Bistable Fronts in Discrete Inhomogeneous Media

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Abstract

Bistable differential-difference equations with inhomogeneous diffusion are considered using McKean's caricature of the cubic. Front solutions are constructed for essentially arbitrary inhomogeneous discrete diffusion and these solutions correspond, in the case of homogeneous diffusion, to monotone traveling front solutions, or stationary front solutions in the case of propagation failure. Explicit conditions reveal relationships between zero wave speed and defects in the medium, and changes in wave speed and shape are analyzed as fronts propagate.

This poster summarizes results published in the article: A.R. Humphries, B.E. Moore, & E.S. Van Vleck, Front Solutions for Bistable Differential-Difference Equations with Inhomogeneous Diffusion, *SIAM Journal of Applied Mathematics*, 71(4):1374-1400, 2011.

We seek solutions of $\dot{u}_j(t) = Lu_j(t) - f(u_j(t))$.

$u_j(t)$ maps $\mathbb{R}^+ \cup \{0\} \to \mathbb{R}$, $j \in \mathbb{Z}$ indicates an element of a one-dimensional lattice, $\dot{u} = \frac{du}{dt}$,

$$Lu_j(t) = \alpha_j[u_{j+1}(t) - u_j(t)] + \alpha_{j-1}[u_{j-1}(t) - u_j(t)],$$

and using $\alpha_j \in \mathbb{R}^+$ and $m, n \in \{0\} \cup \mathbb{Z}^+$, the inhomogeneous medium is defined by

$$\alpha_j = \alpha \quad \text{for} \quad j < -m \text{ or } j > n \qquad \text{and} \qquad \alpha_j \neq \alpha \quad \text{for} \quad -m \le j \le n.$$

$f : \mathbb{R} \to \mathbb{R}$ is the derivative of a double-well potential. We use a piecewise linear f known as McKean's caricature of the cubic

$$f(u) = u - h(u - a)$$

where h is the Heaviside function

$$h(u) = \begin{cases} 0, & u < 0 \\ 1, & u > 0 \end{cases}$$

with $h(u) = [0, 1]$ for $u = 0$.

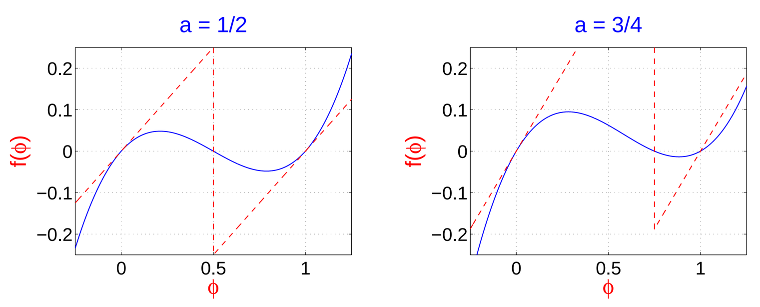

FIGURE 1: Blue: Cubic $f(u) = u(u-1)(u-a)$, Red: McKean $f(u) = u - h(u-a)$.

Problem Set-Up for Propagating Fronts

Make the traveling wave ansatz

$$u_j(t) = \varphi(\xi_j; \xi^*), \qquad \xi_j = j - c_j(t). \tag{1}$$

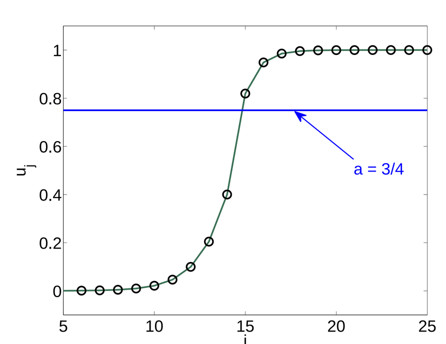

FIGURE 2: Assume the solution crosses the value of a only once $\forall j$.

We choose a particular wave form by setting

$$a = \varphi(\xi^*; \xi^*), \tag{2}$$

so that ξ^* is the spatial location at which φ takes the value a, and we seek solutions that satisfy

$$\begin{array}{lll} \varphi(x;\xi^*) < a & \text{for} & x < \xi^*, \\ \varphi(x;\xi^*) > a & \text{for} & x > \xi^*. \end{array} \tag{3}$$

Thus, the nonlinearity may be written as a linear inhomogeneous term

$$f(\varphi(x;\xi^*)) = \varphi(x;\xi^*) - h(x - \xi^*), \tag{4}$$

which is independent of a, and it is natural to impose the boundary conditions

$$\lim_{x \to -\infty} \varphi(x;\xi^*) = 0, \qquad \lim_{x \to \infty} \varphi(x;\xi^*) = 1. \tag{5}$$

Solutions with Zero Wave Speed

Steady-state solutions satisfy the linear difference equation

$$\alpha_k(\phi_{k+1} - \phi_k) + \alpha_{k-1}(\phi_{k-1} - \phi_k) - \phi_k = -h_k \qquad \forall k \in \mathbb{Z}. \tag{6}$$

where $h_k := h(k - \xi^*)$. Using Jacobi operator theory, the general solution is

$$\phi_k = \phi_{k^*}\rho(k,k^*) + \phi_{k^*+1}\sigma(k,k^*) + \begin{cases} -\sum_{j=k^*+1}^{k} \frac{1}{\alpha_j}\sigma(k,j) & k > k^* \\ 0 & k = k^* \\ \frac{h_{k^*}}{\alpha_{k^*}}\sigma(k,k^*) & k < k^* \end{cases} \tag{7}$$

for all $\lfloor \xi^* \rfloor = k^* \in \mathbb{Z}$, where ρ and σ denote the fundamental solutions, and satisfy

$$\rho(k^*,k^*) = 1, \quad \rho(k^*+1,k^*) = 0, \quad \sigma(k^*,k^*) = 0, \quad \sigma(k^*+1,k^*) = 1.$$

ϕ_{k^*} and ϕ_{k^*+1} are determined using the boundary conditions (5).

Varying ξ^* allows consideration of fronts at different positions in the media. By (2)-(3)

- $\xi^* \in \mathbb{Z} \iff k^* = \xi^*$ implies $a = \phi_{k^*}$ with $h_{k^*} = [0, 1]$.
- $\xi^* \notin \mathbb{Z} \iff k^* < \xi^* < k^* + 1$ implies $a \in (\phi_{k^*}, \phi_{k^*+1})$ with $h_{k^*} = 0$.

This provides necessary and sufficient conditions on front position ξ^*, detuning paramter a, and diffusion coefficients α_j that guarantee stationary fronts.

Theorem for one defect: If $a \in (0,1)$ yields a traveling front when $\alpha_0 = \alpha$, then there are no corresponding stationary fronts for $\alpha_0 < \alpha$ and $\xi^* \notin (0,1)$, nor for $\alpha_0 > \alpha$ and $\xi^* \in (0,1)$. Still, there exist $a \in (0,1)$ which yield traveling fronts for $\alpha_0 = \alpha$, but yield stationary fronts for $\alpha_0 < \alpha$ and $\xi^* \in (0,1)$ or for $\alpha_0 > \alpha$ and $\xi^* \notin (0,1)$. *(See Fig. 3.)*

Solutions with Non-Zero Wave Speeds

Using a Fourier transform to solve

$$-d_j\varphi'(\xi_j) = \alpha_j(\varphi(\xi_{j+1}) - \varphi(\xi_j)) + \alpha_{j-1}(\varphi(\xi_{j-1}) - \varphi(\xi_j)) - \varphi(\xi_j) + h(\xi_j - \xi^*), \tag{8}$$

with $d_j(t) = \dot{c}_j(t)$, we obtain the solution

$$\varphi(\xi;\xi^*) = \psi(\xi;\xi^*) + \chi(\xi;\xi^*). \tag{9}$$

- $\psi(\xi;\xi^*)$ is the solution in the case $\alpha_j = \alpha\ \forall j$, given by

$$\psi(\xi;\xi^*) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{A(s)\sin(s(\xi-\xi^*))}{s\left(A(s)^2 + c^2s^2\right)}ds + \frac{c}{\pi}\int_0^\infty \frac{\cos(s(\xi-\xi^*))}{A(s)^2 + c^2s^2}ds \tag{10}$$

where $A(s) = 1 + 2\alpha(1 - \cos(s))$, and c is the wave speed in homogeneous media.

- $\chi(\xi;\xi^*)$ is the perturbation from ψ when $\alpha_j \neq \alpha$. Defining $\beta_j = \frac{c}{d_j} - 1$ and $\gamma_j = \alpha_j - \alpha$,

$$\chi(\xi;\xi^*) = \sum_{j\in\mathcal{R}} \beta_j F_j(\xi)B_j(\xi^*) + \alpha\sum_{j\in\mathcal{S}} F_j(\xi)C_j(\xi^*) + \sum_{j\in\mathcal{T}} \gamma_j\left(F_j(\xi) - F_{j+1}(\xi)\right)D_j(\xi^*) \tag{11}$$

where $\mathcal{R} = \{j \in \mathbb{Z} : c_j(t) \neq ct\}$, $\quad \mathcal{S} = \{j \in \mathbb{Z} : j \in \mathcal{R}, \text{ or } j \pm 1 \in \mathcal{R}\}$, $\quad \mathcal{T} = \{j \in \mathbb{Z} : \alpha_j \neq \alpha\}$,

$$B_j(\xi^*) = \alpha_j\varphi(\xi_{j+1};\xi^*) - (1 + \alpha_j + \alpha_{j-1})\varphi(\xi_j;\xi^*) + \alpha_{j-1}\varphi(\xi_{j-1};\xi^*) + h(\xi_j - \xi^*), \qquad D_j(\xi^*) = \varphi(\xi_{j+1};\xi^*) - \varphi(\xi_j;\xi^*),$$

$$C_j(\xi^*) = \varphi(\xi_{j+1};\xi^*) - \varphi(\xi_j + 1;\xi^*) + \varphi(\xi_{j-1};\xi^*) - \varphi(\xi_j - 1;\xi^*), \qquad F_j(\xi) = \frac{1}{\pi}\int_0^\infty \frac{A(s)\cos(s(\xi-\xi_j)) - cs\sin(s(\xi-\xi_j))}{A(s)^2 + c^2s^2}ds.$$

- The position of the front in the media is changed by changing ξ^*, similar to the way ct translates a front defined by $\varphi(j - ct)$. Each ξ^* directly corresponds to $c_j(t)$ and $d_j(t)$, which are estimated based on the position of the front relative to the defect. *(See Fig. 4.)*

Interval of Propagation Failure

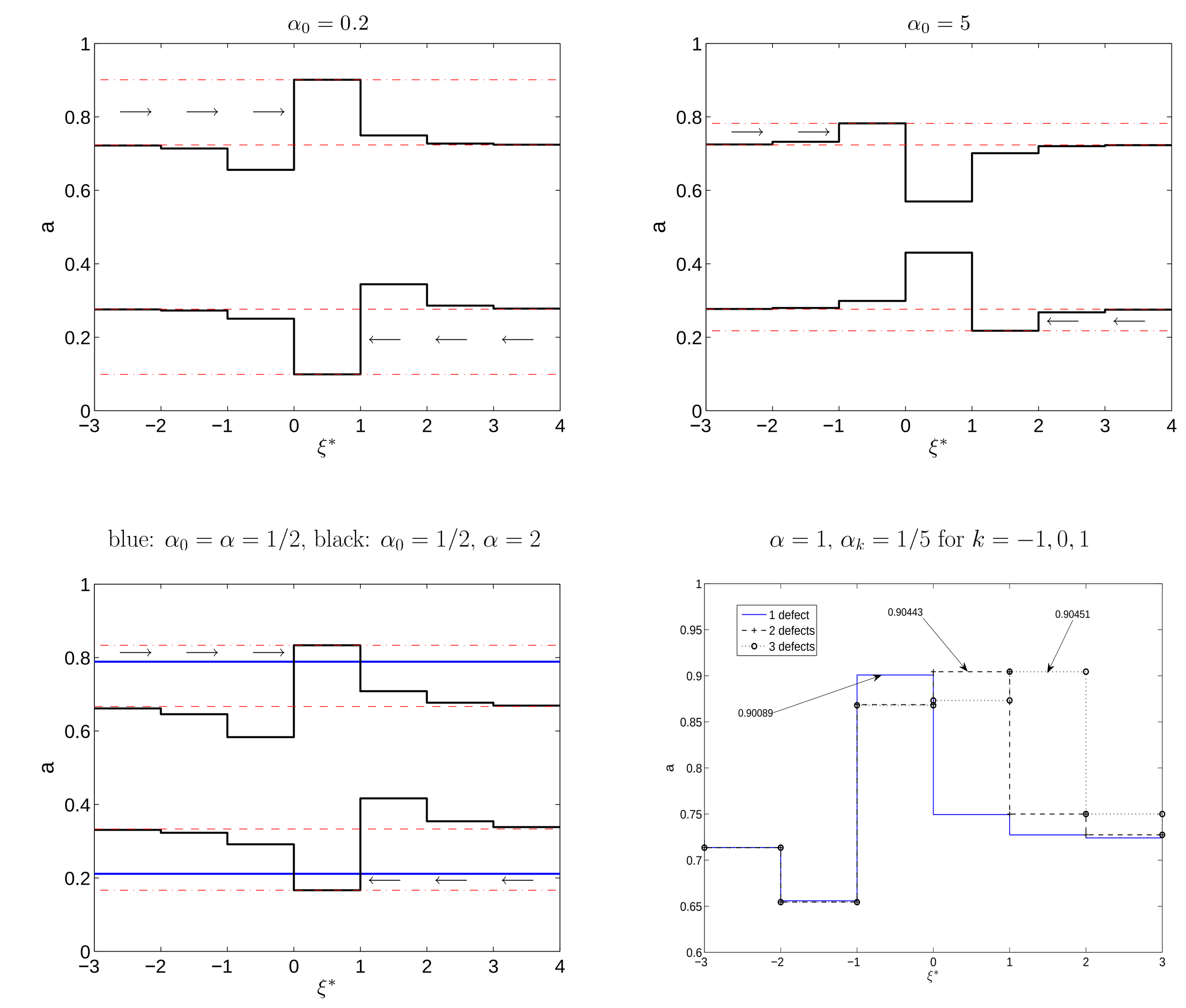

FIGURE 3: Fronts fail to propagate for values of a inside the black lines, depending on the position of the front relative to the defect. (Top plots: $\alpha = 1$.)

Changes in Wave Speeds and Wave Forms

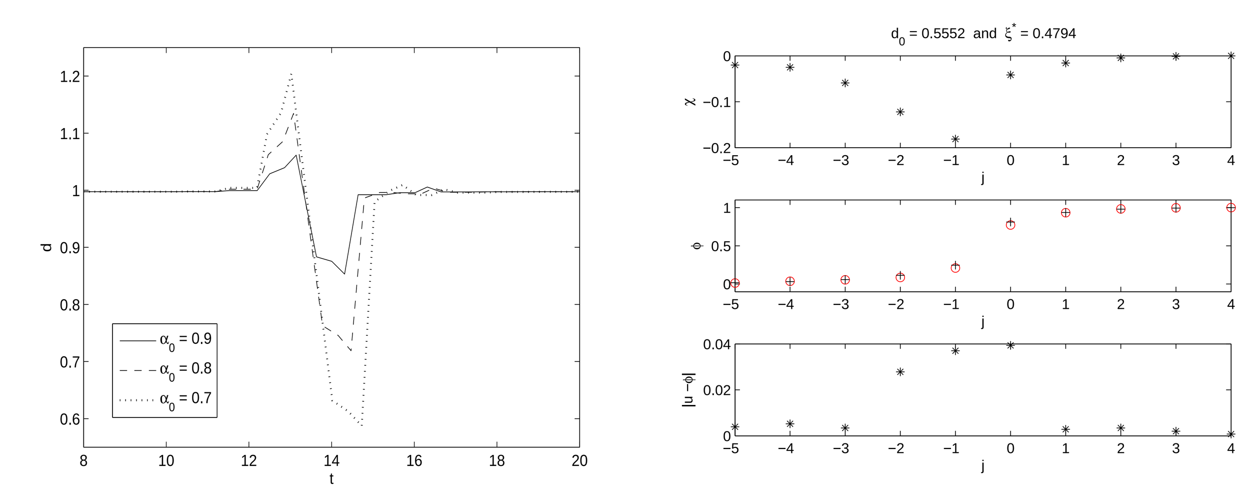

FIGURE 4: Left: Approximated variable wave speed as fonts propagate through a defect. Right: Filon quadrature approx. of (11) and (9) compared to RK approx. of $u_j(t)$.

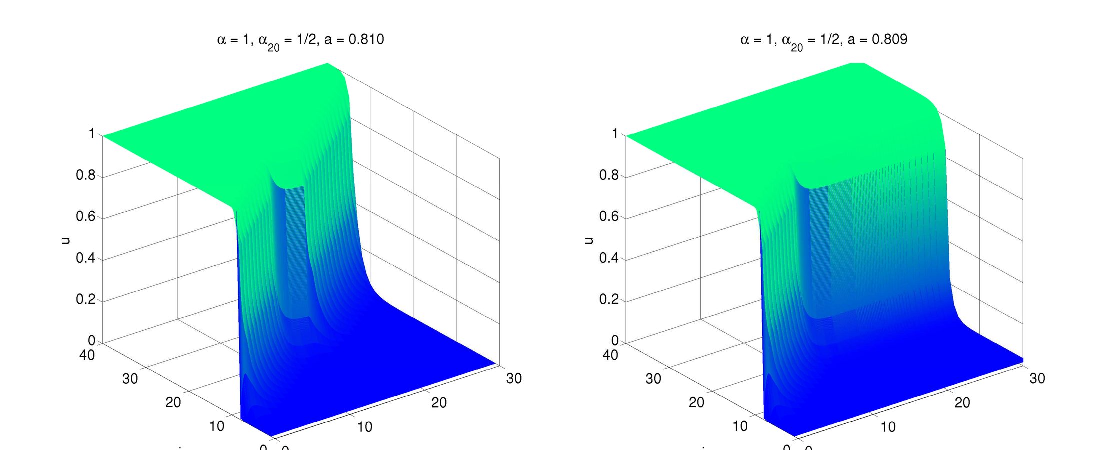

FIGURE 5: Left: Propagation through the defect. Right: Propagation failure at defect.