Bistable Waves in Discrete Inhomogeneous Media

Brian E. Moore

bmoore@math.ucf.edu

Mathematics Seminar at Florida Tech

March 29, 2013

Bistable Waves in Discrete Inhomogeneous Media

Collaborators

- Tony Humphries, McGill University
- Erik Van Vleck, University of Kansas

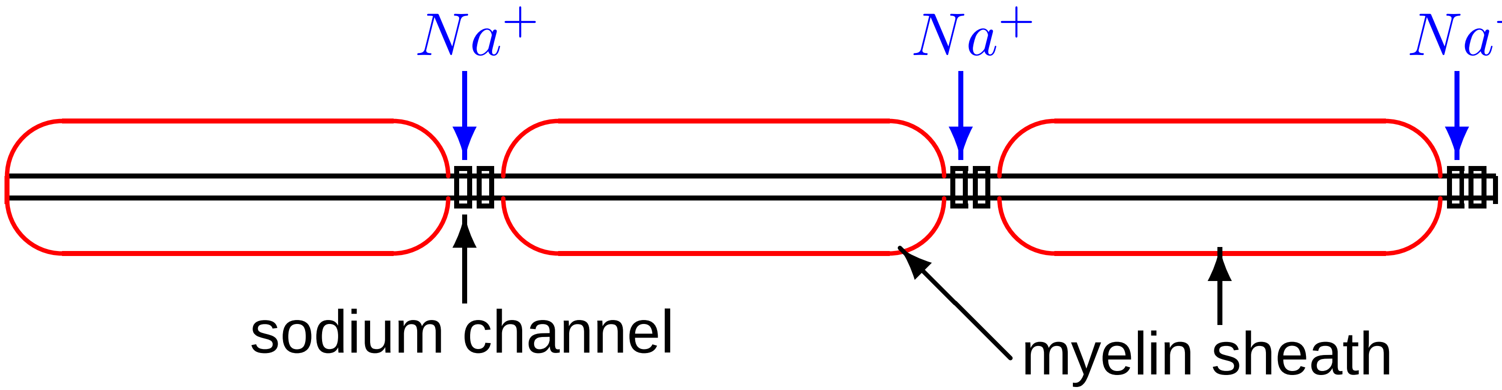
Students

- Lory Ajamian, McGill University
- Jessica Long, University of Iowa
- Joe Segal, University of Central Florida
- Nicole Lopez, University of Central Florida
- Johann Veras, University of Central Florida

Funding

- NSF, NSERC, CRM Applied Math Lab

Our Nervous System

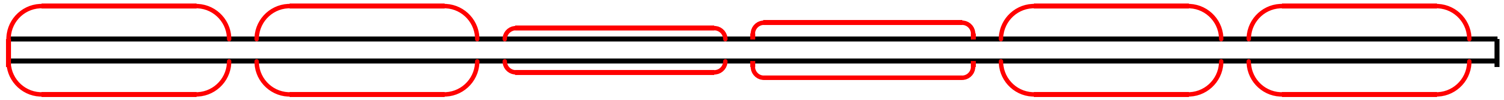


The nervous cells live inside the “Hot Dog Buns” which are called myelin sheath.

The inrush of sodium (Na^+) at the sodium channels causes the electric impulse to jump to the next cell.

Multiple Sclerosis causes the destruction of myelin, inhibiting conduction of electrical signals.

Electrical Impulse with Diseased Cells



Questions

- Where does the electrical impulse stop?
- How much destruction is required to make it stop?
- What happens to wave speed and shape as it passes through deteriorated region?

History

1952 Hodgkin and Huxley model pulse propagation in the nerve axon of a giant squid.

1961 FitzHugh proposed a simplified model

$$u_t = \alpha u_{xx} - v - f(u), \quad v_t = bu - rv$$

1963 Hodgkin and Huxley win the Nobel Prize for their work.

1964 Nagumo et al. proposed a more simplified model

$$u_t = \alpha u_{xx} - f(u)$$

1970 McKean proposed simplifications to the nonlinearity

$$f(u) = u - h(u - a)$$

1980's Various researchers use discrete models instead

$$u_{xx} \to u_{j+1} - 2u_j + u_{j-1}$$

$$(\alpha u_x)_x \to \alpha_j(u_{j+1} - u_j) - \alpha_{j-1}(u_j - u_{j-1})$$

Bistable Equation with Inhomogeneous Diffusion

$$\dot{u}_j = \alpha_j(u_{j+1} - u_j) - \alpha_{j-1}(u_j - u_{j-1}) - f(u_j)$$

with

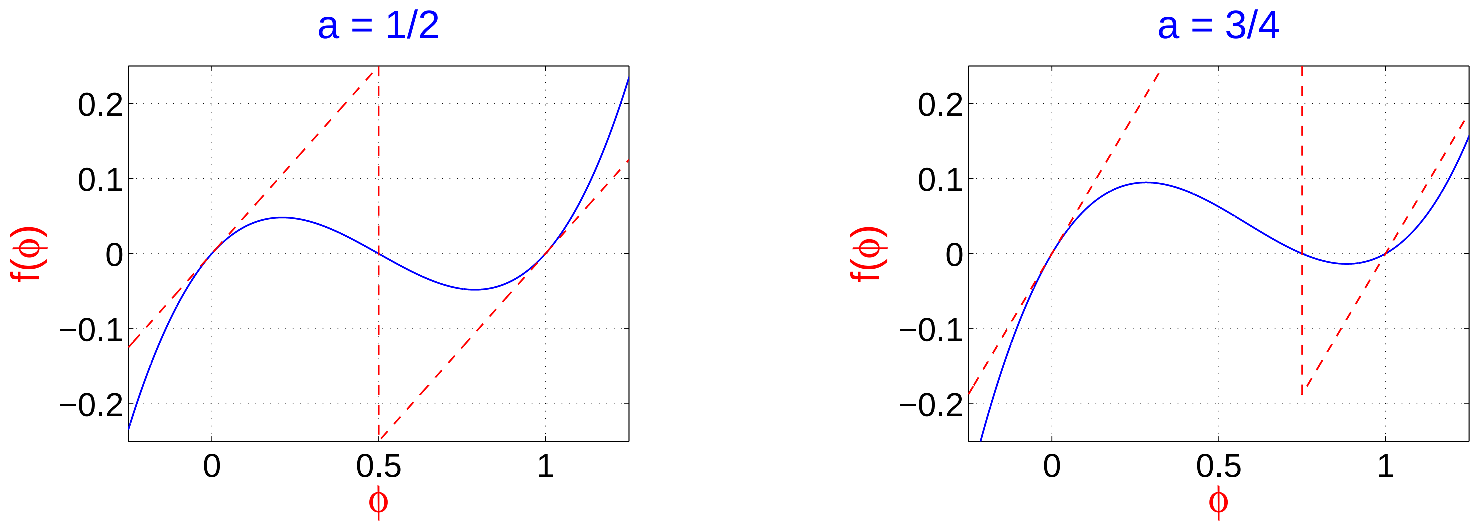
$$\alpha_j = \begin{cases} \alpha_j & -m \leq j \leq n \\ \alpha & j < -m \text{ or } j > n \end{cases}$$

$m, n \in \{0\} \cup \mathbb{N}$

The nonlinearity is the derivative of a double-well potential,

typically $f(u) = u(u-a)(u-1)$ with $a \in (0,1)$.

McKean's Caricature of the Cubic



Dashed red line:

$$f(u) = u - h(u-a) \qquad h(x) = \begin{cases} 1 & x > 0 \\ [0,1] & x = 0 \\ 0 & x < 0 \end{cases}$$

Recent History

1999 Cahn, Mallet-Paret, and Van Vleck derive traveling wave solutions on 2-D lattice with $\alpha_j = \alpha$ using McKean's $f(u)$, and discuss relationship between a and the wave speed.

2000 Lewis and Keener study steady states of the PDE model. Changing parameters m, n, and α_j leads to steady state solutions through a limit point bifurcation.

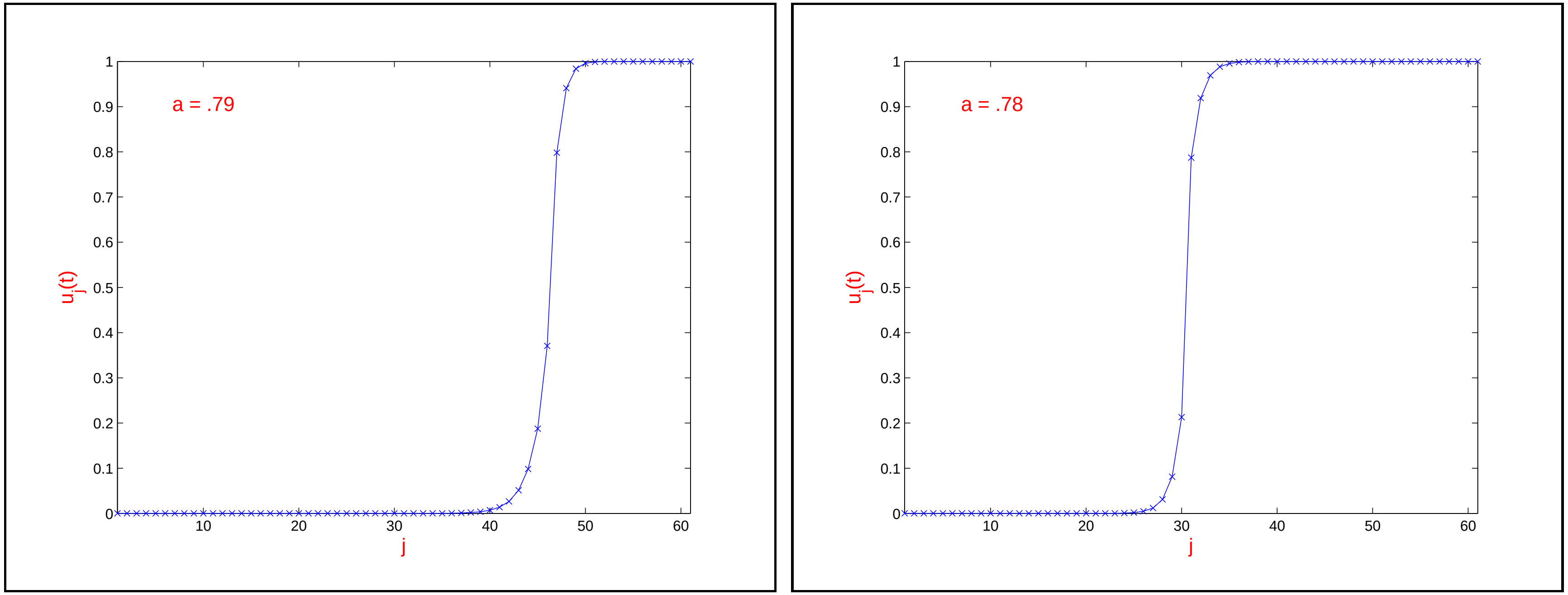
2001 Elmer and Van Vleck consider periodic diffusion and derive solutions using McKean's $f(u)$, and discuss changes in the wave speed as the solution evolves.

2005 Elmer and Van Vleck derive traveling wave solutions for spatially discrete FitzHugh-Nagumo equation with McKean's $f(u)$ and $\alpha_j = \alpha$.

Numerical Simulations for the Evolution Equation

For the case of a single defect

$$\alpha_j = \begin{cases} 0.6 & j = 30 \\ 1 & j \neq 30 \end{cases}$$



A slightly slower wave is stopped by the defect.

Traveling Wave Ansatz

Define $\mathcal{R} = \{j \in \mathbb{Z} : c_j(t) \neq ct\}$ and

$$u_j(t) = \varphi(\xi_j; \xi^*) \qquad \text{for} \qquad \xi_j = \begin{cases} j - c_j(t) & j \in \mathcal{R} \\ j - ct & j \notin \mathcal{R} \end{cases}$$

$\xi^* \in \mathbb{R}$ is a parameter that determines the position of the wave relative to the defect region. We seek solutions with

$$\varphi(-\infty; \xi^*) = 0, \qquad \varphi(\infty; \xi^*) = 1, \qquad \varphi(\xi^*; \xi^*) = a$$

$$\varphi(\xi; \xi^*) < a \quad \text{for} \quad \xi < \xi^* \qquad \varphi(\xi; \xi^*) > a \quad \text{for} \quad \xi > \xi^*,$$

$$\Longrightarrow \qquad h(\varphi(\xi_j; \xi^*) - a) = h(\xi_j - \xi^*)$$

$$\Longrightarrow \qquad f(\varphi(\xi_j; \xi^*)) = \varphi(\xi_j; \xi^*) - h(\xi_j - \xi^*)$$

Solutions with Non-Zero Wave Speed

Solving

$$-d_j\varphi'(\xi_j) = \alpha_j\varphi(\xi_{j+1}) - (\alpha_j + \alpha_{j-1} + 1)\varphi(\xi_j) + \alpha_{j-1}\varphi(\xi_{j-1}) + h(\xi_j - \xi^*),$$

with $d_j(t) = \dot{c}_j(t)$ by Fourier transform yields

$$\phi(\xi;\xi^*) = \psi(\xi;\xi^*) + \chi(\xi;\xi^*),$$

where $\psi(\xi;\xi^*)$ is the solution for $\alpha_j = \alpha\ \forall j$, with wave speed c

$$\psi(\xi;\xi^*) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{A(s)\sin(s(\xi-\xi^*))}{s\,(A(s)^2 + c^2s^2)}ds + \frac{c}{\pi}\int_0^\infty \frac{\cos(s(\xi-\xi^*))}{A(s)^2 + c^2s^2}ds$$

for $A(s) = 1 + 2\alpha(1-\cos(s))$

(Cahn, Mallet-Paret, Van Vleck 1999)

Solutions with Non-Zero Wave Speed

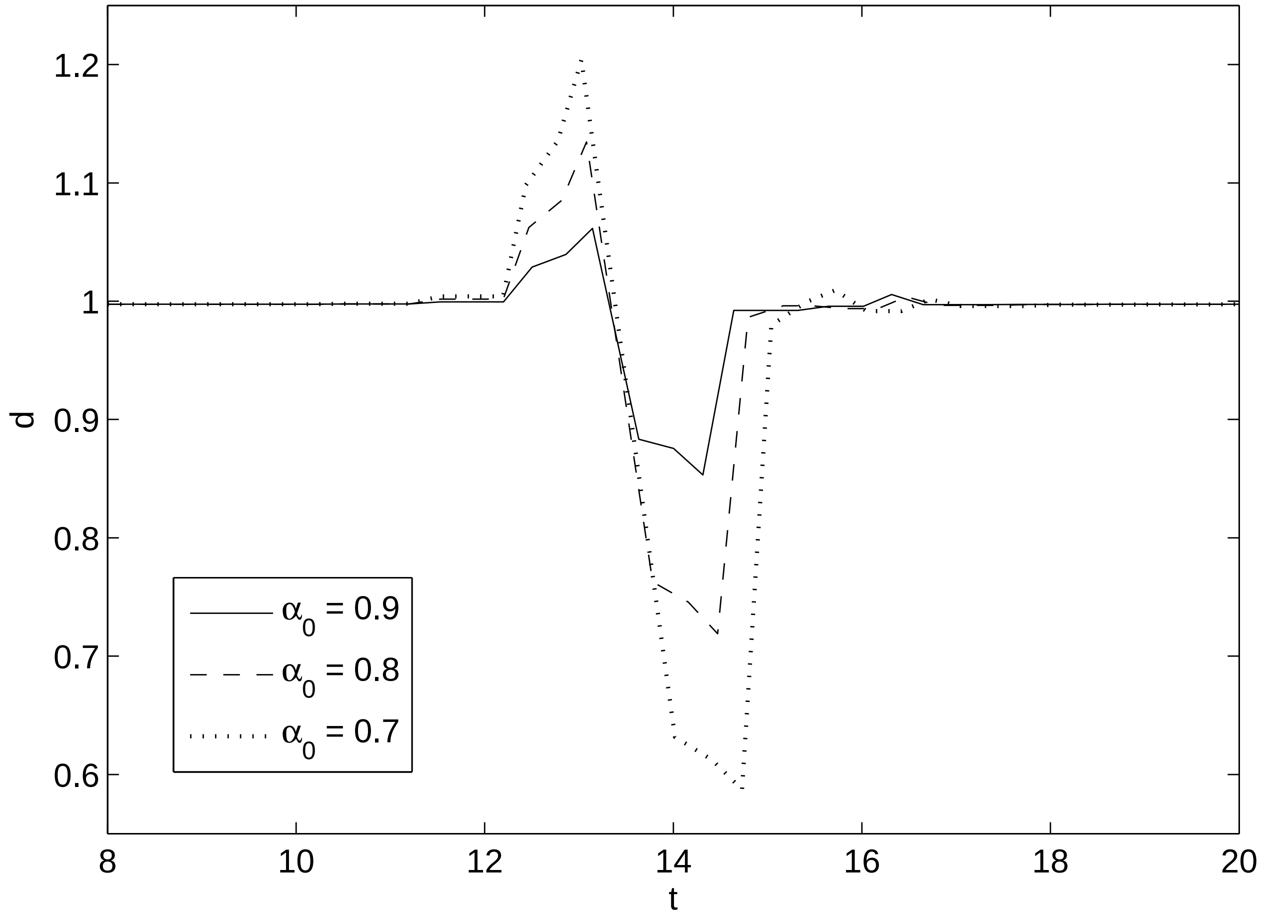
$\chi(\xi;\xi^*)$ is the perturbation from ψ when $\alpha_j \neq \alpha$.
Defining $\beta_j = \frac{c}{d_j} - 1$ and $\gamma_j = \alpha_j - \alpha$,

$$\begin{aligned}\chi(\xi;\xi^*) &= \sum_{j\in\mathcal{R}} \beta_j F_j(\xi) B_j(\xi^*) + \alpha \sum_{j\in\mathcal{S}} F_j(\xi) C_j(\xi^*) \\ &\quad + \sum_{j\in\mathcal{T}} \gamma_j \left(F_j(\xi) - F_{j+1}(\xi)\right) D_j(\xi^*)\end{aligned}$$

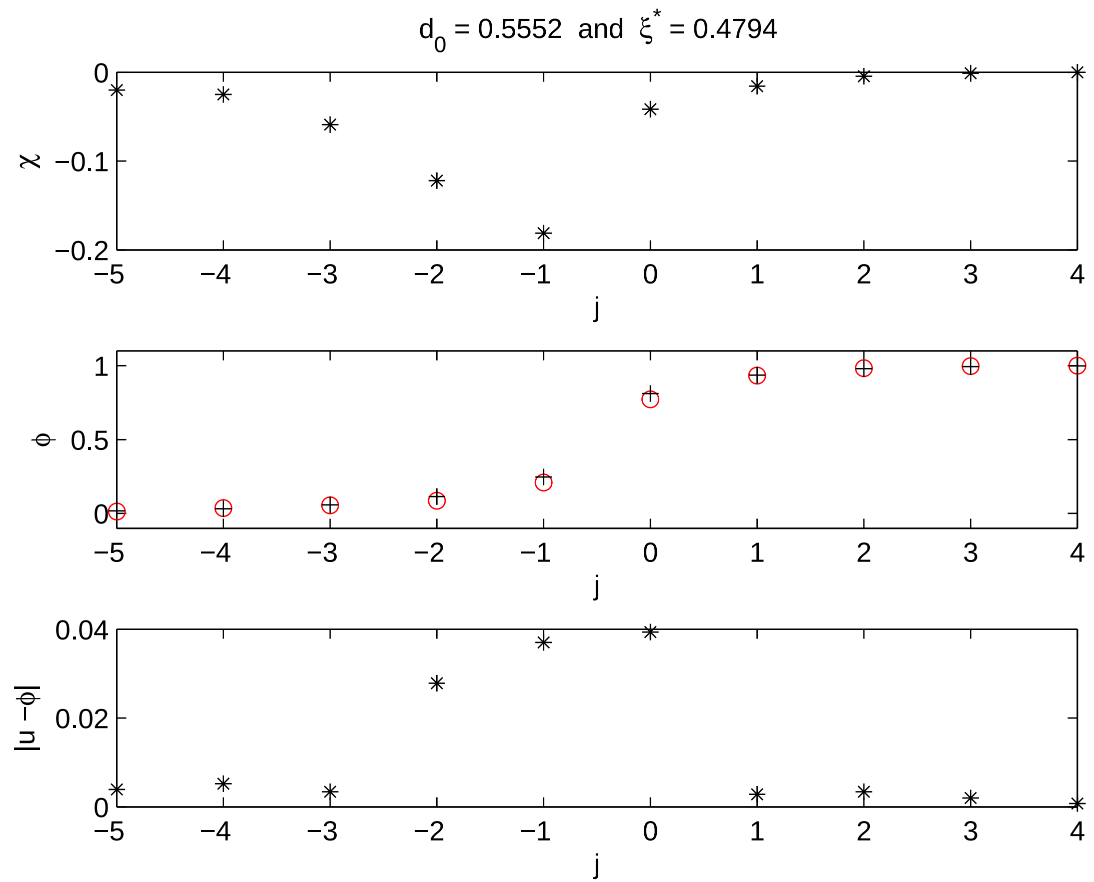
$\mathcal{R} = \{j \in \mathbb{Z} : c_j(t) \neq ct\}$, $\mathcal{S} = \{j \in \mathbb{Z} : j \in \mathcal{R}$, or $j \pm 1 \in \mathcal{R}\}$,
$\mathcal{T} = \{j \in \mathbb{Z} : \alpha_j \neq \alpha\}$,

$$\begin{aligned}B_j(\xi^*) &= \alpha_j \varphi(\xi_{j+1};\xi^*) - (1 + \alpha_j + \alpha_{j-1})\,\varphi(\xi_j;\xi^*) + \alpha_{j-1}\varphi(\xi_{j-1};\xi^*) + h(\xi_j - \xi^*), \\ C_j(\xi^*) &= \varphi(\xi_{j+1};\xi^*) - \varphi(\xi_j + 1;\xi^*) + \varphi(\xi_{j-1};\xi^*) - \varphi(\xi_j - 1;\xi^*), \\ D_j(\xi^*) &= \varphi(\xi_{j+1};\xi^*) - \varphi(\xi_j;\xi^*), \quad F_j(\xi) = \frac{1}{\pi}\int_0^\infty \frac{A(s)\cos(s(\xi-\xi_j)) - cs\sin(s(\xi-\xi_j))}{A(s)^2 + c^2 s^2}\,ds.\end{aligned}$$

Change in Speed as Front Passes Through a Defect



Change in Form as Front Passes Through a Defect



Filon quadrature approx. compared to RK approx. of $u_j(t)$.

Steady State Solutions

Definition: The range of a values that yield standing waves is called the *interval of propagation failure*.

Standing waves are solutions of $\dot{u}_j = 0$ or equivalently

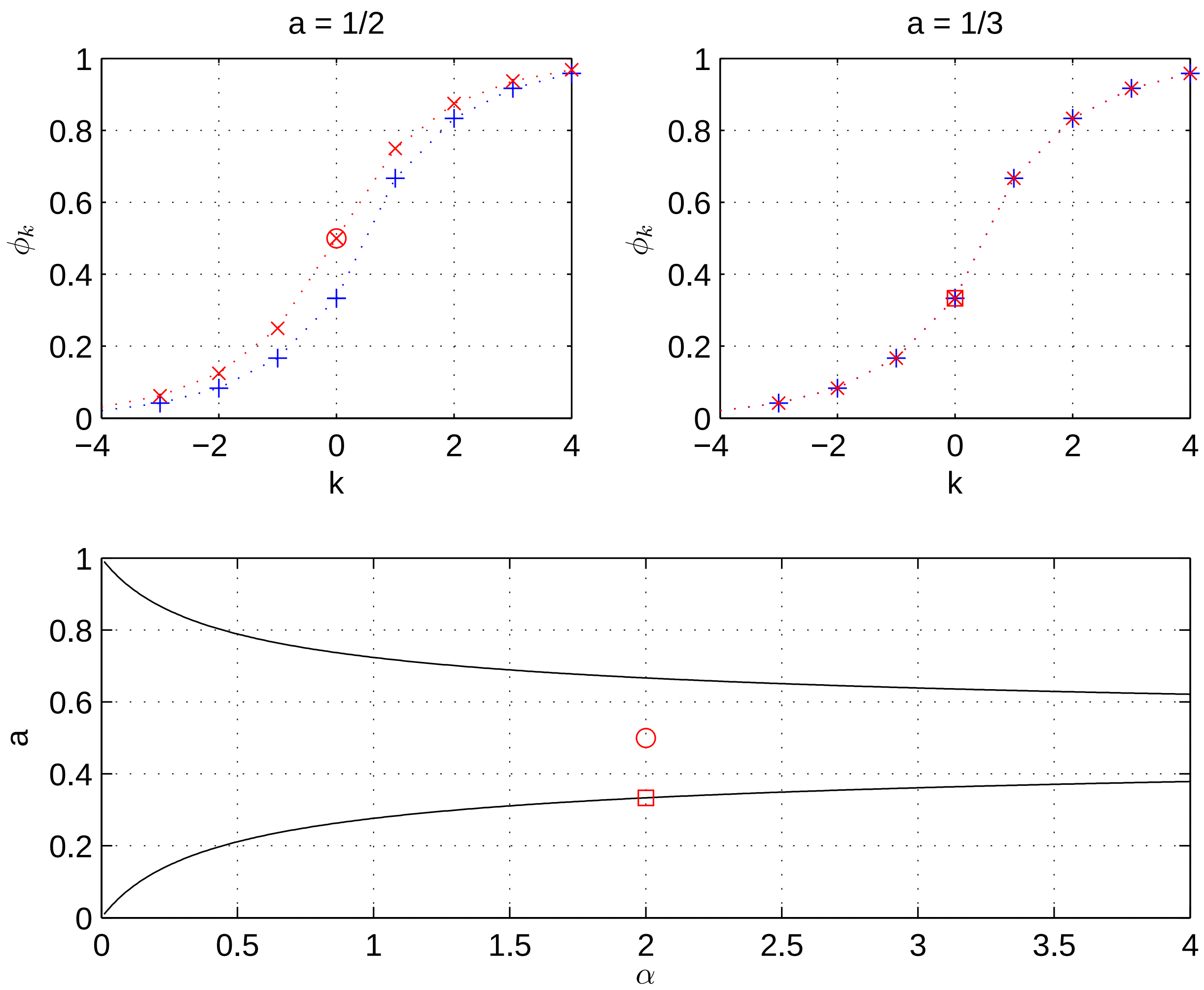
$$\alpha_j(u_{j+1} - u_j) - \alpha_{j-1}(u_j - u_{j-1}) = f(u_j)$$

$$\lim_{j\to\infty} u_j = 1 \qquad \lim_{j\to-\infty} u_j = 0$$

where $f(u_j) = u_j - h(u_j - a) = u_j - h(j - \xi^*)$.

Note: Solutions are not translationally invariant.

Standing Waves for $\alpha_j = \alpha, \forall j$



Derivation of Solutions

To solve the difference equation

$$\alpha_j(u_{j+1} - u_j) - \alpha_{j-1}(u_j - u_{j-1}) - u_j = -h_j,$$

with $h_j = h(j - \xi^*)$, use method of undetermined coefficients.
General Solution = Homogeneous Solution + Particular Solution

$$u_j = u_{j^*}\rho_j + u_{j^*+1}\sigma_j + \begin{cases} -\sum_{k=j^*+1}^{j} \frac{h_k}{\alpha_k}\sigma_{j-k} & j > j^* \\ 0 & j = j^* \\ \frac{h_{j^*}}{\alpha_{j^*}}\sigma_{j-j^*} & j < j^* \end{cases}$$

Fundamental solutions satisfy
$(\rho_{j^*}, \rho_{j^*+1}) = (1, 0)$, and $(\sigma_{j^*}, \sigma_{j^*+1}) = (0, 1)$.
The particular solution can be found in Teschl (2000).
The coefficients are determined by the boundary conditions.

Standing Waves for $\alpha_j(u_{j+1} - u_j) - \alpha_{j-1}(u_j - u_{j-1}) - u_j = -h_j$

Theorem

Suppose

$$\alpha_j = \begin{cases} \alpha_j & -m \leq j \leq n \\ \alpha & j < -m \text{ or } j > n \end{cases}$$

and that $j^* \in [-m, n]$. Then the existence of solutions is guaranteed by the necessary and sufficient conditions:

1. $a \in (u_{j^*}, u_{j^*+1})$ with $h_{j^*} = 0$, if $j^* < \xi^* < j^* + 1$
2. $a \in u_{j^*}$ with $h_{j^*} = [0, 1]$, if $j^* = \xi^*$.

Explicit solutions are provided in the publication.

Interval of Propagation Failure

Theorem

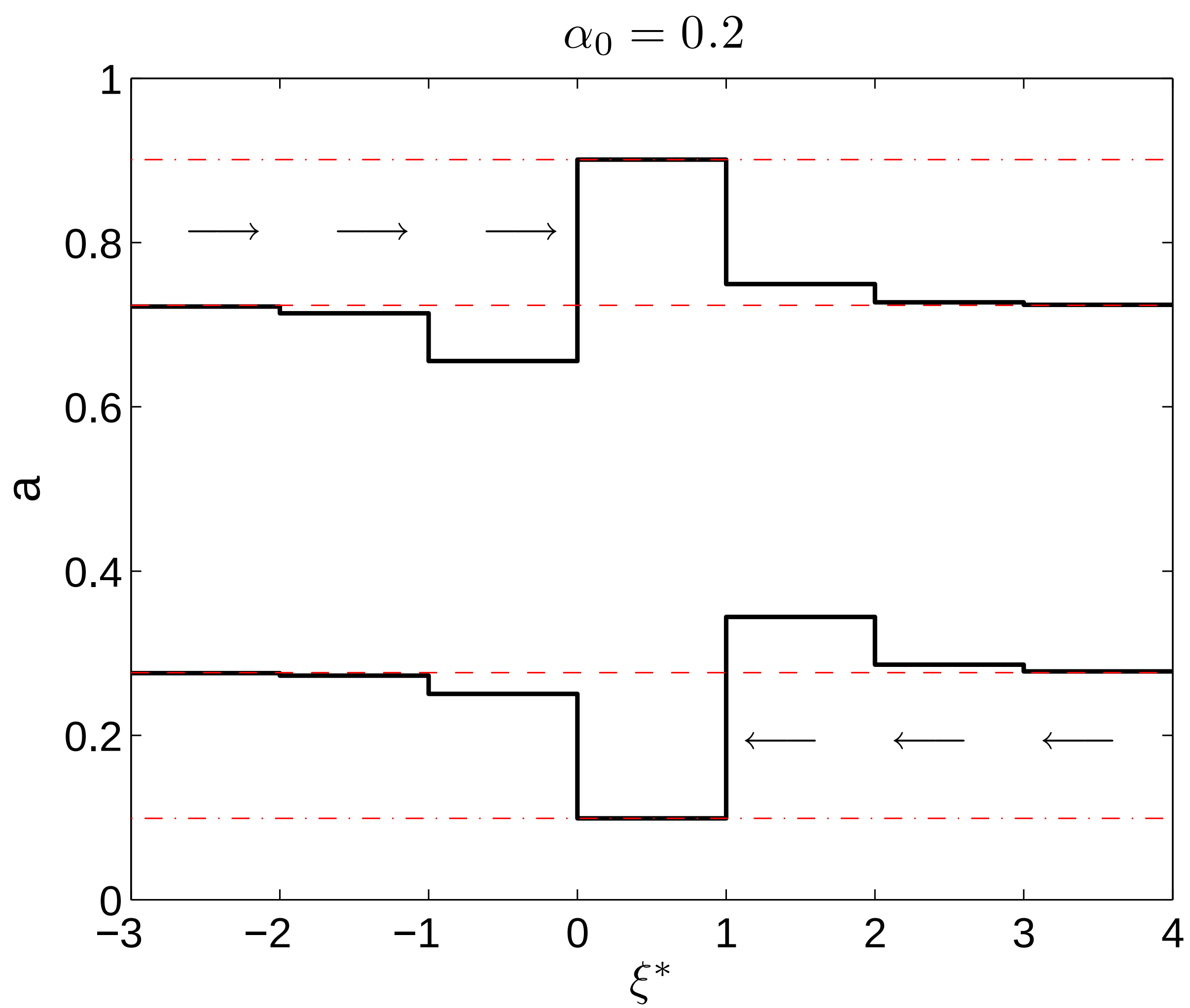
If a yields a traveling wave for $\alpha_0 = \alpha$, then

- Either $a \in (0, 1/(\lambda + 2))$ or $a \in ((\lambda + 1)/(\lambda + 2), 1)$ with $\lambda = (1 + \sqrt{1 + 4\alpha})/2\alpha$
- There are no corresponding standing waves for $\alpha_0 < \alpha$ and $\xi^* \notin (0, 1)$, nor for $\alpha_0 > \alpha$ and $\xi^* \in (0, 1)$.
- There exist standing waves for $\alpha_0 < \alpha$ and $\xi^* \in (0, 1)$, and for $\alpha_0 > \alpha$ and $\xi^* \notin (0, 1)$, provided

$$a \in \left[\frac{\alpha_0/\alpha}{\lambda + 2(\alpha_0/\alpha)}, \frac{\lambda + \alpha_0/\alpha}{\lambda + 2(\alpha_0/\alpha)} \right].$$

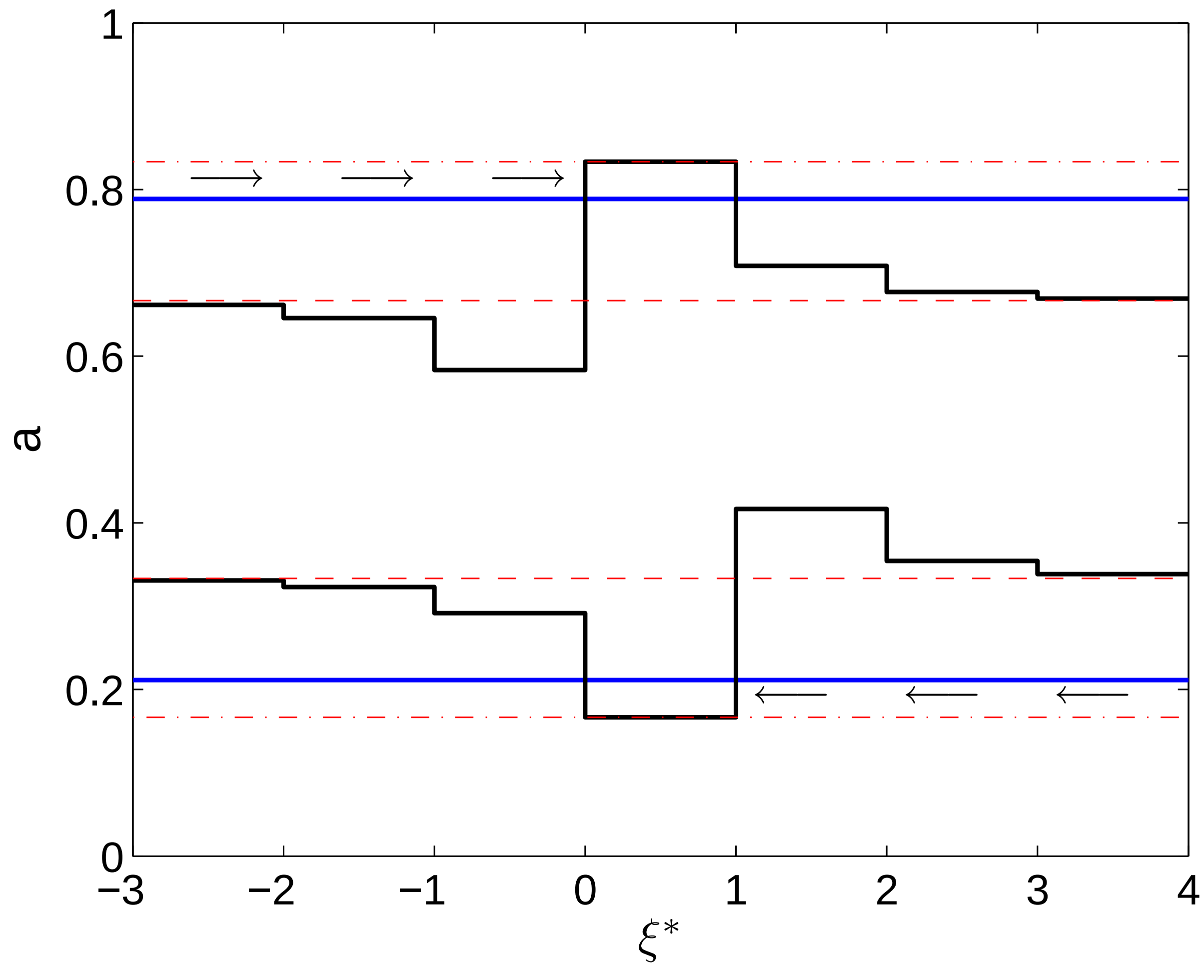
Interval of Propagation Failure

$\alpha = 1$



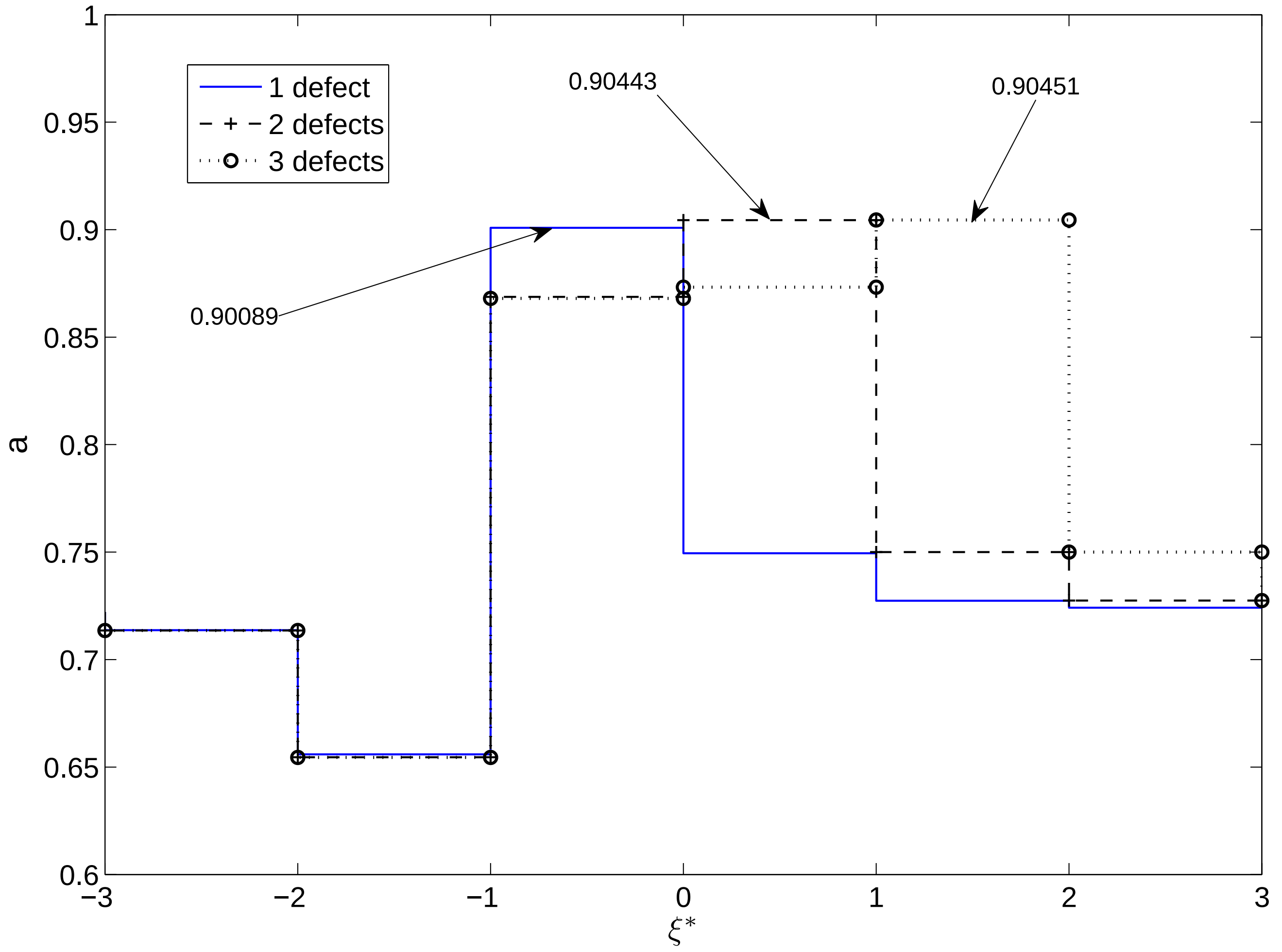
Interval of Propagation Failure

Blue: $\alpha_0 = \alpha = 1/2$ **Black:** $\alpha_0 = 1/2, \alpha = 2$



Interval of Propagation Failure

$\alpha = 1, \alpha_{defect} = 0.2$



Spatially Discrete FitzHugh-Nagumo Equations

$$
\begin{aligned}
\dot{u}_j &= \alpha_j(u_{j+1} - u_j) - \alpha_{j-1}(u_j - u_{j-1}) - v_j - f(u_j), \\
\dot{v}_j &= bu_j - rv_j
\end{aligned}
$$

The recovery term v_j gives pulse solutions.

Stationary Pulses $\quad \Longrightarrow \quad v_j = \frac{b}{r} u_j$

$$
\Longrightarrow \quad \alpha_j(u_{j+1} - u_j) - \alpha_{j-1}(u_j - u_{j-1}) - \frac{b}{r} u_j = f(u_j)
$$

with boundary conditions

$$
\lim_{j \to \pm\infty} u_j = 0
$$

Derivation of Stationary Pulses

We seek solutions that satisfy

$$u_j < a \quad \text{for} \quad j < \xi^* \qquad \text{and} \qquad j > \xi^{**}$$

and

$$u_j > a \quad \text{for} \quad \xi^* < j < \xi^{**}.$$

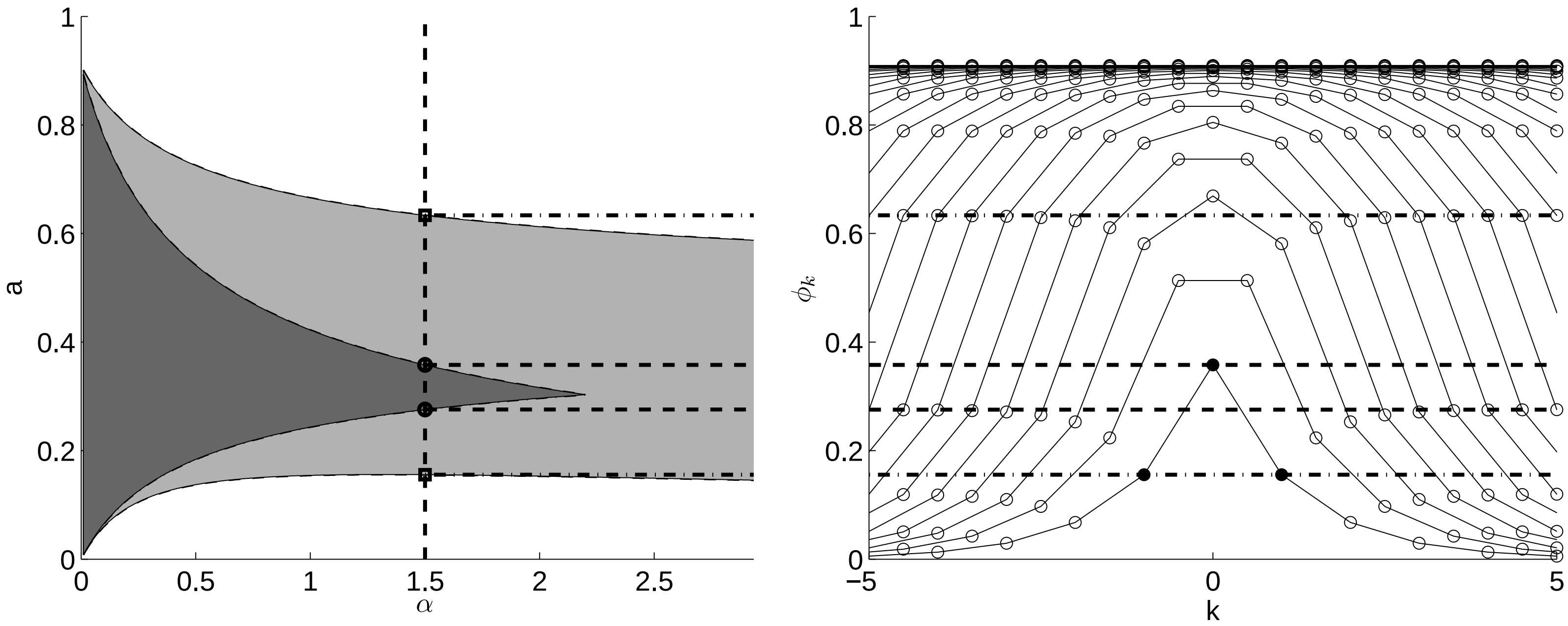
Rewriting the piecewise linear f, yields the equation

$$\alpha_j(u_{j+1} - u_j) - \alpha_{j-1}(u_j - u_{j-1}) - \left(1 + \frac{b}{r}\right) u_j = g_j$$

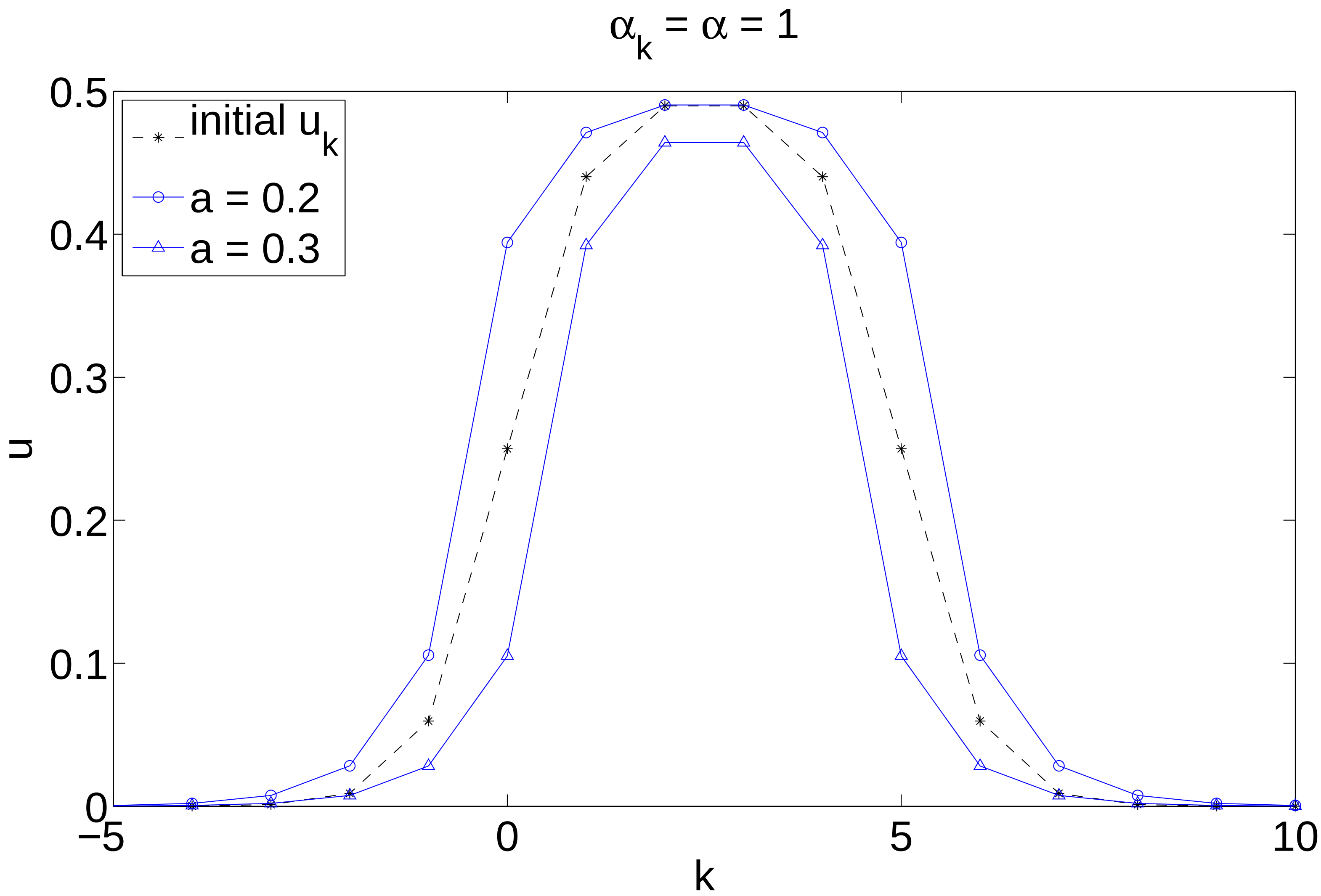
where $j^* = \lfloor \xi^* \rfloor$ and $j^{**} = \lceil \xi^{**} \rceil$ implies

$$g_j = h(j - j^*) - h(j - j^{**})$$

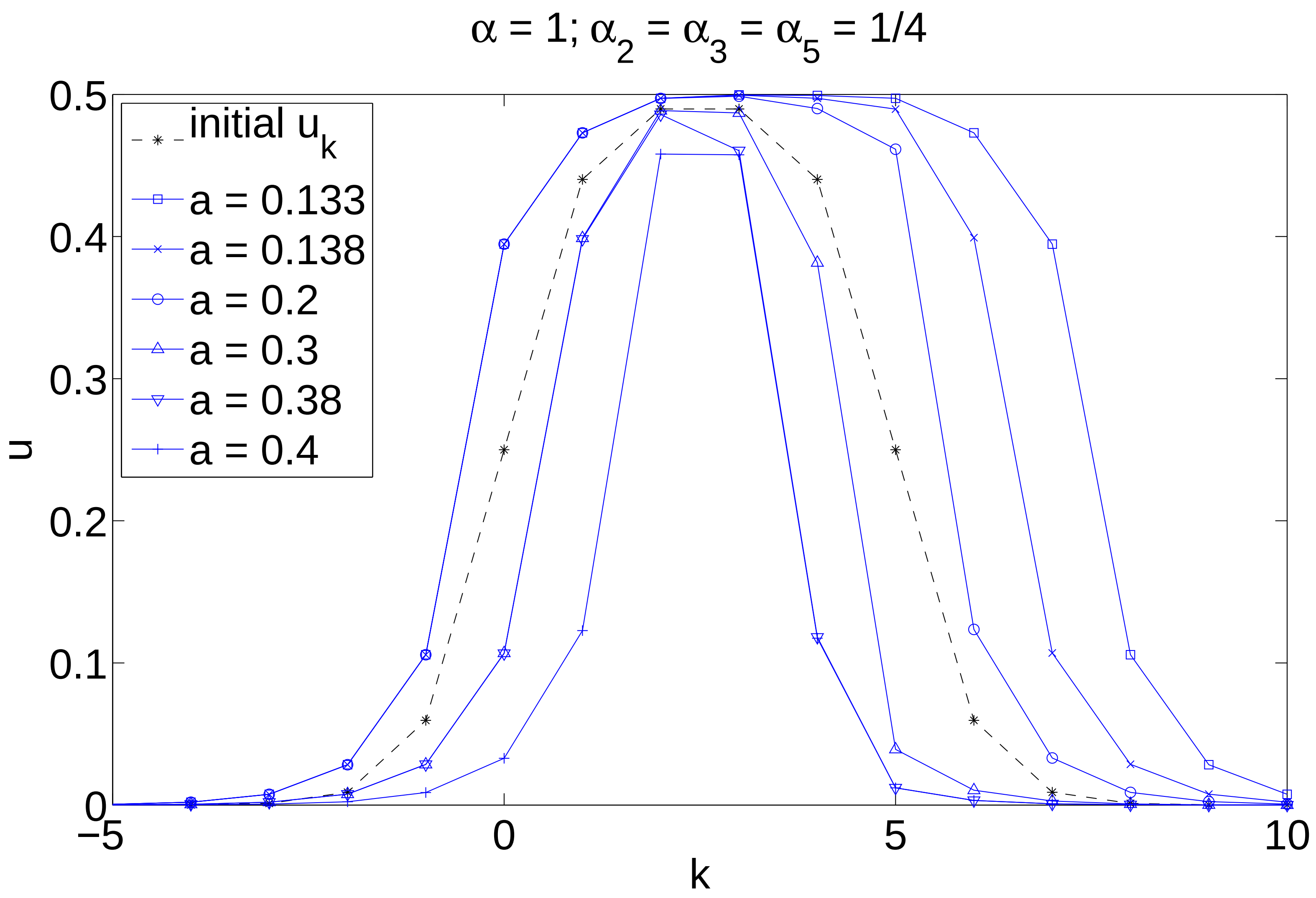
Interval of Stationarity and Pulses



Stable 1-Pulse with $\alpha_k = \alpha, \forall k$

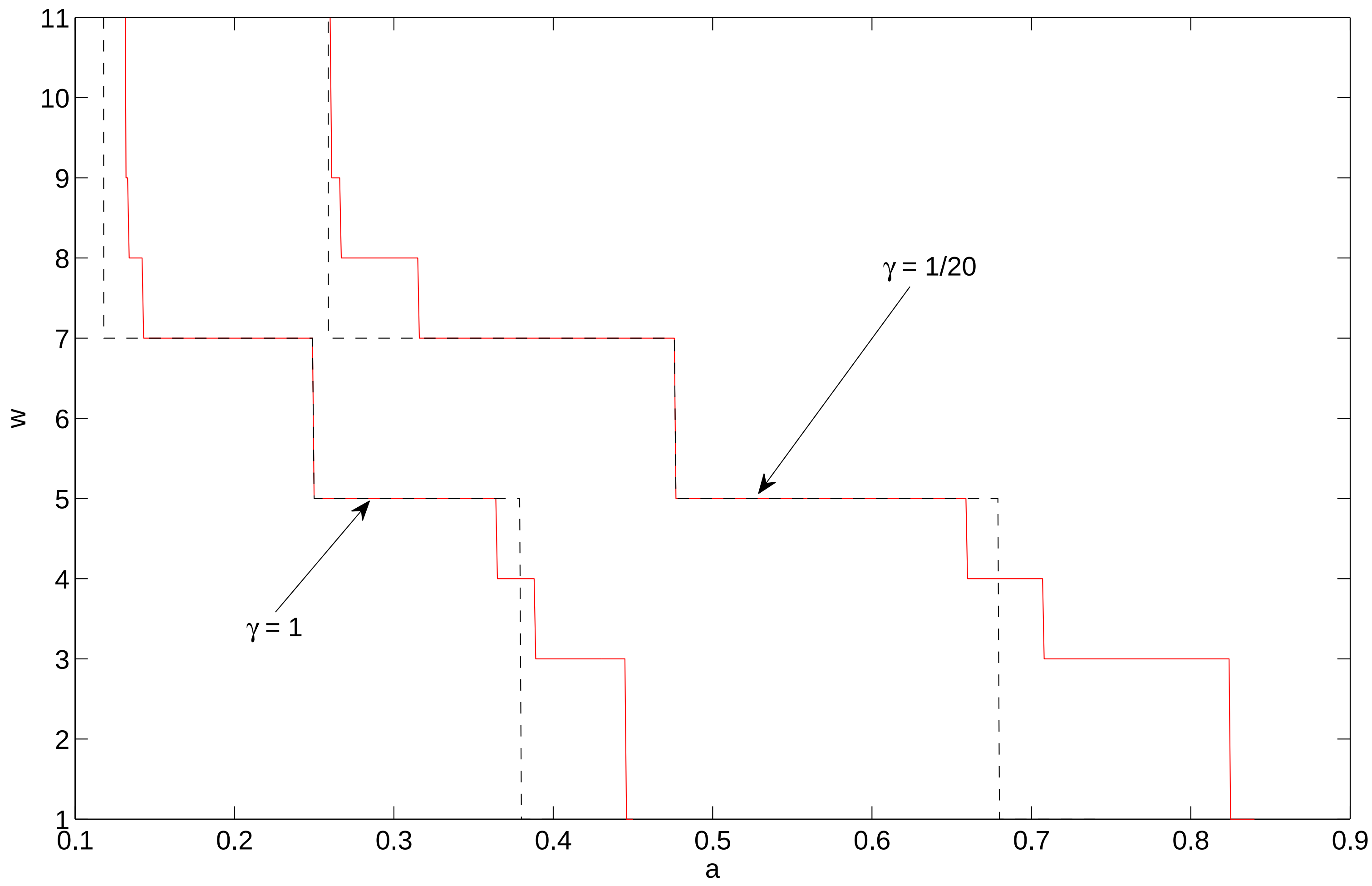


Stable 1-Pulse in the Presence of 3 Defects



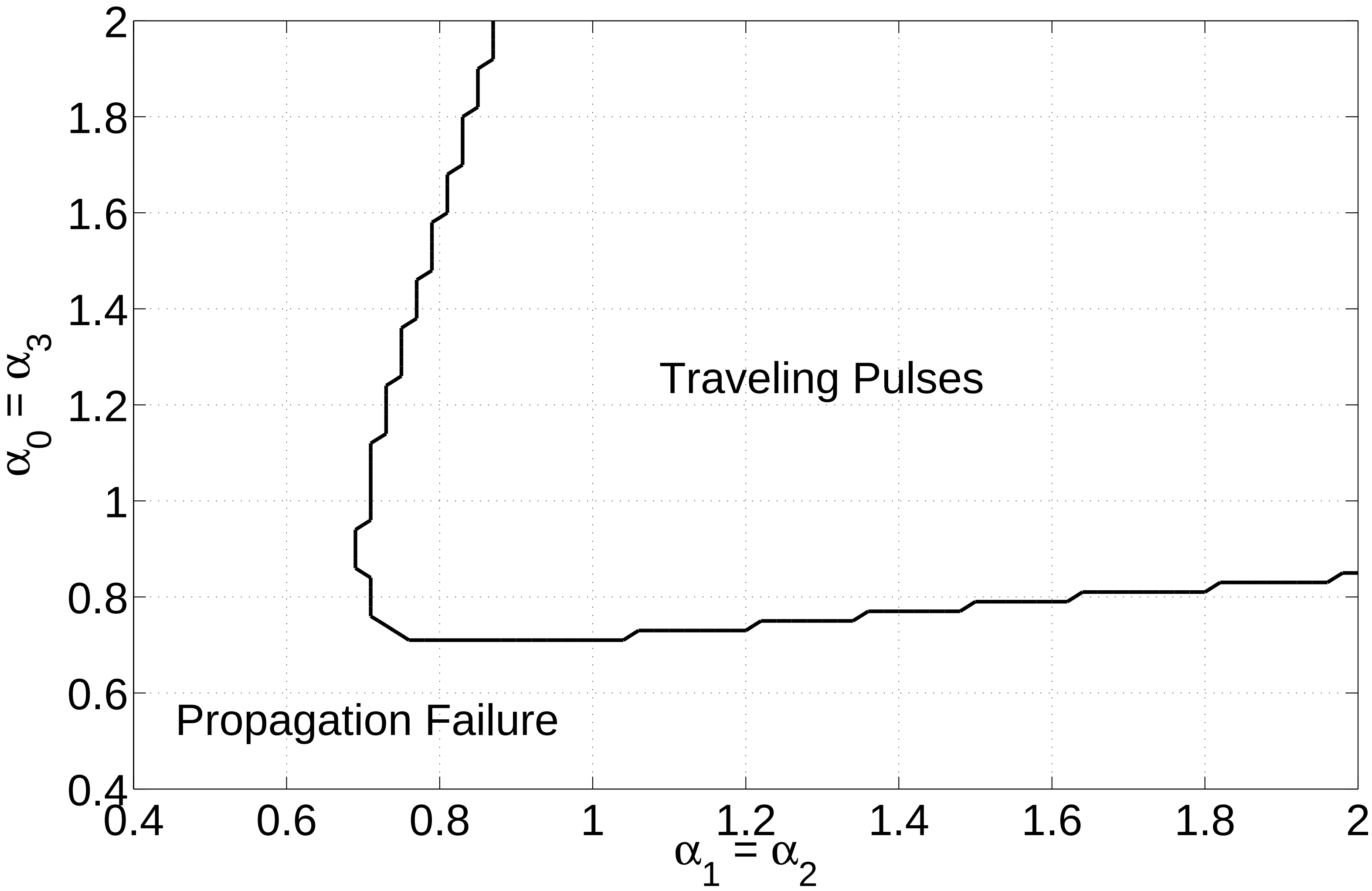
Stable Pulse Width Depending on a

Dashed: No Defects; Solid: 3 Defects



Passing Traveling Pulses Through Defects

Numerical Results with $\alpha = 2$, $a \approx 0.1252$



Conclusions and Future Work

Conclusions

- Results on where an electrical impulse stops.
- Results on destruction required to make it stop.
- Results on wave speed and shape in the defect region.

Future Work

- sufficient conditions for FHN stationary pulses
- explicit propagating pulse solutions
- rigorous analysis of pulse stability
- other nonlinearities
- 2-dimensional lattice