\[
    \nabla(v |u|^{m-1}) = u \nabla(|v|^{m-1} v) + u^{-1} |\nabla u|^{m+1}
\]

\[
    \geq u \nabla(|v|^{m-1} v), \quad v_i = \frac{m-1}{m} v \nabla(|v|^{m-1} v) + |\nabla v|^{m+1}.
\]
To prove (2.1) it is sufficient to prove that \( \text{div}(|\nabla v|^{m-1}\nabla v) \geq -\alpha/t \) by (2.3).

Denote \( w = \text{div}(|\nabla v|^{m-1}\nabla v) \) and let \( z = -v \), we find \( z' > 0 \) and

\[
\begin{align*}
 z_i &= -\frac{m-1}{m} z \text{ div}(|\nabla z|^{m-1}\nabla z) - |\nabla z|^{m+1} \\
 &= -\frac{m-1}{m} z \left( m(z')^{m-2} z'' + \frac{n-1}{r} (z')^m \right) - (z')^{m+1} \\
 &= \frac{m-1}{m} z w - (z')^{m+1},
\end{align*}
\]

By (2.3),

\[
 w = -\left( m(z')^{m-2} z'' + \frac{n-1}{r} (z')^m \right).
\]

By direct computation we have

\[
\begin{align*}
 -w_i &= m(z')^{m-1} z'' + m(m-1)(z')^{m-2} z' z'' + m \frac{n-1}{r} (z')^{m-1} z', \\
 z'_i &= \frac{m-1}{m} z' w + \frac{m-1}{m} z w' - (m+1)(z')^{m} z'', \\
 z''_i &= \frac{m-1}{m} z'' w + \frac{2(m-1)}{m} z' w' + \frac{m-1}{m} z w'' \\
 &- (m+1)(z')^{m} z'' - m(m+1)(z')^{m-1} (z'')^2.
\end{align*}
\]

By a series of calculations we have

\[
\begin{align*}
 -w_i &= (m-1) z(z')^{m-1} \Delta w + m(m+1)(z')^{m-1} z'' \\
 &+ (1-m) w^2 + 2(m-1)(z')^m w' + (m-1)^2 z z'' (z')^{m-2} w' \\
 &+ m(1-m^2)(z')^{2m-2} (z'')^2 - m(m+1)(z')^{2m-1} z''.
\end{align*}
\]

It follows from (2.4) that

\[
\begin{align*}
 -w_i &= m(z')^{m-1} z'' + m(m-1)(z')^{m-2} (z'')^2 \\
 &- \frac{n-1}{r^2} (z')^m + m \frac{n-1}{r} (z')^{m-1} z''.
\end{align*}
\]

Substituting the above into (2.5) we get

\[
\begin{align*}
 -w_i &= (m-1) a(r, t) \Delta w + b(r, t) w' + (1-m) w^2 + m(m+1) z'' (z')^{m-1} w \\
 &+ m(m+1) \frac{n-1}{r} (z')^{2m-1} z'' - (m+1) \frac{n-1}{r^2} (z')^{2m},
\end{align*}
\]

where \( a(r, t), b(r, t) \) are functions produced by \( z(r, t) \) and \( a(r, t) > 0 \). By use of (2.4),

\[
\begin{align*}
 -w_i &= (m-1) a(r, t) \Delta w + b(r, t) w' - 2m w^2 \\
 &- 2(m+1) \frac{n-1}{r} (z')^m w - m(m+1) \frac{n-1}{r^2} (z')^{2m}.
\end{align*}
\]
Taking into account the Cauchy inequality

\[-2(z')^m w \leq (z')^{2m} + w^2,\]

we find

\[-w_t \leq (m - 1)a(r, t)\Delta w + b(r, t)w' - 2mw^2 + (m + 1)\frac{n - 1}{n}w^2\]

\[= (m - 1)a(r, t)\Delta w + b(r, t)w' - \frac{1 + m - n(1 - m)}{n}w^2;\]

i.e.,

\[w_t \geq (1 - m)a(r, t)\Delta w - b(r, t)w' + \frac{1}{\alpha}w^2.\]

Let \(y(r, t) = -\alpha/t\). It is obvious that \(y_t = (1 - m)a(r, t)\Delta y - b(r, t)y' + (1/\alpha)y^2\). Since \(y(r, 0) = -\infty\), it follows by the comparison principle that \(w \geq -\alpha/t\); i.e., \(\text{div}(|\nabla v|^{m-1}\nabla v) \geq -\alpha/t\).

Remark. The above proof is similar to an argument of Aronson and Benilan [1] of the porous media equation \(u_t = \Delta u^m\). But, it seems to us that there is no direct way of transforming their result to the present case.

To prove Proposition 3 we first state a comparison lemma which can be proved by using the methods of [5, Chap. 6] or [12].

**Lemma 2.1.** Let \(0 \leq \tau < +\infty\) and \(S = \{x \in \mathbb{R}^n, |x| > 1\} \times [\tau, +\infty)\). Assume that \(v, w\) are non-negative functions and satisfy

\[v_t = \text{div}(|\nabla v|^{m-1}\nabla v), \quad w_t = \text{div}(|\nabla w|^{m-1}\nabla w) \quad \text{in} \ S,\]

\[v(x, t) \leq w(x, t), \quad |x| = 1, \quad \tau < t < +\infty,\]

\[v(x, \tau) \leq w(x, \tau), \quad |x| \geq 1.\]

Then

\[v(x, t) \leq w(x, t) \quad \text{in} \ S.\]

**Proof of Proposition 3.** Since \((n - 1)/(n + 1) < m < 1\), by the results of [24] we have that problem (II) has the similarity solutions

\[U_\mu(x, t) = \mu^\theta U(\mu x, t), \quad \theta = (1 + m)/(1 - m),\]

where \(\mu > 0\) is a parameter,

\[U(x, t) = U_1(x, t) = t^{-\alpha}[1 + b|x|^{(1+m)/m}t^{-\beta}]^{-m/(1-m)},\]

\(\beta\) is given in the statement Proposition 3, and \(b = \frac{1-m}{1+m}(1 + m - n(1 - m))^{-1/m}\). Since \(U(x, t)\) can be written as

\[U(x, t) = t^{-\alpha + m\beta/(1-m)}[t^{\beta} + b|x|^{(1+m)/m}]^{-m/(1-m)},\]
and $-\alpha + m\beta/(1 - m) = 1/(1 - m) > 0$, we see that
\begin{equation}
U_\mu(x, t - \varepsilon) = 0 \quad \text{for } |x| \geq 1, \ t = \varepsilon, \ \text{where } \ 0 < \varepsilon \ll 1. \quad (2.6)
\end{equation}

By Proposition 2,
\begin{equation}
\begin{aligned}
u(1, t) &\geq \varepsilon^\alpha u(1, \varepsilon) t^{-\alpha} \quad \text{for all } \ t \geq \varepsilon.
\end{aligned} \quad (2.7)
\end{equation}

Now we estimate $U_\mu(1, t - \varepsilon)$. When $t/(t - \varepsilon) \geq K > 1$, it follows that $t \leq K\varepsilon/(K - 1)$, $t - \varepsilon \leq \varepsilon/(K - 1)$. Therefore,
\begin{equation}
u(1, t) \geq \varepsilon^\alpha u(1, \varepsilon) t^{-\alpha} \geq (K/(K - 1))^{-\alpha} u(1, \varepsilon), \quad (2.8)
\end{equation}
and
\begin{equation}
\begin{aligned}
U_\mu(1, t - \varepsilon) &= \mu^\theta(t - \varepsilon)^{-\alpha + m\beta/(1 - m)}(t - \varepsilon)^\beta + b\mu^{(1 + m)/m} t^{-\alpha} \\
&\leq \mu^\theta(t - \varepsilon)^{-\alpha + m\beta/(1 - m)} b^{-m/(1 - m)} \mu^{-\theta} \\
&= b^{-m/(1 - m)}(t - \varepsilon)^{1/(1 - m)} \\
&\leq b^{-m/(1 - m)} e^{1/(1 - m)}(K - 1)^{-1/(1 - m)} \\
&\leq (K/(K - 1))^{-\alpha} u(1, \varepsilon) 
\end{aligned} \quad (2.9)
\end{equation}
if $K$ is suitably large, say $K \geq K_0$, for some $K_0 \gg 1$ independent of $\mu$.

When $t/(t - \varepsilon) \leq K$,
\begin{equation}
\begin{aligned}
U_\mu(1, t - \varepsilon) &= \mu^\theta(t - \varepsilon)^{-\alpha} [1 + b\mu^{(1 + m)/m} (t - \varepsilon)^{-\beta}]^{-m/(1 - m)} \\
&\leq \mu^\theta(t - \varepsilon)^{-\alpha} \leq \mu^\theta K^\alpha t^{-\alpha} \\
&\leq \varepsilon^\alpha u(1, \varepsilon) t^{-\alpha} 
\end{aligned} \quad (2.10)
\end{equation}
if $\mu > 0$ is suitably small. Combining (2.7)--(2.10) we see that
\begin{equation}
U_\mu(1, t - \varepsilon) \leq u(1, t) \quad \text{for all } \ t \geq \varepsilon. \quad (2.11)
\end{equation}

Equations (2.6) and (2.11), when combined with Lemma 2.1, yield
\begin{equation}
u(x, t) \geq U_\mu(x, t - \varepsilon) \quad \text{for all } |x| \geq 1 \ \text{and } \ t \geq \varepsilon.
\end{equation}
Consequently (2.2) holds.  □
In this section we study problem (III) and prove a blow-up result.

**Theorem 2.** Let \( m, p, \) and \( \sigma \) be as in Theorem 1. If \( 1 < p \leq \tilde{p}_c = m + (m + 1 + \sigma)/n \), then every non-trivial solution of (III) blows up in finite time.

Let \( \psi(x) \) be a smooth, radially symmetric and non-increasing function which satisfies

\[
0 \leq \psi(x) \leq 1, \; \psi(x) \equiv 1 \quad \text{for} \quad |x| \leq 1 \quad \text{and} \quad \psi(x) \equiv 0 \quad \text{for} \quad |x| \geq 2.
\]

Let \( \psi_0(x) \) be a smooth radially symmetric and non-decreasing function which satisfies

\[
0 \leq \psi_0(x) \leq 1, \; \psi_0(x) \equiv 0 \quad \text{for} \quad |x| \leq 1 \quad \text{and} \quad \psi_0(x) \equiv 1 \quad \text{for} \quad |x| \geq 2.
\]

Set \( \psi_\ell(x) = \psi(x/\ell) \). It follows that for \( \ell \geq 1 \), \( \psi_\ell(x) \) is a smooth, radially symmetric, and non-increasing function which satisfies

\[
0 \leq \psi_\ell(x) \leq 1, \; \psi_\ell(x) \equiv 1 \quad \text{for} \quad |x| \leq \ell \quad \text{and} \quad \psi_\ell(x) \equiv 0 \quad \text{for} \quad |x| \geq 2\ell.
\]

Denote \( \phi_\ell(x) = \psi_0(x)\psi_\ell(x) \). Then \( \phi_\ell(x) \) is a smooth and radially symmetric function and satisfies for \( \ell > 2 \),

\[
0 \leq \phi_\ell(x) \leq 1, \; \phi_\ell(x) \equiv 0 \quad \text{for} \quad |x| \leq 1, \\
\phi_\ell(x) \equiv 1 \quad \text{for} \quad 2 \leq |x| \leq \ell \quad \text{and} \quad \phi_\ell(x) \equiv 0 \quad \text{for} \quad |x| \geq 2\ell.
\]

Moreover, \( \phi_\ell(x) \) is non-decreasing for \( 1 \leq |x| \leq 2 \) and non-increasing for \( \ell \leq |x| \leq 2\ell \).

Denote

\[
\omega_\ell(t) = \int_\Omega u(x, t)\phi_\ell(x) \, dx,
\]

where \( \Omega = \mathbb{R}^n \setminus B_{1/2} \) with \( B_{1/2} \) being the ball with radial 1/2 and center at the origin. By the remark at the end of Section 1, we may assume, without loss of generality, \( u \) is radially symmetric. Then \( \omega_\ell \) is an increasing function of \( \ell \) and

\[
\frac{d}{dt} \int_\Omega u \phi_\ell \, dx = -\int_\Omega |\nabla u|^{m-1} \nabla u \cdot \nabla \phi_\ell + \int_\Omega |x|^{\sigma} \phi_\ell(x) u^p \, dx
\]

\[
= -\omega_n \int_1^{2\ell} |u'|^{m-1} u' \phi_\ell r^{n-1} \, dr + \int_\Omega |x|^{\sigma} \phi_\ell(x) u^p \, dx
\]

\[
= \omega_n \int_1^{2\ell} |u'|^{m} \phi_\ell r^{n-1} \, dr + \int_\Omega |x|^{\sigma} \phi_\ell(x) u^p \, dx
\]

\[
\geq -\omega_n \int_1^{2\ell} |u'|^{m} |\phi_\ell'| r^{n-1} \, dr + \int_\Omega |x|^{\sigma} \phi_\ell(x) u^p \, dx, \quad (3.1)
\]
where \( \omega_n \) is the area of the unit sphere in \( \mathbb{R}^n \).

\[
\begin{align*}
\int_{1}^{2^\ell} |u'|^m |\phi'|^{r-1} &\, dr = \int_{1}^{2^\ell} r^{m(n-1)} |u'|^m |\phi'|^{r-1} \, dr \\
&\leq \left( \int_{1}^{2^\ell} |u'|^m |\phi'|^{r-1} \, dr \right)^m \left( \int_{1}^{2^\ell} |\phi'|^{r-1} \, dr \right)^{1-m}. 
\end{align*}
\]  
(3.2)

\[
\begin{align*}
\int_{1}^{2^\ell} r^{n-1} |u'| |\phi'| &\, dr = \int_{1}^{2^\ell} r^{n-1} |u'| |\phi'| \, dr + \int_{1}^{2^\ell} r^{n-1} |u'| |\phi'| \, dr \\
&= - \int_{1}^{2^\ell} r^{n-1} u' \phi' \, dr + \int_{1}^{2^\ell} r^{n-1} u' \phi' \, dr \\
&= - \frac{1}{\omega_n} \int_{\Omega_1} \nabla u \cdot \nabla \phi \, dx + \frac{1}{\omega_n} \int_{\Omega_2} \nabla u \cdot \nabla \phi \, dx \\
&= \frac{1}{\omega_n} \int_{\Omega_1} u \Delta \phi \, dx - \frac{1}{\omega_n} \int_{\Omega_2} u \Delta \phi \, dx \\
&\leq \frac{1}{\omega_n} \int_{\Omega} |u \Delta \phi| \, dx,
\end{align*}
\]  
(3.3)

where \( \Omega_1 = \{x \mid 1 \leq |x| \leq 2\} \), \( \Omega_2 = \{x \mid \ell \leq |x| \leq 2\ell\} \).

\[
\int_{1}^{2^\ell} r^{n-1} |\phi'| \, dr = \int_{1/\ell}^{2} r^{n-1} |\phi'| \, dr \leq C\ell^{n-1},
\]  
(3.4)

where \( \phi(x) = \psi(x)\psi_0(x) \). It is easy to see that \( C \) is independent of \( \ell \). Substituting (3.2)–(3.4) into (3.1) we obtain

\[
\frac{d}{dt} \int_{\Omega} u\phi_\ell \, dx \geq -C\ell^{(1-m)(n-1)} \left( \int_{\Omega} |u|^{\Delta \phi_\ell} \, dx \right)^m + \int_{\Omega} |x|^\sigma \phi_\ell u^p \, dx.
\]  
(3.5)

By use of the Hölder inequality we get

\[
\int_{\Omega} u|\Delta \phi_\ell| \, dx \leq \left( \int_{\Omega} |x|^{-(k-1)|\Delta \phi_\ell|/k} \right)^{1/k} \phi_\ell^{-k} \left( \int_{\Omega} x^{\sigma \phi_\ell u^p} \, dx \right)^{1/p},
\]  
(3.6)

where \( k = p/(p-1) \). Thus, we have by (3.5)

\[
\frac{d}{dt} \int_{\Omega} u\phi_\ell \, dx \geq -C\ell^{(1-m)(n-1)} C(\phi_\ell, \ell)^m/k \left( \int_{\Omega} |x|^\sigma \phi_\ell u^p \, dx \right)^{m/p} + \int_{\Omega} |x|^\sigma \phi_\ell u^p \, dx,
\]  
(3.7)
where

\[ C(\phi_\ell, \ell) = \int_\Omega |x|^{-\sigma/(p-1)} \Delta \phi_\ell |p/(p-1) \phi_\ell^{-1/(p-1)} \, dx \leq C\ell^{(p-1)(n-(\sigma+2p))/(p-1)}. \]

Therefore

\[
\frac{d}{dt} \int_\Omega u \phi_\ell \, dx \geq \left\{ \begin{array}{l}
-C\ell^{(n-m-1)-(m+n)/p} + \left( \int_\Omega |x|\sigma \phi_\ell u^p \, dx \right)^{(p-m)/p} \\
\times \left( \int_\Omega |x|\sigma \phi_\ell u^p \, dx \right)^{m/p}
\end{array} \right. 
\]

(3.8)

By Hölder’s inequality we have

\[
\int_\Omega |x|\sigma \phi_\ell u^p \, dx \geq \left( \int_\Omega u \phi_\ell \, dx \right)^p \left( \int_\Omega |x|^{-\sigma/(p-1)} \phi_\ell \, dx \right)^{-(p-1)}.
\]

Hence

\[
\int_\Omega |x|\sigma \phi_\ell u^p \, dx \geq \left\{ \begin{array}{l}
C w_\ell^p \ell^{\sigma-n(p-1)} \quad \text{if } \sigma < n(p-1), \\
w_\ell^p (\log \ell)^{-(p-1)} \quad \text{if } \sigma = n(p-1), \\
w_\ell^p \quad \text{if } \sigma > n(p-1).
\end{array} \right. 
\]

(3.9)

Next, we quote a result from [23].

**Lemma 3.1** [23]. Let \( w(t) \) be a positive continuous function which satisfies the inequality

\[
\frac{dw}{dt} \geq C w^n
\]

in the distributional sense, where \( C > 0 \) is a constant and \( q > 1 \). Then \( w(t) \) is an increasing function, and there exists a finite \( T > 0 \) such that \( w(t) \to + \infty \) as \( t \to T^- \).

**Proof of Theorem 2.** First we consider the case \( \sigma < n(p-1) \). It follows from (3.8) and (3.9) that

\[
\frac{dw_\ell}{dt} \geq \left\{ -C_1 \ell^{(n-m-1)-(m+n)/p} + C_2 w_\ell^{p-m} \ell^{(\sigma-n(p-1))(p-m)/p} \right\} \\
\times \left( \int_\Omega |x|\sigma \phi_\ell u^p \, dx \right)^{m/p}.
\]

(a) \( p < \bar{p}_c = m + (m + 1 + \sigma)/n \).

Under this assumption, \( [\sigma-n(p-1)](p-m)/p > n-m-1-m(n+\sigma)/p \), and consequently

\[
\frac{\ell^{(\sigma-n(p-1))(p-m)/p}}{\ell^{(n-m-1)-(m+n+\sigma)/p}} \to +\infty \quad \text{as } \ell \to +\infty.
\]

(3.11)
Using the fact that \( w_\ell \) is an increasing function of \( \ell \), we find from (3.10) and (3.11) that there exists \( \ell \gg 1 \) and \( \delta > 0 \) such that
\[
\frac{dw_\ell}{dt} \geq \delta u_\ell^p(t)\ell^{\sigma - n(p - 1)} \quad \text{for all} \quad t > 0.
\]
Thus \( w_\ell \), and consequently \( u \), blows up in finite time by Lemma 3.1.

(b) \( p = \tilde{p}_\ell = m + (m + 1 + \sigma)/n \).
In this case \( \frac{\sigma - n(p - 1)}{p} = n - m - 1 - m(n + \sigma)/p \). If we can prove that for any \( M > 0 \), there exists \( l > 0 \) and \( t > 0 \) such that
\[
\int_\Omega u(x, t)\phi_\ell(x) \, dx > M,
\]
then it can be shown just as in the above case that \( w_\ell \), and hence \( u \), blows up in finite time. Otherwise, \( u(\cdot, t) \in L^1(\Omega) \) for all \( t > 0 \) and there exists an \( M_0 > 0 \) such that
\[
\|u(t)\|_{L^1(\Omega)} \leq M_0 \quad \text{for all} \quad t > 0. \quad (3.12)
\]
We prove (3.12) is impossible. Suppose the contrary; it is clear from (3.6) that if \( \int_\Omega |x|^\sigma u^p \, dx < +\infty \) then
\[
\int_\Omega u|\Delta \phi_\ell| \, dx \to 0 \quad \text{and} \quad \ell^{(1-m)(n-1)} \left( \int_\Omega u|\Delta \phi_\ell| \, dx \right)^m \to 0 \quad \text{as} \quad \ell \to +\infty,
\]
because \( n - m - 1 - m(n + \sigma)/p < 0 \). By (3.5) we get \( w'_\ell(t) \geq \frac{1}{2} \int_\Omega |x|^\sigma \phi_\ell u^p \, dx \). If \( \int_\Omega |x|^\sigma u^p \, dx = +\infty \), then \( w'_\ell(t) \geq 1 \) by (3.7). Hence
\[
w'_\ell(t) \geq \kappa_\ell(t) \geq \min \left\{ 1, \frac{1}{2} \int_\Omega |x|^\sigma \phi_\ell u^p \, dx \right\}, \quad \ell \gg 1.
\]
Upon integration we have
\[
w_\ell(t) - w_\ell(0) \geq \int_0^t \kappa_\ell(\tau) \, d\tau.
\]
Let \( w(t) = \int_\Omega \psi_0(x)u(x, t) \, dx \), and take \( \ell \to +\infty \) in the above inequality to obtain
\[
w(t) - w(0) \geq \int_0^t \kappa(\tau) \, d\tau, \quad (3.13)
\]
where \( \kappa(t) = \min \{1, \frac{1}{2} \int_\Omega |x|^\sigma \psi_0(x)u^p \, dx\} \).
Using (2.2), by direct computation we have
\[
\int_\Omega |x|^\sigma \psi_0(x)u^p \, dx \geq \delta^p(t - \varepsilon)^{-1} \int_{|y| \geq (t - \varepsilon)^{-\delta}} |y|^\sigma \psi_0(y(t - \varepsilon)^\delta)
\times (1 + C|y|^{1+m}/m - mp/(1-m)) \, dy
\geq C(t - \varepsilon)^{-1} \quad \text{as} \quad t \gg 1, \quad (3.14)
\]
where \( \theta = 1/[1 + m - n(1 - m)] > 0 \). In view of (3.13) and (3.14) it yields

\[
\lim_{t \to +\infty} w(t) = +\infty; \quad \text{i.e.,} \quad \lim_{t \to +\infty} \int_{\Omega} \psi_{0}(x) u(x, t) \, dx = +\infty.
\]

Since \( \psi_{0}(x) \leq 1 \), this shows that (3.12) is impossible. And hence \( u(x, t) \) blows up in finite time.

Next, we consider the case \( \sigma \geq n/(p-1) \). Since \( m > (n-1)/(n+\sigma) \), it follows that \( n - m - 1 - m(n + \sigma)/p < 0 \).

For the case \( \sigma = n/(p-1) \), combining (3.8) and (3.9), we find

\[
\frac{dw_{\ell}}{dt} \geq \left\{ -C\ell^{(n-m-1)-m(n+\sigma)/p} + C w_{\ell}^{p-m} (\log \ell)^{(m-p(p-1)/p)} \right\}
\]

\[
\times \left( \int_{\Omega} |x|^{\sigma} \phi_{\ell} u^{p} \, dx \right)^{m/p}.
\] (3.15)

Using the fact that \( w_{\ell} \) is an increasing function of \( \ell \), we find from (3.15) that there exist \( \ell \gg 1 \) and \( \delta > 0 \) such that

\[
\frac{dw_{\ell}}{dt} \geq \delta (\log \ell)^{1-p} w_{\ell}^{p}(t) \quad \text{for} \quad t > 1.
\]

Thus \( w_{\ell} \), and consequently \( u \), blows up in finite time.

The case of \( \sigma > n/(p-1) \) can be handled similarly using the third inequality of (3.9). This completes the proof of Theorem 2. \( \blacksquare \)

4. PROOF OF THEOREM 1

In this section we shall prove Theorem 1 for the general case (I).

When \( 1 < p \leq p_{c} = m + (m + 1 + 2s + \sigma)/n \), using the methods similar to those of the last section and [23], it can be proved that every non-trivial solution of (I) blows up in finite time. We omit the details.

When \( p > p_{c} \), we shall prove that (I) has global positive solutions for small initial data. By the comparison principle, it is enough to prove this conclusion for the following problem (since \( s \geq 0 \))

\[
\begin{align*}
\frac{d}{dt} u_{t} &= \text{div}(\nabla u^{[m-1]}) + (1 + t)^{s} |x|^{\sigma} u^{p}, \quad x \in \mathbb{R}^{n}, \quad t > 0, \\
u(x, 0) &= u_{0}(x) \geq 0, \quad x \in \mathbb{R}^{n}, \quad (4.1)
\end{align*}
\]

where the constants \( m, s, \sigma \), and \( p \) are as in problem (I). We shall show the existence of global solutions of (4.1) by constructing global similarity solutions. They take the form

\[
u(x, t) = (1 + t)^{-\alpha} w(r) \quad \text{with} \quad r = |x|(1 + t)^{-\beta},
\]
where \( \alpha = [(1 + m)(1 + s) + \sigma]/K, \beta = [p - 1 + (1 - m)(1 + s)]/K, K = (p - 1)(1 + m) - \sigma(1 - m) \). It is easy to verify that \( \alpha, \beta, \) and \( K \) satisfy the following relations:

\[
m(\alpha + \beta) + \beta = \alpha + 1 = p\alpha - \beta\sigma - s, \quad K > 0.
\]

The resulting ODE for \( w \) is

\[
m|w'|^{m-1}w'' + \frac{n-1}{r}|w'|^{m-1}w' + \alpha w + B r w' + r^\sigma w^p = 0, \quad r > 0,
\]

\( w(0) = \eta > 0, |w'|^{m-1}w'(0) = \lim_{r \to 0^+} (-r^{\sigma+1}w^p(r)/(n + \sigma)) \). (4.2)

We observe that a function \( \bar{u}(x, t) = (1 + t)^{-\alpha}v(|x|(1 + t)^{-\beta}) \) is an upper solution of (4.1) if and only if \( v(r) \) satisfies the inequality

\[
m|v'|^{m-1}v'' + \frac{n-1}{r}|v'|^{m-1}v' + \alpha v + B r v' + r^\sigma v^p \leq 0, \quad r > 0.
\]

We first discuss the case \( \sigma \geq 0 \). In this case, we try to find an upper solution of (4.1), i.e., the solution of (4.3). Let

\[
v(r) = \varepsilon(1 + br^k)^{-q},
\]

where \( k = (1 + m)/m, q = m/(1 - m) \), and \( \varepsilon \) and \( b \) are positive constants to be determined later. By direct computation we have

\[
\begin{align*}
    v' & = -\varepsilon q b k r^{k-1}(1 + br^k)^{-q-1}, \\
    v'' & = \varepsilon q(b + 1)b^2 r^{2k-2}(1 + br^k)^{-q-2} - \varepsilon q b k(k - 1)r^{k-2}(1 + br^k)^{-q-1}.
\end{align*}
\]

\( v(r) \) satisfies (4.3) if and only if

\[
\varepsilon q b k [\varepsilon^{m-1}(q b k)^m - \beta] r^{k(1 + br^k)^{-q-1}} + \varepsilon(\nu - \gamma(1 + br^k)^{-q} + \varepsilon^\mu
\]

\[
\leq 0.
\]

(4.4)

By \( p > p_c \) we find \( \sigma + q(1 - p)k = \sigma + (1 - p)(1 + m)/(1 - m) < 0 \). It follows that there exists \( a > 0 \) such that

\[
r^\sigma(1 + br^k)^{(1-p)} \leq a \quad \text{for all} \quad r \geq 0
\]

(4.5)

since \( \sigma \geq 0 \). Choose \( b = b(\varepsilon) \) such that

\[
\varepsilon^{m-1}(q b k)^m = \beta; \quad \text{i.e.,} \quad b = \beta^{1/m} \varepsilon^{(1-m)/m(qk)}.
\]

For this choice of \( b \), (4.4) is equivalent to

\[
\alpha - n\beta + \varepsilon^{p-1}r^\sigma(1 + br^k)^{-pq} \leq 0.
\]

(4.6)

By (4.5) we see that (4.6) is true if the following inequality holds

\[
\alpha - n\beta + a\varepsilon^{p-1} \leq 0.
\]

(4.7)
In view of \( p > p_c = m + (m + 1 + 2s + \sigma)/n \) it follows that \( \alpha < n\beta \). Hence, there exists \( \epsilon_0 > 0 \) such that (4.7) holds for all \( 0 < \epsilon \leq \epsilon_0 \). These arguments show that \( u(r) = \epsilon(1 + b(\epsilon))^{-q} \) satisfies (4.3) for all \( 0 < \epsilon \leq \epsilon_0 \). Using the comparison principle we get that the solution \( u(x, t) \) of (4.1) exists globally provided that \( u(x, 0) \leq u(|x|) \). And hence, so does the solution of (I).

Next, we consider the case \( \sigma < 0 \). For this case, it will be proved that (4.2) has a ground state for small \( \eta \). By a standard argument we can prove that for any given \( \eta > 0 \), there exists a unique solution \( w \) of (4.2), which is twice continuously differentiable where \( w'(r) \neq 0 \); see [21, 22]. Denote \( R(\eta) \) the maximum of \( R \) for which \( \omega(\eta) > 0 \). So, \( 0 < R(\eta) \leq +\infty \), and \( w(R(\eta)) = 0 \) when \( R(\eta) < +\infty \).

We divide the proof into several lemmas.

**Lemma 4.1.** _The solution \( w(r) \) of (4.2) satisfies \( w'(r) < 0 \) in \((0, R(\eta))\). In addition, if \( R(\eta) = +\infty \) then \( w(r) \to 0 \) as \( r \to +\infty \)._

**Proof.** We first prove \( w'(r) < 0 \) for \( 0 < r < R(\eta) \). When \( \sigma + 1 \leq 0 \), we have that \( |w'|^{m-1}w'(0) = \lim_{r \to 0} -r^{\sigma+1}w'(r)/(n + \sigma) \) is finite. Consequently \( w'(r) < 0 \) for \( r \ll 1 \). If there exists \( r_0 < R(\eta) \) such that \( w'(r) > 0 \) in \((0, r_0) \) and \( w'(r_0) = 0 \), then \( w'(r_0) < 0 \), a contradiction.

When \( \sigma + 1 > 0 \), it follows that \( w'(0) = 0 \). Using Eq. (4.2) one has

\[
\int_0^r (|w'|^{m-1}w') = -\alpha w(0) - \lim_{r \to 0} r^{\sigma}w' < 0.
\]

Hence \( w'(r) < 0 \), and consequently \( w'(r) < 0 \) for \( r \ll 1 \). Similar to the case of \( \sigma + 1 \leq 0 \) it follows that \( w'(r) < 0 \) for all \( 0 < r < R(\eta) \).

If \( R(\eta) = +\infty \). Since \( w'(r) < 0 \) and \( w(r) > 0 \) in \((0, +\infty) \), \( w(r) \to L \). If \( L > 0 \), integration of (4.2) gives

\[
r^{\alpha-1}(|w'|^{m-1}w' + \beta rw) = -\int_0^r \{s^{n-1}w(s) + s^{n+\sigma-1}w'(s)\} ds.
\]

Let \( r \to +\infty \) in (4.8) we find

\[
\lim_{r \to +\infty} \frac{|w'|^{m-1}w'}{r} = -\frac{\alpha}{n} A - L.
\]

where \( A = L^p \) when \( \sigma = 0 \), \( A = 0 \) when \( \sigma < 0 \), and \( A = +\infty \) when \( \sigma > 0 \). It follows that

\[
\lim_{r \to +\infty} w'(r) = -\infty,
\]

a contradiction. Thus \( w(r) \to 0 \) as \( r \to +\infty \). 

**Lemma 4.2.** _For any given small \( \eta > 0 \) there exists \( R_0(\eta) > 0 \), which satisfies \( \lim_{\eta \to 0} R_0(\eta) = +\infty \) and such that

\[
w(r) > 0, \quad |w'|^{m-1}w' + \beta rw > 0 \quad \text{on} \quad [2, R_0(\eta)].
\]

(4.9)
Proof. Let \( z = \eta - w \). Then \( z'(r) = -w'(r) > 0, 0 < z(r) < \eta \) and \( z(r) \) satisfies
\[
((z')^m)' + \frac{n-1}{r}(z')^m + \beta rz' = \alpha(\eta - z) + r^\sigma(\eta - z)^p, \quad r > 0,
\]
(4.10)
\[z(0) = 0, \quad |z'|^{m-1}z'(0) = \lim_{r \to 0 } (r^{m+1}(\eta - z)^p/(n + \sigma)).\]

Integration of (4.10) gives
\[
r^{m-1}(z')^m + \beta r^m z = \int_0^r [(n \beta - \alpha)s^{m-1}z(s) + \alpha \eta s^{m-1}] ds
\]
\[+ \int_0^r s^{m+\sigma-1}(\eta - z)^p ds
\]
\[\leq \frac{\alpha \eta}{n} r^m + \left( \beta - \frac{\alpha}{n} \right) r^m z(r) + \frac{1}{n + \sigma} \eta^p r^{n+\sigma},
\]
(4.11)
since \( p > p_c \) implies \( n \beta > \alpha \). Let \( R_0(\eta) \) be the first value of \( r \), where \( z(r) = \eta - \eta^{(p+1)/2} \). Then \( R_0(\eta) > 0 \) and \( z(r) \leq \eta - \eta^{(p+1)/2} < \eta \) for all \( 0 < r \leq R_0(\eta) \). From (4.11) it follows that for \( 0 < r \leq R_0(\eta) \)
\[
r^{m-1}(z')^m < \frac{\alpha \eta}{n} r^m + \left( \beta - \frac{\alpha}{n} \right) \eta r^m + \frac{1}{n + \sigma} \eta^p r^{n+\sigma}
\]
\[= \beta \eta r^m + \frac{1}{n + \sigma} \eta^p r^{n+\sigma},
\]
i.e.,
\[
(z')^m \leq \beta \eta r + \frac{1}{n + \sigma} \eta^p r^{n+1}.
\]

Since \( m < 1 \), it follows that
\[
z'(r) < \left( \beta \eta r + \frac{1}{n + \sigma} \eta^p r^{n+1} \right)^{1/m} \leq C(\eta^{1/m}r^{1/m} + \eta^{p/m}r^{(p+1)/m}).
\]

Integrating this inequality from 0 to \( R_0(\eta) \) we have
\[
\eta \leq \eta^{(p+1)/2} + C(\eta^{1/m}(R_0(\eta))^{1+1/m} + \eta^{p/m}(R_0(\eta))^{(m+1)/(m+\sigma)})
\]
In view of \( p > 1 \) and \( m < 1 \), it is obvious that \( R_0(\eta) \to +\infty \) as \( \eta \to 0^+ \).

And \( z(r) \leq \eta - \eta^{(p+1)/2}, z(R_0(\eta)) = \eta - \eta^{(p+1)/2} \), consequently \( w(r) \geq \eta^{(p+1)/2} \) and \( w(R_0(\eta)) = \eta^{(p+1)/2} \), for all \( 0 \leq r \leq R_0(\eta) \).

Integration of (4.2) gives, for \( 0 \leq r \leq R_0(\eta) \),
\[
r^{m-1}|w^{m-1}w' + \beta r^m w(r)
\]
\[= \int_0^r (n \beta - \alpha) s^{m-1}w(s) ds - \int_0^r s^{m+\sigma-1}w^p(s) ds
\]
\[\geq (n \beta - \alpha) w(R_0(\eta)) \int_0^r s^{m-1} ds - \eta^p \int_0^r s^{m+\sigma-1} ds
\]
\[= \left( \beta - \frac{\alpha}{n} \right) \eta^{(p+1)/2} r^m - \frac{1}{n + \sigma} \eta^{p} r^{n+\sigma}
\]
\[= \eta^{(p+1)/2} r^m \left( \beta - \frac{\alpha}{n} \right) r^{n+\sigma}.
\]
Since $\sigma < 0, p > 1$ and $\eta \ll 1$ (note $n + \sigma > 0, n \beta > \alpha$), it follows that

$$r^{n-1}|w'|^{m-1}w' + \beta r^n w(r) > 0 \quad \text{for} \quad 2 \leq r \leq R_0(\eta).$$

The proof of Lemma 4.2 is completed.

Now we prove that, for the case $\sigma < 0$, (4.2) has a ground state for small $\eta$. Choose $\eta_0 : \eta_0^{p-1} < n \beta - \alpha$ and such that (4.9) holds for all $0 < \eta \leq \eta_0$. Integrating (4.2) from $R_0(\eta)$ to $r(R_0(\eta) < r < R(\eta))$ gives

$$r^{n-1}|w'|^{m-1}w' + \beta r^n w(r) = (r^{n-1}|w'|^{m-1}w' + \beta r^n w(r))|_{r=R_0(\eta)}$$

$$+ (n \beta - \alpha) \int_{R_0(\eta)}^{r} s^{n-1}w(s) \, ds - \int_{R_0(\eta)}^{r} s^{n+\sigma-1}w^p(s) \, ds$$

$$> \int_{R_0(\eta)}^{r} s^{n-1}w(s)(n \beta - \alpha - s^{\sigma}w^{p-1}(s)) \, ds$$

$$\geq \int_{R_0(\eta)}^{r} s^{n-1}w(s)(n \beta - \alpha - \eta^{p-1}) \, ds > 0$$

(4.12)

since $\sigma < 0, R_0(\eta) > 2$, and $w(s) < \eta$. In view of $w(r) > 0$ and $w'(r) < 0$ for $0 < r < R(\eta)$, it follows that $R(\eta) = +\infty$ by (4.12). Therefore (4.2) has a ground state.

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