Conformal Conservation Laws and Geometric Integration for Hamiltonian PDE with Added Dissipation

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Abstract: Conformal conservation laws are defined and derived for multi-symplectic equations with added dissipation. In particular, the conservation laws of symplecticity, energy and momentum are considered, along with others that arise from linear symmetries. Numerical methods that preserve these conformal conservation laws are presented. The nonlinear Schrödinger equation and semi-linear wave equation with added dissipation are used to demonstrate the results. This poster summarizes two research articles:

(1) B.E. Moore, Conformal Multi-Symplectic Integration Methods for Forced-Damped Semi-Linear Wave Equations, *Math. Comput. Simulat.* 80:20-28, 2009.

(2) B.E. Moore, L. Noreña, and C. Schober, Conformal Conservation Laws and Geometric Integration for Hamiltonian PDE with Added Dissipation, *preprint*, 2010.

Background

A **multi-symplectic** partial differential equation has the form

$$\mathbf{K}z_t + \mathbf{L}z_x = \nabla_z S(z)$$

where **K** and **L** are constant skew-symmetric matrices and *S* is smooth.

Many conservative PDE may be written this way, including

- Boussinesq Equations
- Dirac Equations
- Schrödinger Equations
- Korteweg-de Vries Equations
- Nonlinear Wave Equations
- Shallow-Water Equations

Many conservation laws are derived directly from the equation, including

- **Multi-symplectic conservation law:** $\omega_t + \kappa_x = 0$

  where $\omega = dz \wedge \mathbf{K}dz$ and $\kappa = dz \wedge \mathbf{L}dz$.

- **Local energy conservation law:** $\partial_t E + \partial_x F = 0$

  where $E = S(z) + \frac{1}{2}\langle z_x, \mathbf{L}z\rangle$ and $F = \frac{1}{2}\langle z, \mathbf{L}z_t\rangle$

- **Local momentum conservation law:** $\partial_x G + \partial_t I = 0$

  where $G = S(z) + \frac{1}{2}\langle z_t, \mathbf{K}z\rangle$ and $I = \frac{1}{2}\langle z, \mathbf{K}z_x\rangle$

- **Conservation laws arising from linear symmetries:**

$$\partial_t\langle z, \mathbf{KB}z\rangle + \partial_x\langle z, \mathbf{LB}z\rangle = 0.$$

If the system is invariant under a transformation **B**
For NLS, for example, this corresponds to rotational invariance.

Conformal Conservation Laws

Conformal Multi-Symplectic PDE (multi-symplectic PDE with dissipation)

$$\mathbf{K}z_t + \mathbf{L}z_x = \nabla_z S(z) + \mathbf{D}z \quad \text{where} \quad \mathbf{D} = -\frac{a}{2}\mathbf{K} - \frac{b}{2}\mathbf{L}$$

- **Conformal multi-symplectic conservation law:** $\partial_t\omega + \partial_x\kappa = -a\omega - b\kappa$
- **Conformal energy conservation law** (iff $a = 0$): $\partial_t E + \partial_x F = -bF$
- **Conformal momentum conservation law** (iff $b = 0$): $\partial_t I + \partial_x G = -aI$
- **Linear symmetries:** $\partial_t\langle z, \mathbf{KB}z\rangle + \partial_x\langle z, \mathbf{LB}z\rangle = -a\langle z, \mathbf{KB}z\rangle - b\langle z, \mathbf{LB}z\rangle$

Numerical Preservation of Conformal Properties

In most applications the constant *b* is set to zero. $\mathbf{K}z_t + \mathbf{L}z_x = \nabla_z S(z) - \frac{a}{2}\mathbf{K}z$

A general form for a conformal conservation law is $\partial_t P + \partial_x Q = -aP$

Integration over *x* with appropriate boundary conditions implies

$$\partial_t\hat{P} = -a\hat{P} \quad \text{or equivalently} \quad \hat{P}(t) = \exp(-at)\hat{P}(0) \quad \text{for} \quad \hat{P} = \int P\,dx$$

Thus, we say a numerical method preserves this conformal property if it satisfies

$$\hat{P}^{i+1} = \exp(-a\Delta t)\hat{P}^i \quad \text{for} \quad \hat{P}^i = \sum_n P^{n,i}\Delta x$$

Methods satisfying this condition can be constructed by solving the conservative part of the equation with a conservative method, and the dissipative part of the equation exactly.

The exact flow map for $z_t = \mathbf{D}z$ is $\Phi_t(z) = \exp(\mathbf{D}t)z$

<u>**Theorem**</u> If the numerical scheme for solving a Hamiltonian equation (ODE or PDE) has a flow map $\Psi_{\Delta t}$ which preserves the conservation property $\partial_t\hat{P} = 0$, then the numerical scheme with flow map $\Phi_{\Delta t} \circ \Psi_{\Delta t}$ for solving the conformal Hamiltonian system preserves the conformal conservation property $\partial_t\hat{P} = -a\hat{P}$.

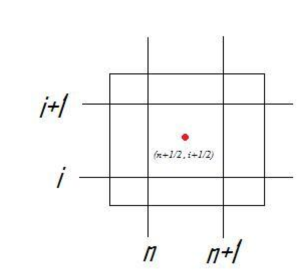
Numerical Methods and Solution Behavior

Define $z^{n,i} \approx z(x_n, t_i)$ such that $t_{i+1} = t_i + \Delta t$ and $x_{n+1} = x_n + \Delta x$

Difference Operators: $D_t^a z = \dfrac{z^{n,i+1} - e^{-(a/2)\Delta t}z^{n,i}}{\Delta t}$ and $D_x^b z = \dfrac{z^{n+1,i} - e^{-(b/2)\Delta x}z^{n,i}}{\Delta x}$

Averaging Operators: $A_t^a z = \dfrac{z^{n,i+1} + e^{-(a/2)\Delta t}z^{n,i}}{2}$ and $A_x^b z = \dfrac{z^{n+1,i} + e^{-(b/2)\Delta x}z^{n,i}}{2}$

<u>**Standard Methods:**</u>



- Preissman Box Scheme (implicit midpoint rule applied to space and time)

$$\mathbf{K}D_t^0A_x^0z + \mathbf{L}D_x^0A_t^0z = \nabla_z S(A_t^0A_x^0z) + \mathbf{D}A_t^0A_x^0z$$

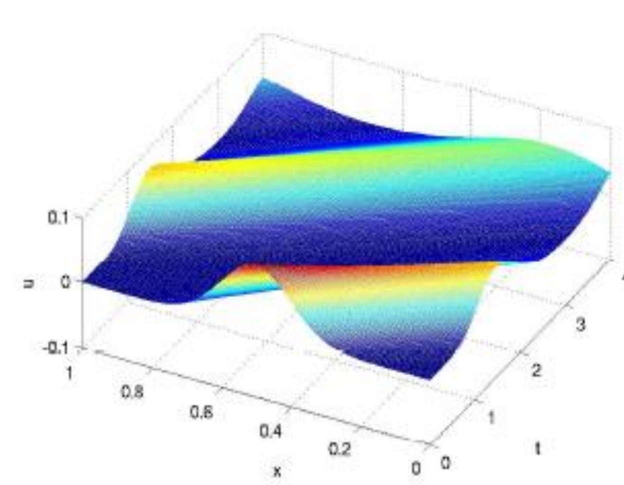
- Euler Box Scheme (symplectic Euler applied in space and time)

$$\mathbf{K}_+D_t^-z + \mathbf{K}_-D_t^0z + \mathbf{L}_+D_x^0z + \mathbf{L}_-D_x^-z = \nabla_z S(z) + \mathbf{D}z$$

matrices $\mathbf{A}_\pm$ are defined by the conditions

$$\mathbf{A} = \mathbf{A}_+ + \mathbf{A}_- \qquad \mathbf{A}_+^T = -\mathbf{A}_-$$

$D_{x,t}^-$ are standard backward difference operators.



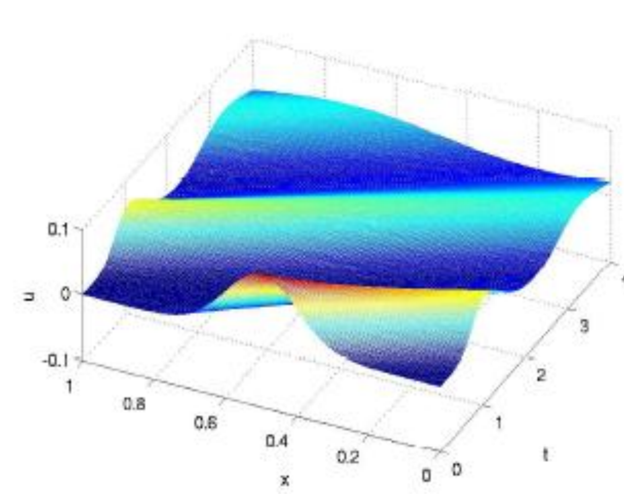
These methods satisfy discrete versions of the conformal conservation laws, but do NOT preserve any conformal properties exactly. Yet, for small dissipation coefficients, they often perform better than their conservative counterparts.

<u>**Splitting Methods:**</u>

Using the Preissman scheme for the conservative part and the exact flow map of the dissipative part, gives

$$\mathbf{K}D_t^aA_x^az + \mathbf{L}D_x^bA_t^bz = \nabla_z S(A_t^aA_x^bz)$$

Methods of this type preserve conformal properties, but may introduce artificial rates of dissipation to other aspects of the solution behavior.



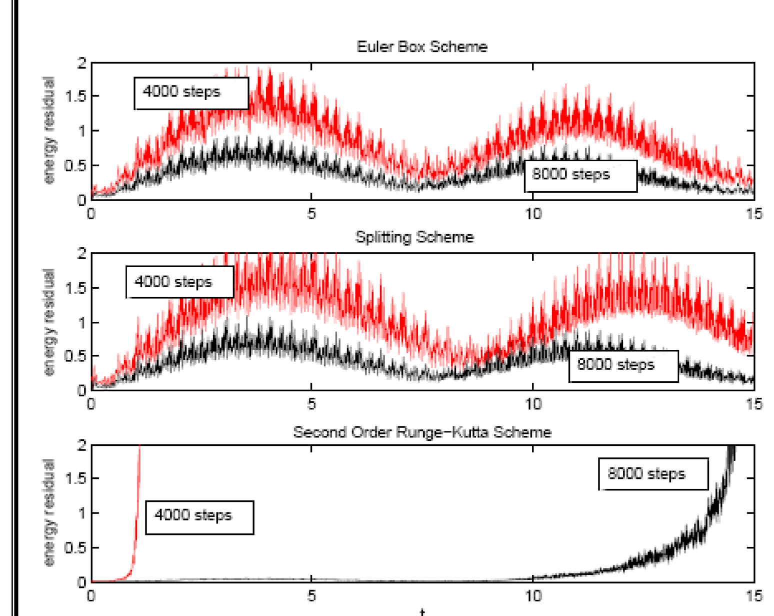
Semi-Linear Wave Equation $u_{tt} = u_{xx} - au_t - f'(u)$

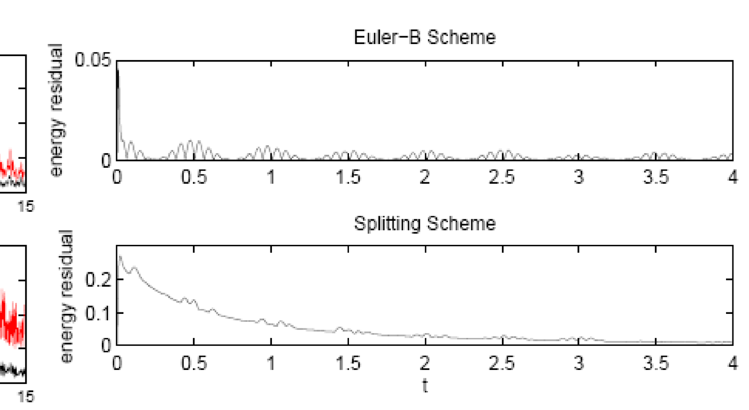
Written in conformal multi-symplectic form

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_t \\ v_t \\ w_t \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_x \\ v_x \\ w_x \end{bmatrix} = \begin{bmatrix} f'(u) + \frac{a}{2}v \\ v + \frac{a}{2}u \\ -w \end{bmatrix} - \frac{a}{2}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Conformal momentum conservation law: $\partial_t(vw) + \partial_x\left(f(u) - \frac{v^2}{2} - \frac{w^2}{2}\right) = -a(vw)$

Conformal multi-symplectic conservation law: $\partial_t(du \wedge dv) + \partial_x(du \wedge dw) = -a(du \wedge dv)$





Experiments: $u_{tt} + au_t = u_{xx} + bu_x - f'(u) + g(x,t)$
Left: $a = b = 1$ and $\Delta x = 0.01$
Right: $a = 150,\ b = 50,\ \Delta x = 0.01,$ and $\Delta t = 0.0008$

Nonlinear Schrödinger $i\psi_t + \psi_{xx} + V'(|\psi|^2)\psi + i\frac{a}{2}\psi = 0$

Define $\psi = v + iw$

and $\psi_x = p + iq$

Written in conformal multi-symplectic form

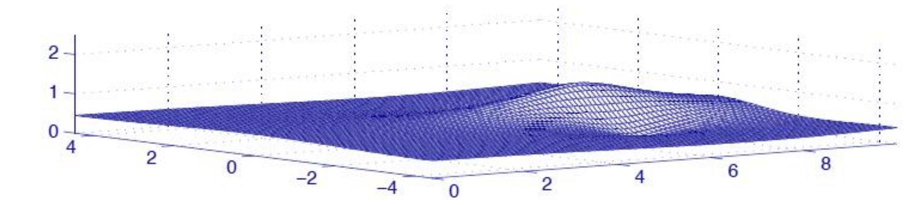
$$z = [v, w, p, q]^T$$

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

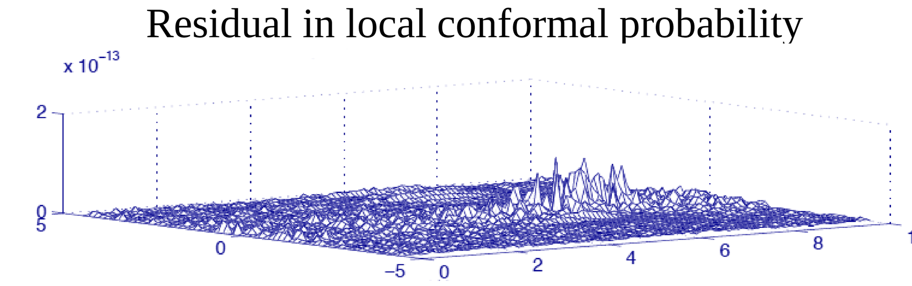
$$K = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} \quad L = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

$$S(z) = \frac{1}{2}(p^2 + q^2 + V(v^2 + w^2))$$

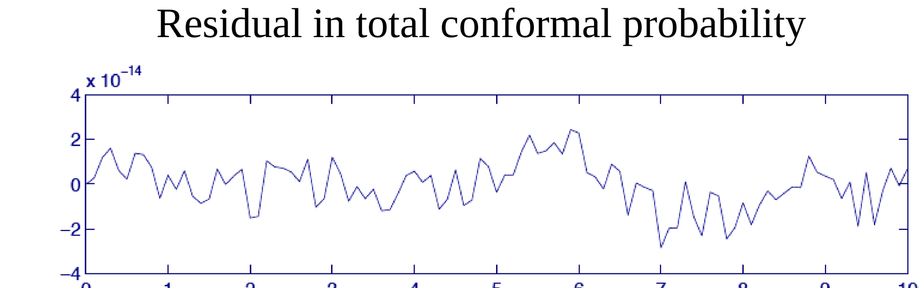
Solution behavior with *a* = 0.01



Residual in local conformal probability



Residual in total conformal probability



The Preissman box scheme is used for simulations, showing conformal properties are very nearly conserved for small *a*.

Conformal probability conservation law: $\partial_t(w^2 + v^2) + 2\partial_x(vw_x - wv_x) = -a(w^2 + v^2)$
The discrete form of this property obtained through the Preissman discretization is

$$D_t^0\left((A_x^0v)^2 + (A_x^0w)^2\right) + 2D_x^0\left(A_t^0vA_t^0q - A_t^0wA_t^0p\right) = -a\left((A_t^0A_x^0v)^2 + (A_t^0A_x^0w)^2\right)$$

Acknowledgments

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