The Erdös-Jacobson-Lehel conjecture on potentially $P_k$-graphic sequence is true*

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Abstract A variation in the classical Turan extremal problem is studied. A simple graph $G$ of order $n$ is said to have property $P_k$ if it contains a clique of size $k+1$ as its subgraph. An $n$-term nonincreasing nonnegative integer sequence $\pi = (d_1, d_2, \cdots, d_n)$ is said to be graphic if it is the degree sequence of a simple graph $G$ of order $n$ and such a graph $G$ is referred to as a realization of $\pi$. A graphic sequence $\pi$ is said to be potentially $P_k$-graphic if it has a realization $G$ having property $P_k$. The problem: determine the smallest positive even number $\sigma(k, n)$ such that every $n$-term graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ without zero terms and with degree sum $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$ at least $\sigma(k, n)$ is potentially $P_k$-graphic has been proved positive.

Keywords: graph, graphic sequence, off-diagonal leftmost matrix, potentially $P_k$-graphic sequence.

An $n$-term nonincreasing nonnegative integer sequence $\pi = (d_1, d_2, \cdots, d_n)$ is said to be graphic if it is the degree sequence of a simple graph $G$ of order $n$ and such a graph is referred to as a realization of $\pi$. A graph $G$ is said to have property $P_k$ if it has a complete subgraph $K_{k+1}$ of order $k+1$ [1]. A graph sequence $\pi$ is potentially (res. forcibly) $P_k$-graphic if it has a realization $G$ having property $P_k$ (res. all of its realizations have property $P_k$). The degree sequence is one of the basic subjects in the graph theory. As to its progress, please refer to the summarized essays of Rao [2] and Li [3].

It is well known that the classical Turan extremal problem is to determine the smallest positive integer $\text{ex}(k, n)$ such that each graph $G$ of order $n$ with edge number $e(G) \geq \text{ex}(k, n)$ contains a complete subgraph of order $k + 1$. The number $\text{ex}(k, n)$ is called the Turan number. The classical Turan theorem determines the Turan number [4]. As Bollobás pointed out in the preface of his masterpiece "Extremal Graph Theory": "Extremal graph theory, in its strictest sense, is a branch of graph theory developed and loved by Hungarians. Its study, as a subject in its own right, was initiated by Turan in 1940, although a special case of his theorem and several other extremal results had been proved many years earlier. The main exponent has been Paul Erdös who, through his many papers and lectures, as well as uncountably many problems, has virtually created the subject." It serves to show that the Turan theorem is one of the basic theorems in extremal graph theory and that the extremal theory of graphs is carried out around it and also to show Erdős' historical position in the extremal graph theory.

In view of the theory of graphic sequences, the Turán number $\text{ex}(k, n)$ is the smallest

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positive integer such that each graphic sequence \( \pi = (d_1, d_2, \cdots, d_n) \) with degree sum \( \sigma(\pi) = d_1 + d_2 + \cdots + d_n \geq 2 \) is forcibly \( P_k \)-graphic. Erdős, Jacobson and Lehel\(^1\) considered a variation of the classical Turán extremal problem: determine the smallest positive even number \( \sigma(k, n) \) such that each \( n \)-term graphic sequence \( \pi = (d_1, d_2, \cdots, d_n) \) without zero terms and with degree sum \( \sigma(\pi) \geq \sigma(k, n) \) is potentially \( P_k \)-graphic. They showed that the graphic sequence \( D_k = ((n-1)^{k-1}, (k-1)^{n-k+1}) \) is not potentially \( P_k \)-graphic, where \( (n-1)^{k-1} \) means \( n-1 \) repeats \( k-1 \) times in \( \pi \). So \( \sigma(k, n) \geq (k-1)(2n-k)+2 \). In ref. \([1]\), they also pointed out:

"We feel that these are the extremal examples, that is, any degree sequence giving more edges than \( D_k \) is potentially \( P_k \)-graphical."

In other words, ref. \([1]\) raised the following conjecture (abbrev. EJLC): \( \sigma(k, n) = (k-1)(2n-k)+2 \). This shows that we only need to prove that the EJLC is positive for any given \( k \geq 2 \) and \( n \) which is large enough. Erdős et al. also proved the following\(^1\).

\[ \text{Theorem A.} \quad \text{Let } n \geq 6 \text{ and } \pi = (d_1, d_2, \cdots, d_n) \text{ be a graphic sequence, where } d_1 \geq d_2 \geq \cdots \geq d_n \geq 1. \text{ If } \sigma(\pi) \geq 2n, \text{ then } \pi \text{ is potentially } P_2 \text{-graphic.} \quad \text{In other words, } \sigma(2, n) = 2n \text{ for } n \geq 6. \]

Recently, Li and Song\(^2\) proved the following result.

\[ \text{Theorem 0.1.} \quad \sigma(k, 2k+1) = 4k^2 - 2k. \]

Li and Song\(^3\) and Gould et al.\(^1\) proved the following result independently.

\[ \text{Theorem 0.2.} \quad \sigma(3, n) = 4n - 4 \text{ for } n \geq 8. \]

Li and Song\(^3\) also proved the following theorem.

\[ \text{Theorem 0.3.} \quad \text{For } n \geq 10, \sigma(4, n) = 6n - 10. \]

The above results show that the EJLC is true for \( 2 \leq k \leq 4 \) and \( n \geq 2k + 2 \). The purpose of this paper is to prove the EJLC is true for \( k \geq 5 \) and \( n \geq \left( \frac{k}{2} \right) + 3. \)

1 Preparation

We need the following terms and notations. Let \( \pi = (d_1, d_2, \cdots, d_n) \) be an integer sequence with \( n-1 \geq d_1 \geq d_2 \geq \cdots \geq d_n \geq 0. \) The set of all such sequences is denoted by \( NS_n \). For a given \( \pi = (d_1, d_2, \cdots, d_n) \in NS_n, \) let \( \sigma_i(\pi) = d_1 + d_2 + \cdots + d_i \) for each \( 1 \leq i \leq n \) and \( \sigma_n(\pi) = \sigma(\pi) \). In addition, let \( f(\pi) = \max \{ i : d_i \geq i, \ 1 \leq i \leq n \} \). \( \sigma(\pi) \) and \( f(\pi) \) are called the degree sum and trace of \( \pi \), respectively. Let us define an \( n \times n \) matrix \( A(\pi) = (a_{ij}) \) as follows:

For \( 1 \leq i \leq f(\pi) \),

\[ a_{ij} = \begin{cases} 1, & \text{if } 1 \leq j \neq i \leq d_i + 1, \\ 0, & \text{otherwise,} \end{cases} \]

\(^1\) Gould, R. J., Cacobson, M. S., Lehel, J., Potentially \( G \)-graphical degree sequences, to appear.


\(^3\) Li Jong-sheng, Song Zi-xia, The smallest degree sum that yields potentially \( P_k \)-graphical sequence, submitted.
and for \( f(\pi) + 1 \leq i \leq n \),

\[
a_{ij} = \begin{cases} 
1, & \text{if } 1 \leq j \leq d_i, \\
0, & \text{otherwise}.
\end{cases}
\]

Then matrix \( A(\pi) \) is called the off-diagonal leftmost matrix of \( \pi \). Clearly, the row sum vector of \( A(\pi) \) is \( \pi \). The column sum vector of \( A(\pi) \) is called the corrected conjugate vector of \( \pi \), and denoted by \( \tilde{\pi} \). Obviously, \( \sigma(\pi) = \sigma(\tilde{\pi}) \). The set of all \( n \)-term graphic sequences are denoted by \( GS_n \). The following is a criterion for a nonincreasing sequence being graphic.

**Theorem B**[5]. Let \( \pi = (d_1, d_2, \ldots, d_n) \in NS_n \) with even \( \sigma(\pi) \). Then \( \pi \in GS_n \) if and only if for each \( t = 1, 2, \ldots, n \), \( \sigma_t(\pi) \leq \sigma_t(\tilde{\pi}) \), or in equivalent words, if and only if for each \( t = 1, 2, \ldots, f(\pi) \), \( \sigma_t(\pi) \leq \sigma_t(\tilde{\pi}) \).

Let \( \pi' = \left( (d_1 - 1, \ldots, d_{k+1} - 1, \ldots, d_{d_k+1} - 1, d_{d_k+2}, \ldots, d_n) \right) \), if \( d_k \geq k \),

\[
\pi' = \left( (d_1 - 1, \ldots, d_{k-1} - 1, d_{k+1} - 1, \ldots, d_{d_k+1} - 1, d_{d_k+2}, \ldots, d_n) \right),
\]

if \( d_k \leq k - 1 \).

Then \( \pi' \) is the residual sequence obtained by laying off \( d_k \) from \( \pi \).

**Theorem C**[6]. Let \( \pi \in NS_n \). Then \( \pi \in GS_n \) if and only if \( \pi' \in GS_{n-1} \).

Let \( G \) be a simple graph with vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and let \( \pi = (d_1, d_2, \ldots, d_n) \) be the degree sequence of \( G \), where \( d_i \) is the degree of \( v_i \) and \( \pi \) is not necessarily nonincreasing. Then \( G \) is said to have property \( A_k \) if its subgraph induced by \( v_1, v_2, \ldots, v_{k+1} \) is complete, i.e. if its first \( k+1 \) vertices form a clique of size \( k+1 \). A degree sequence \( \pi \) is said to be potentially \( A_k \)-graphic if there exists a graph \( G \) having property \( A_k \) that realizes \( \pi \). Rao[7] proved the following.

**Theorem D.** Let \( \pi \) be the degree sequence of graph \( G \). Then \( \pi \) is potentially \( P_k \)-graphic if and only if \( \pi \) is potentially \( A_k \)-graphic.

Let \( \pi = (d_1, d_2, \ldots, d_n) \) be a sequence of nonnegative integers. For any given \( s \) and \( t \), \( 0 \leq s \leq k + 1 \) and \( 0 \leq t \leq n - k - 1 \), denote

\[
L(s, t) = \sum_{i=1}^{k+1} d_i + \sum_{j=1}^{k+1} d_{k+1+j},
\]

\[
R(s, t) = (s + t)(s + t - 1) + \sum_{i=s+1}^{k+1} \min\{s + t, d_i - k + s\} + \sum_{j=k+2+t}^{n} \min\{s + t, d_j\}.
\]

Rao[1] also gave a criterion for a sequence \( \pi \) being potentially \( A_k \)-graphic as follows.

**Theorem E.** Let \( \pi = (d_1, d_2, \ldots, d_n) \) be a sequence of nonnegative integers in which \( d_1 \geq d_2 \geq \cdots \geq d_{k+1} \) and \( d_{k+2} \geq d_{k+3} \geq \cdots \geq d_n \). Then \( \pi \) is potentially \( A_k \)-graphic if and only if the following conditions hold:

(1) \( d_{k+1} \geq k \),

(2) \( \sigma(\pi) \) is even,

(3) for any \( s \) and \( t \), \( 0 \leq s \leq k + 1 \), \( 0 \leq t \leq n - k - 1 \),

\[
L(s, t) \leq R(s, t).
\]

But we have to point out with regret that Theorem E has never been published though it is mentioned in Rao[7]. We have carefully looked through his original proof and make sure that his

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1) Rao, A. R., An Erdős-Gallai type result on the clique number of a realization of a degree sequence, unpublished.
In order to prove the main results of this paper, we will use double induction on \( k \) and \( n \), Kleitman and Wang's laying off technique, Theorems D and E. We will also use the following simple fact repeatedly: Let \( \pi' = (d'_1, d'_2, \ldots, d'_{n-1}) \) be the residual sequence obtained by laying off \( d_i \) from \( \pi \). If \( \pi' \) is potentially \( A_{k-1} \)-graphic and the first \( k \) terms \( d'_1, \ldots, d'_k \) of \( \pi' \) are obtained by subtracting one from the \( k \) terms of \( \pi \), respectively, then \( \pi \) is potentially \( P_k \)-graphic.

In addition, we need the following facts.

**Theorem 1.1.** Let \( \pi = (d_1, d_2, \ldots, d_n) \in GS_n \) with \( d_{k+1} \geq k \) and

\[
\begin{align*}
 n-2 \geq d_1 \geq \cdots \geq d_k = d_{k+1} = \cdots = d_{d_1+2} = \cdots \geq d_n \geq k-1.
\end{align*}
\]

If \( n = 2k+2 \) and \( d_i \geq k \), or \( n = 2k+3 \) and \( d_i \geq k-1 \), then \( \pi \) is potentially \( P_k \)-graphic.

**Proof.** By Theorem E, we only need to verify that (1) holds for any \( s \) and \( t \), \( 0 \leq s \leq n \) and \( 0 \leq t \leq n - k - 1 \). We consider two cases as follows.

**Case 1.** \( d_k \leq s + t - 1 \). Suppose \( n \geq 2k+2 \) and \( d_i \geq k-1 \). If \( s \geq k \), then \( s + t \geq s \geq k \) and \( d_i - k + s \geq (d_k - k) + s \geq s \geq k \) for \( 1 \leq i \leq k + 1 \). Hence \( \min\{s + t, d_i - k + s\} \geq k \) for \( 1 \leq i \leq k + 1 \). In addition, \( s + t > d_k \geq \cdots \geq d_n \geq k - 1 \), hence \( \min\{s + t, d_j\} \geq d_j \geq k - 1 \) for \( k \leq j \leq n \). Therefore

\[
R(s, t) \geq (s + t)(s + t - 1) + (k + 1 - s)k + (n - k - 1 - t)(k - 1)
\]

\[
= (k - 1)(n - 2) + (s + t + 1 - k)(s + t - 1) + (2k - s)
\]

\[
\geq (k - 1)(n - 2) + (s - k + 1) + t(s + t - 1)
\]

\[
\geq (k - 1)d_1 + (s - k + 1)d_k + td_k \geq L(s, t).
\]

If \( 0 \leq s \leq n - k - 1 \), then \( d_i - k + s \geq (d_i - k) + s \geq s \) for \( 1 \leq i \leq k + 1 \). Hence

\[
R(s, t) \geq (s + t)(s + t - 1) + (k + 1 - s) + (n - k - 1 - t)(k - 1),
\]

\[
\geq 0 + (s + t + 1 - k)(s + t - 1) + (2k - s)
\]

\[
\geq (k - 1)(n - 2) + (s - k + 1) + t(s + t - 1)
\]

\[
\geq (k - 1)d_1 + (s - k + 1)d_k + td_k \geq L(s, t).
\]

So (1) holds for \( d_k \leq s + t - 1 \) 1 if \( n \geq 2k+2 \) and \( d_n \geq k-1 \).

**Case 2.** \( d_k \geq s + t \). In this case, \( d_k = d_{k+1} = \cdots = d_{d_1+2} = s + t \). Let \( n \geq 2k+2 \) and \( d_n \geq k-1 \). If \( d_k \geq t + k \), then \( d_i - k + s \geq (d_i - k) + s \geq s + t \) for \( 1 \leq i \leq k + 1 \). Hence

\[
R(s, t) \geq (s + t)(s + t - 1) + \sum_{i=1}^{k+1} \min\{s + t, d_i - k + s\} + \sum_{j=k+2}^{d_1+2} \min\{s + t, d_j\}
\]

\[
= (s + t)(s + t - 1) + (k + 1 - s)(s + t) + (d_1 + 2 - k - 1 - t)(s + t)
\]

\[
= (s + t)(d_1 + 1) \geq sd_k + td_k \geq L(s, t).
\]

If \( s + t \leq d_k \leq t + k - 1 \), then \( 0 \leq s \leq k - 1 \) and \( d_k - k + s \leq s + t \). Moreover \( d_k - k + s \leq d_i - k + s \) for \( 1 \leq i \leq k + 1 \), so \( \min\{s + t, d_i - k + s\} \geq d_k - k + s \) for \( 1 \leq i \leq k + 1 \). Denote \( d_k = t + m \), where \( 0 \leq s \leq m \leq k - 1 \). Then \( s + t = d_k - (m - s) \geq k - m + s \). If \( d_n \geq k \), then \( d_n \geq k - (m - s) \). Hence

\[
R(s, t) \geq (s + t)(s + t - 1) + (k + 1 - s)(d_k - k + s) + (n - k - 1 - t)(k - m + s)
\]

\[
= s(n - 2) + td_k + (k - m)(n - 2k - 2) + s(k + 1 - m)
\]

\[
\geq (s + t + 1 - k)(s + t - 1) + (2k - s)
\]

\[
\geq (k - 1)(n - 2) + (s - k + 1) + t(s + t - 1)
\]

\[
\geq (k - 1)d_1 + (s - k + 1)d_k + td_k \geq L(s, t).
\]


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So (1.2) holds for \( n \geq 2k + 2 \) and \( d_n \geq k \).

Combining Cases 1 and 2, we have proved that (1.1) holds for \( n \geq 2k + 2 \) and \( d_n \geq k \).

Now suppose \( n \geq 2k + 3 \) and \( d_n = k - 1 \). Then \( d_1 \leq n - 3 \) since \( d_{d+2} = d_{k+2} \geq k > k - 1 = d_n \). In the proof of Case 2, if \( s + t < d_k \leq t + k - 1 \), then \( 0 \leq s \leq k - 1 \) and the \( m \) in \( d_k = t + m \) satisfies \( 0 \leq s < m \leq k - 1 \). So \( d_n = k - 1 \geq k - (m - s) \) still holds. Hence (1.2) holds for \( s + t < d_k \leq t + k - 1 \) if \( n \geq 2k + 3 \) and \( d_n = k - 1 \). If \( s + t = d_k \leq t + k - 1 \), then \( 0 \leq s < k - 1 \) and

\[
R(s, t) = (s + t)(s + t - 1) + (d_k - k + s)(k + 1 - s) + (n - k - 1 - t)(k - 1)
\]

\[
= (s + t)(s + t - 1) + (s + t - k + s)(k + 1 - s) + (n - k - 1 - t)(k - 1)
\]

\[
= s(n - 3) + t(s + t) + (n - 2k - 3)(k - 1 - s) + s(k - s) + (s + t - 2)
\]

\[
\geq sd_1 + td_1 = L(s, t).
\]

Hence (1.1) holds for \( d_k \geq s + t \) if \( n \geq 2k + 3 \) and \( d_n \geq k - 1 \).

**Theorem 1.2.** Let \( n \geq 2k + 2 \) and \( \pi = (d_1, d_2, \ldots, d_n) \in GS_n \) where \( d_{k+1} \geq k \),

\[
n - 3 \geq d_1 \geq \cdots \geq d_k = d_{k+1} = \cdots = d_{d+2} = \cdots = d_p > d_{p+1} \geq \cdots \geq d_n, \quad (1.3)
\]

\[
1 \leq d_k \leq k - 1 \text{ and } d_1 \geq d_n \geq n - 2. \text{ Denote } \pi' = (d_1 - 1, \ldots, d_i - 1, d_{i+1} - 2, \ldots, d_p - 2, d_{p+1} - 1, \ldots, d_n - 1), \text{ where } s \geq k + 1. \text{ Then } \pi' \text{ is graphic.}
\]

**Proof.** Denote \( \pi' = (d_1', d_2', \ldots, d_{n-2}') \). Clearly \( n - 4 \geq d_1' - 1 \geq d_2' \geq d_2 \geq \cdots \geq d_{n-2}' \). In order to prove that \( \pi' \) is graphic, we only need to verify by Theorem B that

\[
\sigma_i(\pi') \leq \sigma_i(\pi)
\]

(1.4) for each \( t, \ 1 \leq t \leq f(\pi') \).

First, \( f(\pi) = d_k \) because \( d_k = \cdots = d_{d_k} = \cdots = d_p \geq k \). So \( f(\pi') \leq f(\pi) - 1 = d_k - 1 \) by the definition of \( \pi' \).

Second, from (1.3), the off-diagonal leftmost matrix \( \overline{A(\pi)} \) of \( \pi \) has the following form:

\[
\begin{bmatrix}
1 & 2 & \cdots & k - 1 & k & k + 1 & \cdots & d_k - 1 & d_k & d_k + 1 & d_k + 2 & \cdots & n - 3 & n - 2 & n - 1 & n
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & * & \cdots & * & * & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
2 & 1 & 0 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & * & \cdots & * & * & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\]

\[
\begin{bmatrix}
k - 1 & 1 & 1 & \cdots & 0 & 1 & 1 & 0 & 1 & 1 & 1 & * & \vdots & \vdots & \vdots & \vdots & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
k & 1 & 1 & \cdots & 1 & 0 & 1 & \cdots & 1 & 1 & 1 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\]

\[
\begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\]

\[
\begin{bmatrix}
d_k - 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 0 & 1 & 1 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\]

\[
\begin{bmatrix}
d_k & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 0 & 1 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\]

\[
\begin{bmatrix}
d_k + 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\]

\[
\begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\]

\[
\begin{bmatrix}
d_{k+1} & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\]

\[
\begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\]

\[
\begin{bmatrix}
d_{k+2} & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\]

\[
\begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\]

\[
\begin{bmatrix}
p & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\]

\[
\begin{bmatrix}
p + 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\]

\[
\begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\]

\[
\begin{bmatrix}
n - 1 & 1 & 1 & \cdots & 1 & 1 & * & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\]

\[
\begin{bmatrix}
n & 1 & 1 & \cdots & 1 & 1 & * & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\]

While the off-diagonal leftmost matrix \( \overline{A(\pi')} \) of \( \pi' \) is obtained from \( \overline{A(\pi)} \) by deleting the first and \( n \)-th rows and columns and deleting the \( p - s \) ones of the \( d_k \)-th column from the \( p \)-th row to
(s + 1)-th row if s >= d_k + 1 and if k + 1 <= s <= d_k, deleting p - d_k ones of the d_k-th column from the p-th row to (d_k + 1)-th row, and then deleting d_k - s ones of the (d_k + 1)-th column from the d_k-th row to (s + 1)-th row. Hence \( d'_i = p - 2 \geq d_1 \) for 1 <= i <= d_k - 2 and d'_{d_k-1} >= d_k - 2. So \( \sigma_1(\pi', \bar{\pi}) \geq \sigma_1(\pi', \bar{\pi}) \) for 1 <= t <= d_k - 2. Notice that \( d'_i - 1 <= d_{d_k - 1} = d_k - 1 \). Hence \( \sigma_{d_k-1}(\pi') = \sigma_{d_k-2}(\pi') + d_{d_k-1} >= (d_k - 2)d_1 + (d_k - 2) >= \sigma_{d_k-1}(\pi') \). This shows that (1.4) holds.

2 Main results

We need the following lemmas.

**Lemma 2.1.** If \( k >= 5 \), \( \pi = (d_1, d_2, \ldots, d_{2k+2}) \in GS_{2k+2} \), and \( \sigma(\pi) >= 4k^2 - 4k \), then \( d_{k+1} >= k \) and \( d_{2k} >= 2 \).

**Proof.** If \( d_{k+1} <= k - 1 \) then we know from the off-diagonal leftmost matrix \( \bar{A}(\pi) \) of \( \pi \) that \( d_i <= 2k + 1 \) for 1 <= i <= k - 1 and \( d_k <= k - 1 \). Hence by Theorem B,

\[
4k^2 - 4k <= \sigma(\pi) <= \sigma_k(\pi) + (d_{k+1} + \cdots + d_{2k+2}) \\
<= \sigma_k(\pi) + d_{k+1} + \cdots + d_{2k+2} \\
<= \sigma_{d_k-1}(\pi') + d_k + (d_{k+1} + \cdots + d_{2k+2}) \\
<= (k-1)(2k+1) + (k-1) + (k-1)(k+2) \\
= (k-1)(3k+4) = 3k^2 + k - 4,
\]
a contradiction. Thus \( d_{k+1} >= k \).

If \( d_{2k} = d_{2k+1} = d_{2k+2} = 1 \), then from the off-diagonal leftmost matrix \( \bar{A}(\pi) \) of \( \pi \), we know \( d_1 = 2k + 1 \) and \( d_i <= 2k - 2 \) for 2 <= i <= 2k + 2. By Theorem B,

\[
4k^2 - 4k <= \sigma(\pi) = \sigma_{2k-1}(\pi) + d_{2k} + d_{2k+1} + d_{2k+2} \\
<= \sigma_{2k-1}(\pi) + 3 \\
<= 2k + 1 + (2k - 2)(2k - 2) + 3 \\
= 4k^2 - 6k + 8,
\]
i.e. \( k <= 4 \), a contradiction.

**Remark.** In a similar way, we can prove that \( d_{k+1} >= k \) if \( \pi = (d_1, d_2, \ldots, d_n) \in GS_n \) and \( \sigma(\pi) >= (k-1)(2n-k) + 2 \).

**Lemma 2.2.** Let \( \pi \in GS_{2k+2} \) without zero terms and \( \sigma(\pi) >= 4k^2 - 4k \). If \( k = 5 \), or \( \sigma(k - 1, 2k) <= 4k^2 - 12k + 8 \) and \( \sigma(k - 1, 2k + 1) <= 4k^2 - 10k + 2 \) for \( k >= 6 \), then \( \pi \) is potentially \( P_k \)-graphic.

**Proof.** By Lemma 2.1, \( d_1 >= d_{k+1} >= k \). Denote \( \pi' = (d_2 - 1, \ldots, d_{k+1} - 1, d_{d_k+2}, \ldots, d_{2k+2}) \). Then by Theorem C, \( \pi' \) is graphic and \( \sigma(\pi') = \sigma(\pi) - 2d_1 >= 4k^2 - 4k - 2(2k + 1) = 4k^2 - 8k - 2 \). By Lemma 2.1, \( d_{2k} >= 2 \), i.e. \( \pi' \) has at most two ones. So \( \pi' \) has at most two zeros. The sequence obtained by deleting the zero terms of \( \pi' \) is denoted by \( \pi'' \). Then \( \sigma(\pi'') = \sigma(\pi') >= 4k^2 - 8k - 2 \) and \( \pi'' \in GS_{2k+1} \cup GS_{2k} \cup GS_{2k-1} \). If \( k = 5 \), \( \sigma(\pi'') >= 58 \) and \( \pi'' \in GS_9 \cup GS_{10} \cup GS_{11} \). By Theorem 0.1, \( \sigma(4, 9) = 56 \). By Theorem 0.3, \( \sigma(4, 10) = 50 \) and \( \sigma(4, 11) = 56 \); therefore, \( \sigma(\pi'') >= \max(\sigma(4, 9), \sigma(4, 10), \sigma(4, 11)) \). If \( k >= 6 \), by Theorem 0.1, \( \sigma(k - 1, 2k - 1) <= 4k^2 - 10k + 6 \). By the assumption, we have \( \sigma(\pi'') = \max(\sigma(k - 1, 2k - 1), \sigma(k - 1, 2k), \sigma(k - 1, 2k + 1)) \). Therefore \( \pi'' \) is potentially \( A_{k-1} \)-graphic. Hence if \( \pi' \) has zero
terms, then \( \pi \) is potentially \( P_k \)-graphic. Suppose that \( \pi' \) has no zero terms. If \( d_1 = 2k + 1 \) or there exists \( t, k + 1 \leq t \leq d_1 + 1 \) such that \( d_i > d_{i+1} \), then the first \( k \) largest terms of \( \pi' \) are obtained by subtracting one from \( k \) terms of \( \pi \), respectively; therefore \( \pi \) is potentially \( P_k \)-graphic. Hence we may assume

\[
2k \geq d_1 \geq \cdots \geq d_k \geq d_{k+1} = \cdots = d_{d_1+1} = d_{d_1+2} \geq \cdots \geq d_{2k+2}.
\]

If \( d_k > d_{k+1} \), the sequence obtained by laying off \( d_{d_1+1} = 1 \) from \( \pi \) is denoted by \( \pi'' \). By Theorem C, \( \pi'' \in \text{GS} \). Moreover \( \pi'' \) has no zero terms and \( \sigma(\pi'') \geq 4k^2 - 8k - 2 \). By the assumption, \( \sigma(\pi'') \geq \sigma(k - 1, 2k + 1) \). So \( \pi'' \) is potentially \( A_{k-1} \)-graphic, and therefore \( \pi \) is potentially \( P_k \)-graphic. Hence we may assume

\[
2k \geq d_1 \geq \cdots \geq d_k = d_{k+1} = \cdots = d_{d_1+2} \geq \cdots \geq d_{2k+2}.
\]

If \( d_{k+2} \geq k \), then by Theorem 1.1, \( \pi \) is potentially \( P_k \)-graphic. Hence assume \( 1 \leq d_{k+2} \leq k - 1 \). Since \( d_{d_1+2} = d_{k+1} \geq k - 1 \geq d_{k+2} \), we have \( d_1 + 2 \leq 2k + 1 \), i.e., \( d_1 \leq 2k - 1 \). If \( d_1 \leq 2k - 4 \), then

\[
4k^2 - 4k \leq \sigma(\pi) \leq (2k + 1)(2k - 4) + (k - 1) = 4k^2 - 5k - 5,
\]
a contradiction. Therefore \( 2k - 3 \leq d_1 \leq 2k - 1 \). We consider two cases as follows.

**Case 1.** \( d_1 + d_{2k+1} \geq 2k \). Denote

\[
2k - 1 \geq d_1 \geq \cdots \geq d_k = d_{k+1} = \cdots = d_{d_1+1} = d_{d_1+2} = \cdots = d_{2k+2}.
\]

where \( d_1 + 2 \leq p \leq 2k + 1 \) and denote

\[
\pi' = (d_2 - 1, \ldots, d_{s-1}, d_{s+1} - 2, \ldots, d_{p-1} - 2, d_p - 2, d_{p+1} - 1, \ldots, d_{2k+1} - 1),
\]

where \( d_1 + d_{2k+2} = 2k + p - s \). Since \( 2k + p - s = d_1 + d_{2k+2} \leq 2k - 1 + k - 1 = 3k - 2 \), we have \( s \geq p + 2 - k \geq d_1 + 4 - k \geq 2k - 3 + 4 - k = k + 1 \). By Theorem 1.2, \( \pi' \) is graphic. Notice that \( d_p - 2 = d_{s+1} - 2 \geq k > 2 \). In addition, by Lemma 2.1, \( d_{2k+2} > 1 \). Therefore \( \pi' \) has at most one zero. If \( \pi' \) has exactly one zero, then \( d_{2k+1} = d_{2k+2} = 1 \). Since \( d_1 + d_{2k+2} = d_1 + 1 \geq 2k \), we have \( d_1 \geq 2k - 1 \); therefore \( d_1 = 2k - 1 \). Hence \( \pi'' = (d_2 - 1, d_3 - 1, \ldots, d_{2k - 1}, d_{2k+1}, d_{2k+2}) \) \( \in \text{GS} \) has no zero terms and \( \sigma(\pi'') = \sigma(\pi) - 2(2k - 1) \geq 4k^2 - 8k + 2 \). If \( k = 5 \), then by Theorem 0.3, \( \sigma(4, 11) = 56 < 58 = 4 \times 5^2 - 8 \times 5 - 2 \leq \sigma(\pi'') \), so \( \pi'' \) is potentially \( A_4 \)-graphic. Therefore, \( \pi \) is potentially \( P_5 \)-graphic. If \( k \geq 6 \), then by the assumption, \( \sigma(k - 1, 2k + 1) \leq 4k^2 - 10k + 2 < 4k^2 - 8k - 2 \leq \sigma(\pi'') \); therefore \( \pi'' \) is potentially \( A_{k-1} \)-graphic, so \( \pi \) is potentially \( P_k \)-graphic. If \( \pi' \) has no zero terms, then \( \sigma(\pi') = \sigma(\pi) - 2(d_1 + d_{2k+2}) \geq 4k^2 - 4k - 2(3k - 2) = 4k^2 - 10k + 4 \). If \( k = 5 \), by Theorem 0.3, \( \sigma(4, 10) = 50 < 54 = 4k^2 - 10k + 4 \leq \sigma(\pi') \). If \( k \geq 6 \), by the assumption, \( \sigma(k - 1, 2k) \leq 4k^2 - 10k + 4 < 4k^2 - 10k + 4 \leq \sigma(\pi') \). Therefore \( \pi' \) is potentially \( A_{k-1} \)-graphic. Notice that \( \pi' \) is obtained by deleting term operations from \( \pi \) twice: one is to lay off \( d_1 \) from \( \pi \) and the other is to subtract one from the \( (2k + 1 - d_{2k+2}) \)-th term to the term before the last of the residual sequence and then delete the last term. Thus \( \pi \) is potentially \( P_k \)-graphic.

**Case 2.** \( d_1 + d_{2k} \leq 2k - 1 \). In this case, \( 1 \leq d_{2k+2} \leq 2 \) because \( 2k - 3 + d_{2k+2} \leq d_1 + d_{2k+2} \leq 2k - 1 \). If \( d_{2k+2} = 2 \), then \( d_1 = 2k - 3 \). Therefore \( 4k^2 - 4k \leq \sigma(\pi) \leq (2k + 1)(2k - 3) + 2 = 4k^2 - 4k - 1 \), a contradiction. Hence \( d_{2k+2} = 1 \). So \( 2k - 3 \leq d_1 \leq 2k - 2 \). If \( d_1 = 2k - 3 \), then \( 4k^2 - 4k \leq \sigma(\pi) \leq (2k + 1)(2k - 3) + 1 = 4k^2 - 4k - 2 \), a contradiction. Hence \( d_1 \)
Let $k = 2k - 2$ and $d_{d+2} = d_{2k} = d_{k+1}$. If $d_{k+1} \leq 2k - 4$, then $4k^2 - 4k \leq \sigma(\pi) \leq (k-1)(2k-2) + (k+2)(2k-4) + 1 = 4k^2 - 4k - 5$, a contradiction. Hence $2k - 3 \leq d_{k+1} \leq 2k - 2$. If $d_{k+1} = 2k - 2$, then $\pi = ((2k-2)^2, d_{2k+1}, 1^1)$. Because $\pi \in GS_{2k+2}$, $d_{2k+1}$ is odd. Therefore $d_{2k+1} \leq 2k - 3$. Hence the residual sequence obtained by laying off $d_1 = 2k - 2$ from $\pi$ is $\pi' = ((2k-2)^2, (2k-3)^{2k-2}, d_{2k+1}, 1^1)$, and then the residual sequence obtained by laying off $d_{2k+2} = 1$ from $\pi'$ is $\pi'' = ((2k-3)^{2k-1}, d_{2k+1})$. By Theorem C, $\pi'' \in GS_{2k}$ and has no zero terms. Hence we may assume $n > 2k - 3$. Denote $\pi_1 = ((k - 2)^{2l}, (k - 3)^{k-2l}$ and $\pi_2 = ((k - 1)^{2l-2}, (k - 2)^{k-2l+1}, (k - 3)^1, 1^1)$. Clearly $\sigma(\pi_1) = k^2 - 2k - 3 + 2l$ and the conjugate sequence of $\pi_1$ is $\pi_1^* = ((k + 1)^{k-3}, (2l)^1, 0^1)$. It is easy to verify that $\pi_1^*$ majorizes $\pi_2$. Therefore the sequence pair $(\pi_1, \pi_2)$ is bipartite graphic [3, 8]. Obviously $((2k-2)^{2l}, (2k-3)^{k-2l+1}) - \pi_1 = (k^{k+1})$ is the degree sequence of the complete graph $K_{k+1}$ of order $k + 1$. Denote $\pi_3 = ((2k-3)^k, 1^1) - \pi_2 = (k^1, (k - 1)^{k-2l-1}, (k - 2)^{k-2l+1}, 1^1)$. By the off-diagonal leftover matrix $A(\pi_3)$ of $\pi_3$, we obtain $\pi_3 = (k^1, (k - 1)^{k-2l-1}, (k - 2)^{k-2l+1}, 1^1)$, so $\sigma(\pi_3) \leq \sigma(\pi_3)$ for $t = 1, 2, \cdots, k + 1$. By Theorem B, $\pi_3$ is graphic. Thus $\pi$ is potentially $P_k$-graphic.

**Theorem 2.1.** \(\sigma(5, 12) \leq 80\) and \(\sigma(5, n) \leq 8n - 18\) for $n \geq 13$.

**Proof.** By Lemma 2.2, $(5, 12) \leq 4 \times 5^2 - 4 \times 5 = 80$.

We are going to use induction on $n \geq 13$ to prove that if $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ has no zero terms and $\sigma(\pi) \geq 8n - 18$, then $\pi$ is potentially $P_k$-graphic. If $d_n \leq 3$, then by Theorem C, $\pi' = (d_1 - 1, \cdots, d_{n-1} - 1, d_n, 1^1) \in GS_{n-1}$ without zero terms and $\sigma(\pi') = \sigma(\pi) - 2d_n \geq 8n - 18 - 2 \times 3$. Clearly, if $n = 13$, then $\sigma(\pi) \geq 80 \geq \sigma(5, 12)$. If $n > 13$, then $\sigma(\pi') \geq 8(n - 1) - 18 \geq \sigma(5, n - 1)$. Therefore both $\pi'$ and $\pi$ are potentially $P_5$-graphic. If $d_n \geq 4$, then by Theorem C, $\pi' = (d_1 - 1, \cdots, d_{d+1} - 1, d_{d+2}, \cdots, d_n) \in GS_{n-1}$ without zero terms and $\sigma(\pi') \geq \sigma(\pi) - 2d_1 \geq 8n - 18 - 2(n - 1) = 6n - 16 = 6(n - 1) - 10$. By Theorem 0.3, $\sigma(4, n - 1) = 6(n - 1) - 10$ for $n - 1 \geq 13$; therefore $\pi'$ is potentially $A_4$-graphic. If $d_1 = n - 1$, or there exists $t$, $6 \leq t \leq d_1 + 1$ such that $d_t > d_{d+1}$, then $\pi$ is potentially $P_5$-graphic. Hence we may assume $n - 2 \geq d_1 \geq \cdots \geq d_5 \geq d_6 = \cdots = d_{d+2} \geq \cdots \geq d_n \geq 4$. If $d_5 > d_6$, then by Theorem C, the residual sequence $\pi'' = (d_1 - 1, \cdots, d_{d+1}, \cdots, d_{d+2}, \cdots, d_n)$ obtained by laying off $d_{d+1} = 1$ from $\pi$ belongs to $GS_{n-1}$, and has no zero terms and $\sigma(\pi'') = \sigma(\pi) - 2d_5 \geq 6(n - 1) - 10 = \sigma(4, n - 1)$, where $n - 1 \geq 13$. Therefore $\pi''$ is potentially $A_4$-graphic. Hence $\pi$ is potentially $P_5$-graphic. So we may further assume $n - 2 \geq d_1 \geq \cdots \geq d_5 = d_6 = \cdots = d_{d+2} = \cdots = d_n \geq 4$. By the remark of Lemma 2.1, $d_6 \geq 5$. By Theorem 1.1, $\pi$ is potentially $P_5$-graphic.
Theorem 2.2.  For $k \geq 5$

$$
\sigma(k, n) \leq \begin{cases} 
2n(k - 2) + 8, & \text{if } 2k + 2 \leq n \leq \binom{k}{2} + 3 \\
(k - 1)(2m - k) + 2, & \text{if } n \geq \binom{k}{2} + 3.
\end{cases}
$$

(2.1)

Proof.  We use induction on $k \geq 5$. If $k = 5$, then $2k + 2 = 12$, $\binom{k}{2} + 3 = 13$ and

$$
2 \times 12 \times (5 - 2) + 8 = 80.
$$

$2 \times 13 \times (5 - 2) + 8 = 86 = (5 - 1) \times (2 \times 13 - 5) + 2.$

Hence (2.1) holds for $k = 5$ by Theorem 2.1. Suppose that (2.1) holds for $k - 1 \geq 5$, i.e.

$$
\sigma(k - 1, n) \leq \begin{cases} 
2n(k - 3) + 8, & \text{if } 2k \leq n \leq \binom{k - 1}{2} + 3 \\
(k - 2)(2n - k + 1) + 2, & \text{if } n \geq \binom{k - 1}{2} + 3.
\end{cases}
$$

(2.2)

Now we prove that (2.1) holds for $k > 5$ by induction on $n \geq 2k + 2$. If $n = 2k + 2$, then by (2.2), $\sigma(k - 1, 2k) \leq 4k^2 - 12k + 8$ and $\sigma(k - 1, 2k + 1) \leq 4k^2 - 10k + 2$. By Lemma 2.2, if $\pi = (d_1, d_2, \ldots, d_{2k+2}) \in GS_{2k+2}$ has no zero terms and $\sigma(\pi) \geq 4k^2 - 4k = 2(2k+2)(k - 2) + 8$, then $\pi$ is potentially $P_k$-graphic, i.e. $\sigma(k, 2k+2) \leq 2(2k+2)(k - 2) + 8$; therefore (2.1) holds for $n = 2k + 2$.

Suppose (2.1) holds for $n - 1 \geq 2k + 2$, i.e.

$$
\sigma(k, n - 1) \leq \begin{cases} 
2n(k - 1)(k - 2) + 8, & \text{if } 2k + 2 \leq n - 1 \leq \binom{k}{2} + 3 \\
(k - 1)(2n - k - 2) + 2, & \text{if } n - 1 \geq \binom{k}{2} + 3.
\end{cases}
$$

(2.3)

Now we will prove that (2.1) holds for $n \geq 2k + 3$. We only need to prove that if $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ without zero terms and

$$
\sigma(\pi) \geq \begin{cases} 
2n(k - 2) + 8, & \text{if } 2k + 2 < n \leq \binom{k}{2} + 3 \\
(k - 1)(2n - k) + 2, & \text{if } n \geq \binom{k}{2} + 3,
\end{cases}
$$

(2.4)

then $\pi$ is potentially $P_k$-graphic. We consider two cases as follows.

Case 1.  $d_n \leq k - 2$. Denote $\pi' = (d_2, \ldots, d_n - 1, d_{n+1}, \ldots, d_n)$. Then by Theorem C, $\pi' \in GS_{n-1}$, has no zero terms and $\sigma(\pi') = \sigma(\pi) - 2d_n \geq \sigma(\pi) - 2(k - 2)$. If $2k + 2 < n \leq \binom{k}{2} + 3$, then $\sigma(\pi') \geq \sigma(\pi) - 2(k - 2) \geq 2(n - 1)(k - 2) + 8$. By (2.3), we obtain $\sigma(\pi') \geq \sigma(k, n - 1)$. If $n \geq \binom{k}{2} + 3$, then

$$
\sigma(\pi') \geq \sigma(\pi) - 2(k - 2) \geq (k - 1)(2n - k) + 2 - 2(k - 2)
$$

$$
= (k - 1)(2(n - 1) - k) + 2 + 2.
$$

By (2.3), $\sigma(\pi') \geq \sigma(k, n - 1)$; therefore $\pi'$ is potentially $P_k$-graphic. Hence $\pi$ is potentially $P_k$-graphic.

Case 2.  $d_n \geq k - 1$. Denote $\pi' = (d_2, \ldots, d_{n+1} - 1, d_{n+2}, \ldots, d_n)$. By Theorem C, $\pi' \in GS_{n-1}$, has no zero terms and $\sigma(\pi') = \sigma(\pi) - 2d_1$. If $2k + 2 < n \leq \binom{k}{2} + 3$, then by as-
The following is our main result.

**Theorem 2.3.** If \( k \geq 5 \), then \( \sigma(k, n) = (2n - k)(k - 1) + 2 \) for \( n \geq (k \choose 2) + 3 \).

**Proof.** This is an immediate consequence of Theorem 2.2 and the lower bound of \( \sigma(k, n) \).

**Corollary 2.1.** \( \sigma(k, 2k + 2) = 4k^2 - 4k \) for \( k \geq 5 \).

**Proof.** By Theorem 2.2, \( \sigma(k, 2k + 2) \leq 4k^2 - 4k \) for \( k \geq 5 \). Now consider the sequence \( \pi = ((2k - 3)^{2k+1}, 1^1) \). Clearly, \( \sigma(\pi) \) is even and \( f(\pi) = 2k - 3 \). It is easy to obtain \( \overline{\pi} = ((2k + 1)^1 (2k)^{2k-4}, (2k - 3)^1, 0^0) \) from its off-diagonal leftmost matrix \( \overline{A}(\pi) \). Hence \( \sigma_i(\pi) \leq \sigma_i(\overline{\pi}) \) for \( 1 \leq i \leq 2k - 3 \). By Theorem B, \( \pi \not\in GS_{2k+2} \).

Assume that \( G \) is one of the realizations of \( \pi \) and the subgraph induced by the vertex subset \( U = \{ u_1, \ldots, u_{k+1} \} \) of \( G \) is complete. Then the edge number of \( G \) from \( U \) to \( V(G) \setminus U \) is \( r = (k + 1)(k - 3) \). Therefore the edge number of the induced subgraph \( G[V(G) \setminus U] \) of \( G \) is...
\[\sigma(\pi)/2 - r - k(k + 1)/2 = \left(\frac{k}{2}\right) + 2.\] Because \(d_{2k+2} = 1\), the edge number of \(G[V(G) \setminus U]\) is at most \(\left(\frac{k}{2}\right) + 1\), a contradiction. Hence \(\pi\) is not potentially \(P_k\)-graphic; therefore \(\sigma(k,2k+2) \geq \sigma(\pi) + 2 = 4k^2 - 4k\). Thus \(\sigma(k,2k+2) = 4k^2 - 4k\).

Remark. By Theorem 0.3, \(\sigma(4,10) = 6 \times 10 - 10 = 50 > 48 = 4 \times 4^2 - 4 \times 4\), so Corollary 2.1 does not hold for \(k \leq 4\).

3 Conclusion

For any given integer number \(k \geq 2\), denote by \(N(k)\) the smallest positive integer number \(m\) such that for any integer number \(n \geq m\), \(\sigma(k,n) = (k-1)(2n-k) + 2\). It follows from Theorems A, 0.1, 0.2 and 0.3 that \(N(2) = 6\), \(N(3) = 8\) and \(N(4) = 10\). By Theorem 2.3 and Corollary 2.1, \(N(k)\) exists for \(k \geq 5\) and \(2k + 3 \leq N(k) \leq \left(\frac{k}{2}\right) + 3\). In particular if \(k = 5\), \(13 = 2 \times 5 + 3 \leq N(5) \leq \left(\frac{5}{2}\right) + 3 = 13\). So \(N(5) = 13\). The problem of how to determine the exact value of \(N(k)\) for \(k \geq 6\) needs further study.

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References