

Geometric Integration for Damped Hamiltonian PDEs

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SIAM Conference on Nonlinear Waves and Coherent Structures, University of
Washington, Seattle, June 13, 2011

Thanks to the National Science Foundation for partial funding

Multi-Symplectic PDE, Bridges (1997)

These equations can be written

$$\mathbf{K}z_t + \mathbf{L}z_x = \nabla_z S(z)$$

- \mathbf{K} and \mathbf{L} are skew-symmetric matrices
- $z = z(x, t)$ is the vector of state variables
- S is smooth

Examples include KdV, Boussinesq, Zakharov-Kuznetsov, nonlinear Schrödinger, nonlinear wave equations, etc.

Conservation laws are derived directly from the equation. An example is the multi-symplectic conservation law $\partial_t \langle \mathbf{K}U, V \rangle + \partial_x \langle \mathbf{L}U, V \rangle = 0$, where U and V satisfy the variational equation.

Conformal Multi-Symplectic PDE, M. (2009)

These equations can be written

$$\mathbf{K}z_t + \mathbf{L}z_x = \nabla_z S(z) - \frac{a}{2}\mathbf{K}z - \frac{b}{2}\mathbf{L}z + F(x, t)$$

- a and b are positive real constants
- S is smooth and may depend on a and b
- F is a forcing term

Equation satisfies a conformal multi-symplectic conservation law

$$\partial_t \langle \mathbf{K}U, V \rangle + \partial_x \langle \mathbf{L}U, V \rangle = -a \langle \mathbf{K}U, V \rangle - b \langle \mathbf{L}U, V \rangle.$$

Methods that preserve this property behave similarly to standard multi-symplectic methods for small a and b .

General Conformal Conservation Laws

If

$$\mathbf{K}z_t + \mathbf{L}z_x = \nabla_z S(z)$$

has a conservation law

$$\partial_t P + \partial_x Q = 0,$$

then

$$\mathbf{K}z_t + \mathbf{L}z_x = \nabla_z S(z) - \frac{a}{2}\mathbf{K}z - \frac{b}{2}\mathbf{L}z$$

has a conformal conservation law if it satisfies

$$\partial_t P + \partial_x Q = -aP - bQ.$$

Examples of Conservation Laws for

$$\mathbf{K}z_t + \mathbf{L}z_x = \nabla_z S(z)$$

- Energy: inner product of the equation with z_t

$$\partial_t \left(S(z) + \frac{1}{2}(z_x^T \mathbf{L}z) \right) + \partial_x \frac{1}{2}(z^T \mathbf{L}z_t) = 0$$

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- Momentum: inner product of the equation with z_x

$$\partial_x \left(S(z) + \frac{1}{2}(z_t^T \mathbf{K}z) \right) + \partial_t \frac{1}{2}(z^T \mathbf{K}z_x) = 0$$

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- Linear Symmetries: inner product with $\mathbf{B}z$

$$\partial_t (z^T \mathbf{K}\mathbf{B}z) + \partial_x (z^T \mathbf{L}\mathbf{B}z) = 0$$

Examples of Conformal Conservation Laws for

$$\mathbf{K}z_t + \mathbf{L}z_x = \nabla_z S(z) - \frac{a}{2}\mathbf{K}z - \frac{b}{2}\mathbf{L}z$$

- Energy: inner product of the equation with z_t

$$\partial_t \left(S(z) + \frac{1}{2}(z_x^T \mathbf{L}z) \right) + \partial_x \frac{1}{2}(z^T \mathbf{L}z_t) = \frac{-b}{2}(z^T \mathbf{L}z_t)$$

- Momentum: inner product of the equation with z_x

$$\partial_x \left(S(z) + \frac{1}{2}(z_t^T \mathbf{K}z) \right) + \partial_t \frac{1}{2}(z^T \mathbf{K}z_x) = \frac{-a}{2}(z^T \mathbf{K}z_x)$$

- Linear Symmetries: inner product with $\mathbf{B}z$

$$\partial_t (z^T \mathbf{K}\mathbf{B}z) + \partial_x (z^T \mathbf{L}\mathbf{B}z) = -a (z^T \mathbf{K}\mathbf{B}z) - b (z^T \mathbf{L}\mathbf{B}z)$$

Numerical Preservation

Suppose we have the following PDE and Conformal CL

$$\mathbf{K}z_t + \mathbf{L}z_x = \nabla S(z) - \frac{a}{2}\mathbf{K}z \qquad \partial_t P + \partial_x Q = -aP$$

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Integrating with appropriate boundary conditions, yields

$$\partial_t \hat{P} = -a\hat{P} \quad \Longleftrightarrow \quad \hat{P}(t) = \exp(-at)\hat{P}(0) \quad \text{with} \quad \hat{P} = \int P dx.$$

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A numerical method preserves this property if it satisfies

$$\hat{P}^{i+1} = \exp(-a\Delta t)\hat{P}^i \quad \text{with} \quad \hat{P}^i = \sum_n P^{n,i} \Delta x.$$

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Note: $\partial_t P < 0$ is typically considered preserved if $P^{i+1} < P^i$

Splitting Methods, McLachlan and Quispel (2002)

$$z_t = F_H(z) + F_D(z)$$

- Solve $z_t = f_H(z)$ with conservative method $z^{i+1} = \Psi_{\Delta t}(z^i)$.
- Solve $z_t = \mathbf{D}z$ exactly with flow map $\Phi_t(z) = \exp(\mathbf{D}t)z$.
- Original system is solved by composing these flow maps.

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Example: Define non-standard difference/average operators

$$D_t^a z = \frac{z^{i+1} - e^{-a\Delta t} z^i}{\Delta t} \quad \text{and} \quad A_t^a z = \frac{z^{i+1} + e^{-a\Delta t} z^i}{2}$$

Implicit midpoint for $z_t = F_H(z)$ is $D_t^0 z = F_H(A_t^0 z)$.

Splitting method for $z_t = F_H(z) + F_D(z)$ is $D_t^{a/2} z = F_H(A_t^{a/2} z)$

Structure-Preservation

A non-standard 1-step FD method preserves a conformal CL, if it has a discrete product rule and the corresponding CL is preserved by the conservative method upon which it is based.

Example: The conformal box scheme

$$\mathbf{K} \left(D_t^{a/2} A_x^{b/2} z \right) + \mathbf{L} \left(D_x^{b/2} A_t^{a/2} z \right) = \nabla S \left(A_t^{a/2} A_x^{b/2} z \right)$$

satisfies
$$D_t^a \left\langle A_x^{b/2} z, \mathbf{K} B A_x^{b/2} z \right\rangle + D_x^b \left\langle A_t^{a/2} z, \mathbf{L} B A_t^{a/2} z \right\rangle = 0.$$

Sum over spatial index with $b = 0$ and appropriate BCs gives

$$\hat{P}^{i+1} = \exp(-a\Delta t) \hat{P}^i \quad \text{with} \quad \hat{P}^i = \sum_n \left\langle A_x^0 z, \mathbf{K} B A_x^0 z \right\rangle \Delta x.$$

Nonlinear Schrodinger with Dissipation

$$i\psi_t + \psi_{xx} + V'(|\psi|^2)\psi + i\frac{a}{2}\psi = 0$$

Can be written $\mathbf{K}z_t + \mathbf{L}z_x = \nabla_z S(z) - \frac{a}{2}\mathbf{K}z$ with

$$\mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} -\mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix},$$

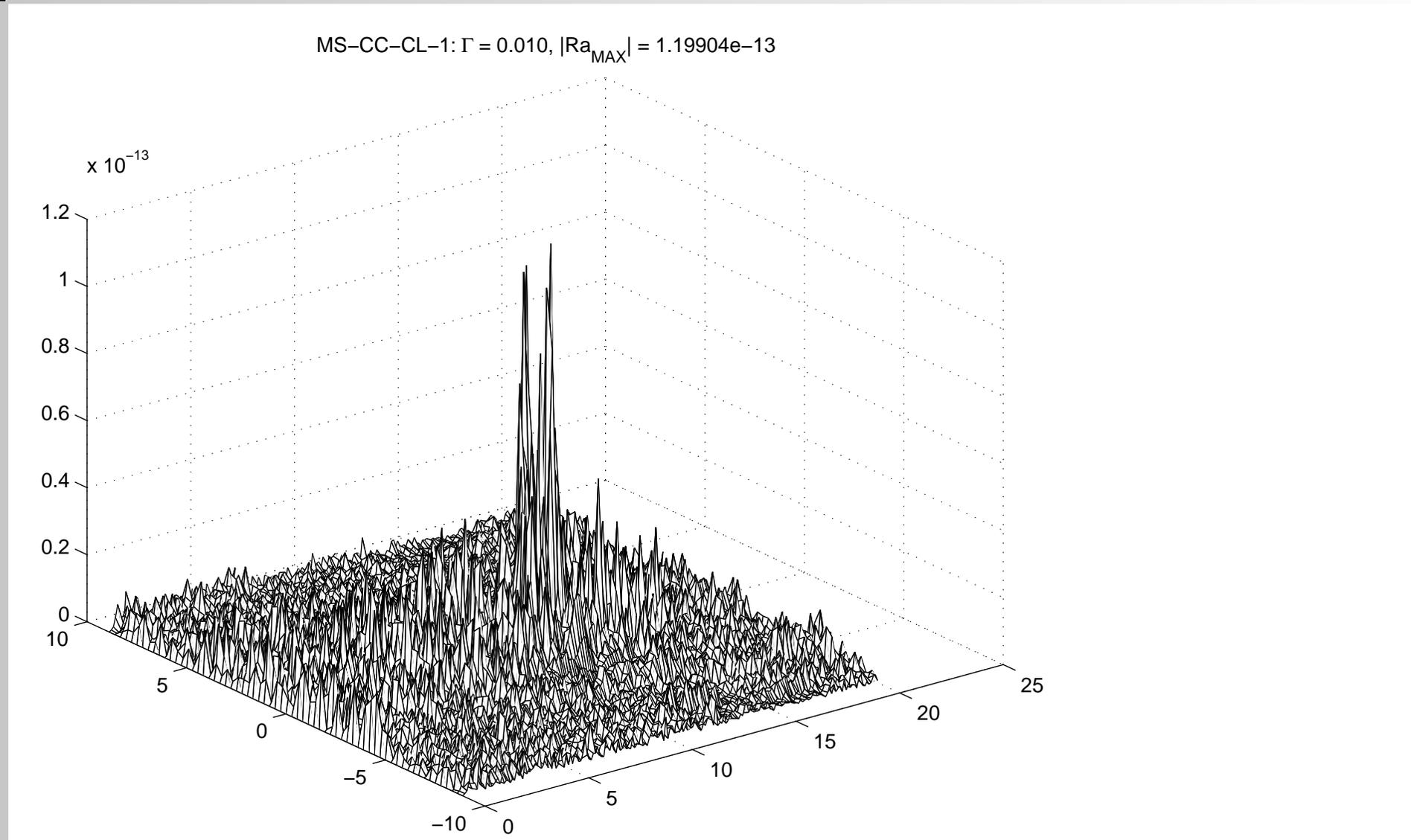
$$z = [v, w, p, q]^T, \quad S(z) = \frac{1}{2}(p^2 + q^2 + V(v^2 + w^2))$$

The conformal norm conservation law

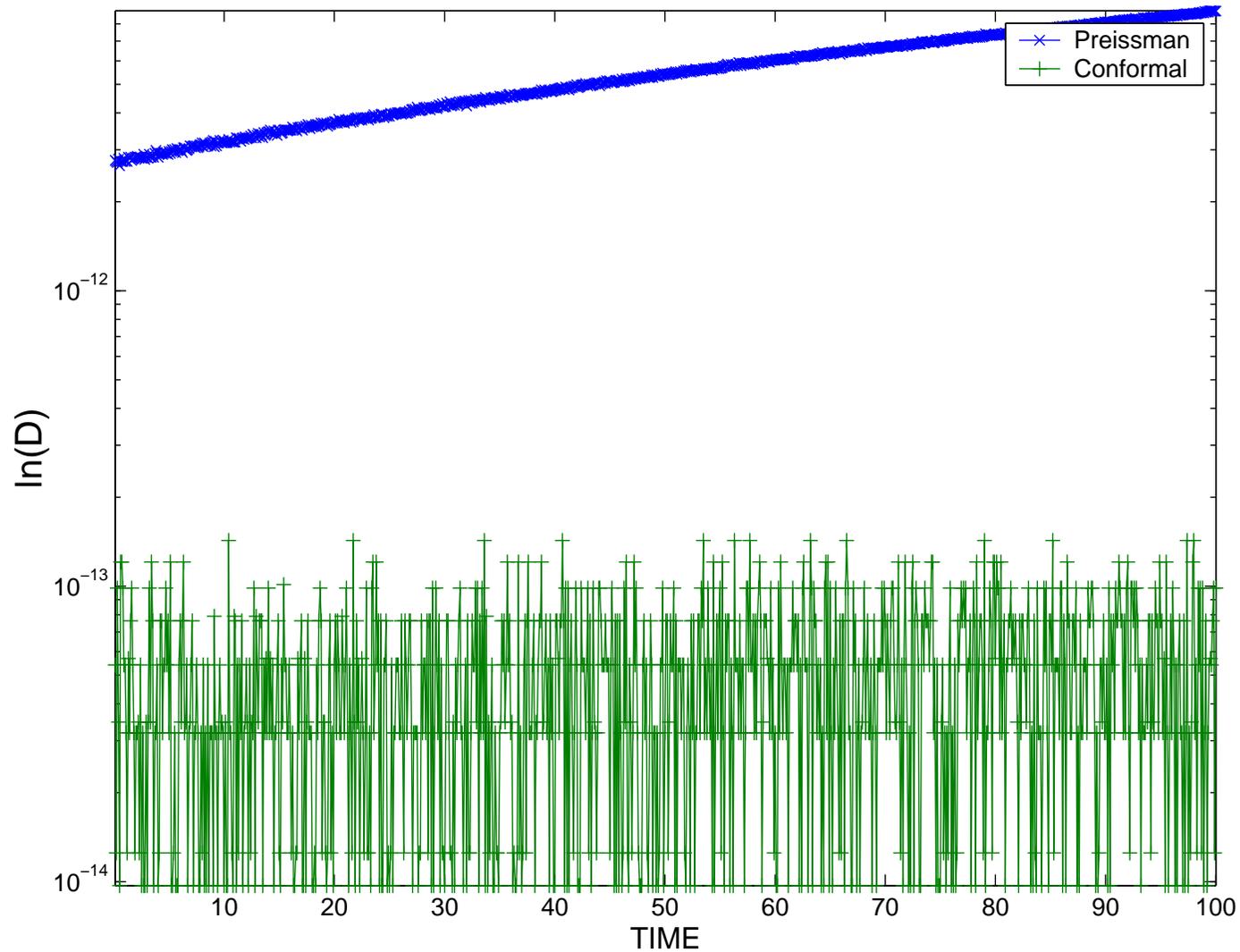
$$\partial_t(w^2 + v^2) + 2\partial_x(vw_x - wv_x) = -a(w^2 + v^2)$$

Residual in Local Conformal Conservation Law

$$a = 0.01, h = 0.01,$$



Residual in Global Conformal Conservation



Dissipative Semi-Linear Wave Equation

$$u_{tt} = u_{xx} - au_t - f'(u)$$

Can be written $\mathbf{K}z_t + \mathbf{L}z_x = \nabla_z S(z) - \frac{a}{2}\mathbf{K}z$ with

$$\mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{J} \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix},$$

$$z = [u, v, w, p]^T, \quad S(z) = \frac{1}{2} (a(uv + wp) + v^2 - w^2 + 2f(u))$$

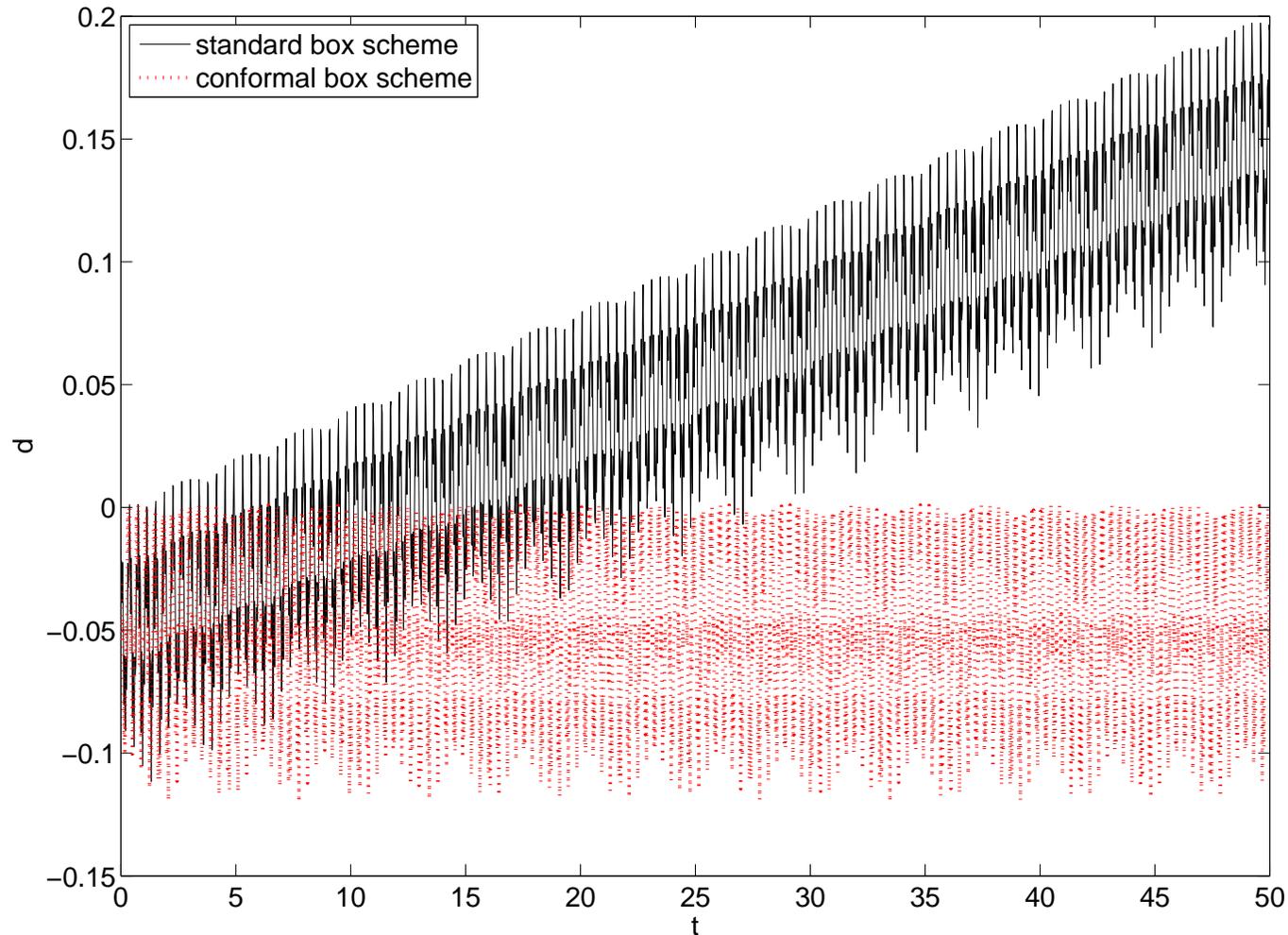
momentum:

$$\partial_t(vu_x + pw_x) + \partial_x \left(f(u) + \frac{v^2}{2} - \frac{w^2}{2} - vu_t - pw_t \right) = -a(vu_x + pw_x)$$

$$\text{angular momentum: } \partial_t(u \times u_t) + \partial_x(u_x \times u) = -a(u \times u_t)$$

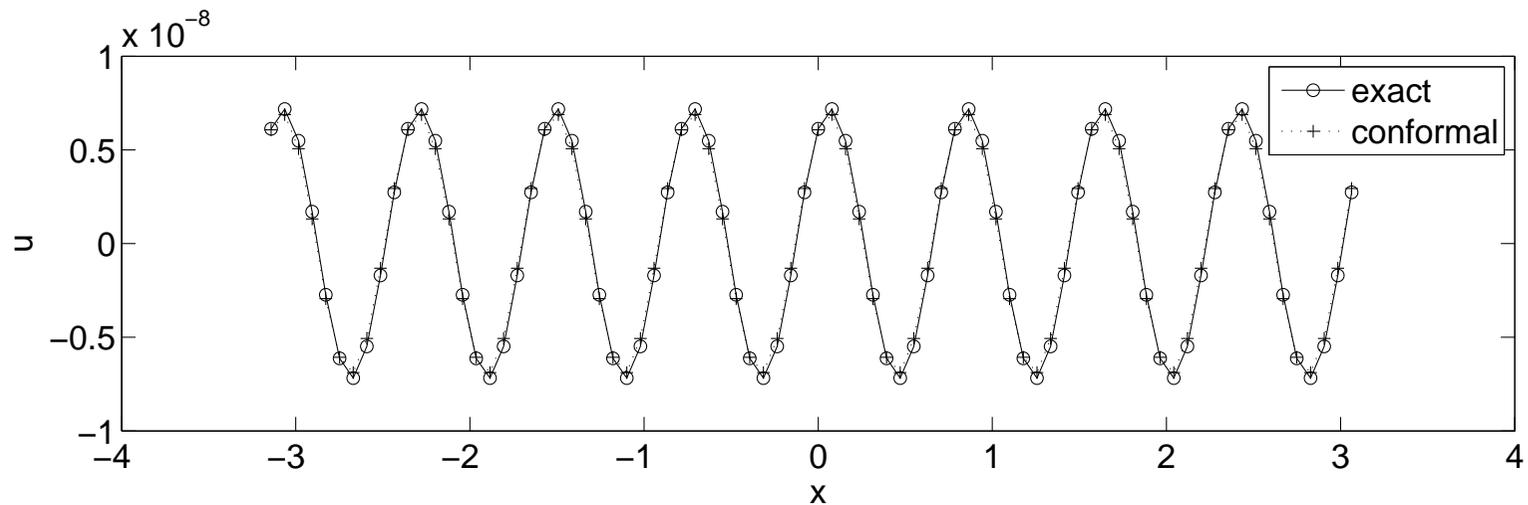
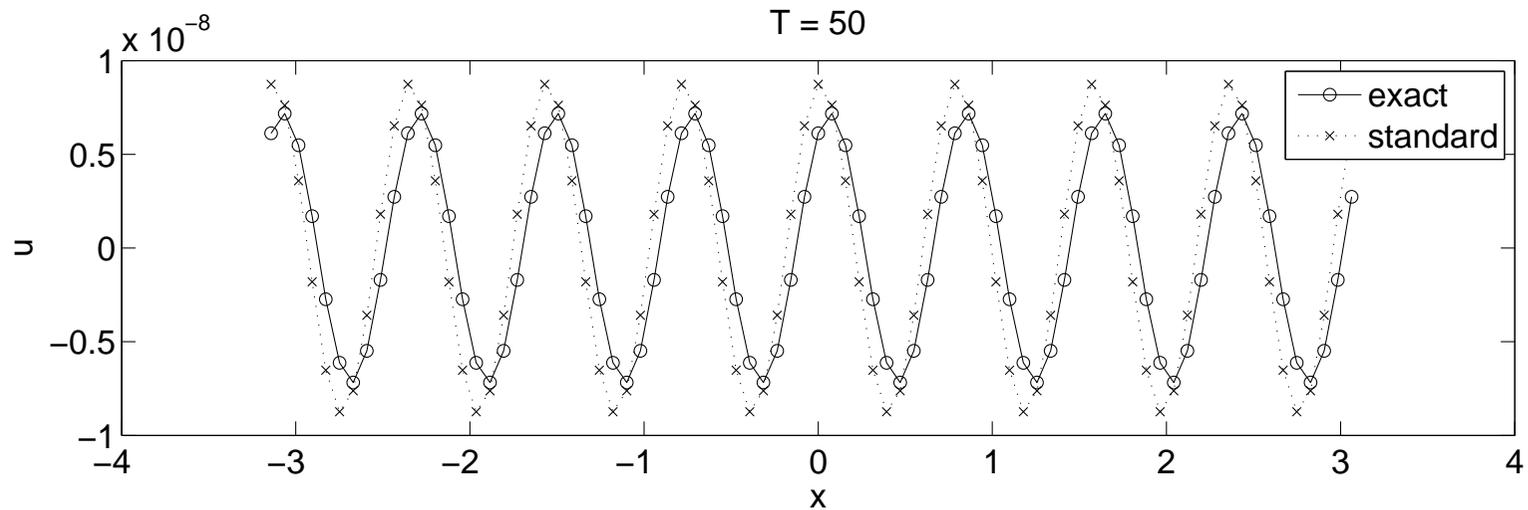
$$\text{others: } \partial_t(vw + cup) + \frac{1}{2}\partial_x(v^2 + w^2 - c(u^2 + p^2)) = -a(vw + cup)$$

Error in dissipation rates for $u_{tt} = u_{xxx} - au_t - cu$ $c = 1, a = \frac{3}{4}, \Delta t = 0.025, 80$ grid points on $[-\pi, \pi]$.



Solution Behavior for $u_{tt} = u_{xx} - au_t - cu$

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