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SPINODAL DECOMPOSITION FOR SPATIALLY DISCRETE CAHN-HILLIARD EQUATIONS

Stanislaus Maier-Paape[1], Brian E. Moore[2], and Erik S. Van Vleck[3]

[1]Institut für Mathematik, RWTH Aachen
[2]Department of Mathematics and Statistics, McGill University
Montreal, Quebec, Canada H3A 2K6.
[3]Department of Mathematics, University of Kansas,
Lawrence, KS 66045 U.S.A..

Abstract. The process of phase separation with a characteristic wavelength is a phenomenon known as spinodal decomposition. In this paper, an extensive and mathematically rigorous analysis is performed for how and when spinodal decomposition occurs for a spatially discretized fourth-order parabolic partial differential equation known as the Cahn-Hilliard equation. First, a linearization is considered for the spatially discrete equation. It is shown that the unstable eigenvalues for the discrete linear equation are almost equal to the eigenvalues of the continuous linear equation for a sufficiently fine discretization of the domain. Then we show that, with a probability close to one, an initial condition, chosen at random inside a particular neighbourhood of an homogeneous equilibrium in the spinodal interval, will lead to spinodal decomposition for the discrete problem. An estimate of the wavelength of spinodally decomposed states is also derived.
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1 Introduction

When a high-temperature homogeneous mixture of two metals is quenched to a lower temperature, the mixture may exhibit a phase separation which will occur in two stages. In the first stage, the mixture quickly becomes inhomogeneous as it decomposes into a fine-grained structure, which exhibits a characteristic length scale. This phenomenon is known as spinodal decomposition. Following this stage, the mixture will go through a coarsening process in which the characteristic length scale grows. Cahn and Hilliard [6, 10] proposed a fourth-order parabolic partial differential equation, which describes this process of phase separation and is given by

$$u_t = -\Delta(\epsilon^2 \Delta u + f(u)), \quad \text{for all } x \in \Omega, \tag{1}$$
$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0, \quad \text{for all } x \in \partial\Omega,$$

where ν is the unit outward normal, ϵ is a small parameter, and $-f$ is the derivative of a double-well potential W, the standard example being the nonlinear cubic function $f(u) = u - u^3$. Here and throughout the paper, Δ denotes the standard Laplacian, and $u_t = \partial u/\partial t$. In general $\Omega \subset \mathbb{R}^d$ is a bounded domain with $d \in \{1,2,3\}$, however, the results in this paper focus on $\Omega = [0,1]^d$. Cahn and Hilliard [6, 10] first derived this equation through Fick's law of diffusion using the van der Waals free energy functional

$$E[u] := \int_\Omega \left(\frac{\epsilon^2}{2} |\nabla u|^2 + W(u) \right) dx,$$

which was introduced in [36], where ∇u denotes the gradient of u, and (1) is a gradient system with respect to this functional. The derivation of (1) can also be found in Elliott [12] or Fife [15].

In the so-called Cahn-Hilliard equation (1), the variable u represents the concentration of one of the two metallic components. For the standard cubic function f from above, typical u values range from -1 to 1, representing concentrations of that component from 0 to 1. Therefore $\int_\Omega u dx$ represents the total mass of that component. Observe that mass is conserved in (1). In the process of spinodal decomposition, the values of u are initially close to a constant solution but start approaching the minima of the potential function $W(u)$. For more on spinodal decomposition refer to [6, 7, 8].

Every constant function is a stationary solution or equilibrium of (1), and if an equilibrium is contained in the spinodal interval, which is the set of all $m \in \mathbb{R}$ such that $f'(m) > 0$, then it is unstable. (Notice that the spinodal interval for $f(u) = u - u^3$ is the open interval $(-1/\sqrt{3}, 1/\sqrt{3})$.) Thus, if $u_0 = m$ is contained in the spinodal interval, almost all orbits starting close to u_0 will soon leave a neighbourhood of that equilibrium. This is when phase separation for (1), and in particular spinodal decomposition, takes place. In fact, understanding how this happens is central to understanding spinodal decomposition. Most orbits leaving this neighbourhood will exit close to some strongly unstable invariant subspace. In fact, the behaviour of these orbits is very similar to that of orbits of the linearization of (1) on that subspace, and it has been shown that these orbits exhibit a very characteristic pattern formation of snake-like pattern cf. [22]. Notice that only this strongly unstable invariant subspace yields the characteristic pattern, whereas the whole unstable subspace does not.

The Cahn-Hilliard equation (1) has been the subject of much study since its derivation, almost 40 years ago. Existence and uniqueness of solutions of (1) have been proven by Elliott and Zheng [14], Nicolaenko and Scheurer [26], Rankin [30], and Temam [35]. Results on steady state solutions of (1) can be found in Novick-Cohen and Segel [27], Modica [24], and Zheng [37]. There are many results concerning the previously mentioned coarsening process, which are due to Alikakos et al. [1], Alikakos et al. [2], Bates and Xun [3, 4], Bronsard and Hilhorst [5], Pego [29], and Stoth [34], among others. Numerical results that portray spinodal decomposition and coarsening for one, two, and three dimensions can be found in Cahn et al. [9], Elliott and French [13], Elliott [12], Rogers, Elder and Desai [31], Nash [25], and Sander and Wanner [32]. Analytic results on spinodal decomposition for the continuous equation (1) have been presented by Grant [16], Maier-Paape and Wanner [22], [23], and Sander and Wanner [33].

Despite this extensive amount of study, there are no analytic results concerning spinodal decomposition of the spatially discrete Cahn-Hilliard equation. An analysis of this sort would be beneficial because numerics are essential for visualizing the behaviour of the solutions, and it is important that we know when and how the equation portrays a given property after discretization. Our aim in this paper is to present an initial step to a mathematically rigorous analysis of when and how spinodal decomposition occurs for the spatially discrete Cahn-Hilliard equation. For simplicity reasons, however we restrict ourselves to the case of cube–shaped regions $\Omega = [0,1]^d$, and neglect more general domains. Also, we only discuss the standard discretization on the cube (finite differences with equal mesh size h) and neglect other discrete versions.

Our approach in this paper is as follows. In § 2, we will consider a linearization for the spatially discrete version of (1)at the homogeneous equilibrium $u_0 = m$. For this linearization, we derive a probability estimate for when an initial condition starting inside a certain ball will lead to spinodal decomposition, and we derive a wavelength estimate of the order $O(\varepsilon)$ for discrete spinodally decomposed states by relating them to continuous spinodally decomposed states.

This is achieved by showing that the eigenvalues of the discrete Laplacian corresponding to unstable eigenvalues of the discrete Cahn-Hilliard equation are almost equal to the eigenvalues of the continuous Laplacian. This enables us to make conclusions about the discrete problem and in particular prove a power law using the results of [22] designed for the continuous problem.

In § 3, a similar probability estimate is proven for the nonlinear discrete equation. In the course of proving this, we restate in Subsection 3.1 an abstract finite-dimensional result of [23] on the dominance of strongly unstable subspaces. Then in Subsection 3.2 we verify the assumptions of that abstract result for the discretized Cahn-Hilliard equation. At last, in Subsection 3.3 the result on spinodal decomposition for the discrete Cahn-Hilliard equation is formulated for the case that the mesh size h is proportional to ε.

We find that the solutions of the discrete Cahn-Hilliard equation are dominated by linear orbits in the strongly unstable subspace up to a neighborhood of the order ε^d in the discrete $L^2(\Omega)$-norm around the constant solution. That result is given in Theorem 3.11. The central assertion is that a certain fraction of volumes

$$\frac{\mathrm{vol}\Big(M_{r,\varepsilon}\Big)}{\mathrm{vol}\Big(B_{r,\varepsilon}(0)\Big)}$$

is in the limit $\varepsilon \to 0$ arbitrarily small. Here $B_{r,\varepsilon}(0)$ represents a small ball of radius r of initial conditions for the considered discrete model centered at a constant solution. On the other hand in $M_{r,\varepsilon} \subset B_{r,\varepsilon}(0)$ are those initial conditions which may not be dominated by the above mentioned strongly unstable subspace, and thus may not show characteristic spinodal decomposition pattern.

The above result is similar to the one for the continuous case (cf. [23]), although estimates needed on the way (spectral gaps, Lipschitz constants) differ from the continuous case.

From our results in Section 2, in particular the result on the closeness of eigenvalues of the discrete Laplacian to those of the continuous Laplacian, Corollary 2.8, it is clear that $h \leq Const \cdot \varepsilon$ is necessary for any kind of result, theory or numerics, on spinodal decomposition. By assuming $h \sim \varepsilon$ in our main result we neglected the case $h \ll \varepsilon$. We needed this restriction, because the abstract result of Subsection 3.1 does not allow this generalization. Nevertheless, we believe that such a generalization should be possible.

2 The Linearized Equation

We begin our analysis with the following assumptions on the nonlinear term f and the domain Ω.

(A1) : Let $\Omega = [0,1]^d \subset \mathbb{R}^d$ with $d \in \{1,2,3\}$ be the unit interval, square or cube.

(A2) : Let $-f : \mathbb{R} \to \mathbb{R}$ be the derivative of a double-well potential and of class C^2. Assume a fixed mass $m \in \mathbb{R}$ and furthermore that $f'(m) > 0$, i.e., m is in the spinodal interval.

Now consider a linearization of (1) at a homogeneous equilibrium $u_0 = m$, where m is chosen according to **(A2)**. This linearization is given by

$$v_t = -\Delta(\epsilon^2 \Delta v + f'(m)v), \quad \text{for all } x \in \Omega, \tag{2}$$

with $x = \left(x^{(1)}, \ldots, x^{(d)}\right)$ and the Neumann-type boundary conditions

$$\frac{\partial v}{\partial \nu} = \frac{\partial \Delta v}{\partial \nu} = 0, \quad \text{for all } x \in \partial\Omega.$$

Though results similar to those obtained in this paper may hold for different boundary conditions, these will be the only boundary conditions considered.

2.1 Spectral Analysis

Because of the previously discussed mass constraint on (1), the linear operator associated with (2) is acting on the space

$$X := \left\{ u \in L^2(\Omega) : \int_\Omega u dx = 0 \right\}. \tag{3}$$

Before we consider the spatially discrete version of (2), we must first understand the Laplace operator for both the linearized discrete and continuous problems.

2.1.1 The Continuous Laplace Operator

We begin with a lemma.

Lemma 2.1 *Suppose* **(A1)** *is satisfied. Then the spectrum of the operator* $-\Delta$ *on* X *consists of eigenvalues of the form* $\kappa = \kappa_{\tilde{j}} = \sum_{i=1}^{d} j_i^2 \pi^2$ *corresponding to* $L^2(\Omega)$*–normalized eigenfunctions* $\psi = \psi_{\tilde{j}}(x) = C_{\tilde{j}} \cdot \prod_{i=1}^{d} \cos(j_i \pi x^{(i)})$, *where* $\tilde{j} = (j_1, \ldots, j_d) \in \mathbb{N}_0^d \setminus \{0\}$. *These eigenfunctions form a complete orthonormal set in* X. *In case* $d = 1$ *we demote these eigenvalues* $\kappa_j = j^2\pi^2$. *Furthermore, if* $N_d(\xi)$ *denotes the number of eigenvalues less than* $\xi \in \mathbb{R}$ *(counting multiplicities), then*

$$\lim_{\xi \to \infty} \frac{N_d(\xi)}{\xi^{d/2}} = c_d, \tag{4}$$

where the constants c_d *are given by* $c_1 = 1/\pi$, $c_2 = 1/4\pi$, *and* $c_3 = 1/6\pi^2$.

Proof. See e.g. Courant and Hilbert [11, page 442] or Pockels [28]. □

This brings us to the eigenvalues of the linear operator associated with the linearized Cahn-Hilliard equation.

Lemma 2.2 *Suppose* **(A1)** *and* **(A2)** *is satisfied, and let* A_ϵ *be the linear operator such that*

$$A_\epsilon v = -\Delta(\epsilon^2 \Delta v + f'(m)v),$$

where $A_\epsilon : X \to X$ *has domain*

$$D(A_\epsilon) = \left\{ u \in X \cap H^4(\Omega) : \frac{\partial u}{\partial \nu}(x) = \frac{\partial \Delta u}{\partial \nu}(x) = 0, \quad \forall x \in \partial\Omega \right\},$$

and X *is given in (3). Then* $-A_\epsilon$ *is a selfadjoint sectorial (cf. [19]) operator. The spectrum of* A_ϵ *consists of real eigenvalues*

$$\lambda_{\tilde{j},\epsilon} = \kappa_{\tilde{j}}(f'(m) - \epsilon^2 \kappa_{\tilde{j}}) = \lambda_\epsilon(\kappa_{\tilde{j}}), \qquad \tilde{j} \in \mathbb{N}_0^d \setminus \{0\}, \tag{5}$$

with corresponding eigenfunctions $\phi_{\tilde{j},\epsilon}$. *Here and in the following we make use of the function* $\lambda_\epsilon(s) := s(f'(m) - \epsilon^2 s)$.

Proof. Compare Henry [19, page 19]. □

Of course the above two lemmas hold similarly for much more general domains. In order to keep things as simple as possible, we have just formulated what we need.

We can now make three important conclusions based on these two lemmas. First, using (5) and **(A2)**, we can conclude that the eigenvalues of A_ϵ are positive when $0 < \kappa_{\tilde{j}} < f'(m)/\epsilon^2$. Thus, the homogeneous equilibrium $u_0 = m$ is unstable whenever

$$0 < \epsilon < \sqrt{\frac{f'(m)}{\kappa_1}}.$$

Using these results and (4), we conclude that as $\epsilon \to 0$, the dimension of the unstable manifold is asymptotically of the order

$$\frac{f'(m)^{d/2} c_d}{\epsilon^d}.$$

Finally, we remark that the largest eigenvalue of A_ϵ is bounded by

$$\lambda_\epsilon^{\max} := \frac{f'(m)^2}{4\epsilon^2}. \tag{6}$$

2.1.2 The Discrete Laplace Operator

We now compare the results for the continuous problem to the discrete Laplace operator under Neumann boundary conditions. In one space dimension ($d = 1$) this operator is given by

$$\Delta_n u(x_k) = \frac{1}{h^2}(u(x_{k+1}) - 2u(x_k) + u(x_{k-1})), \qquad k = 0, 1, \ldots, n,$$

with

$$u(x_{-1}) = u(x_0), \quad u(x_n) = u(x_{n+1}), \tag{7}$$

where $n+1$ is the number of grid points $x_k = (k + 1/2)h$, $k = 0, 1 \ldots, n$, in the interval $\Omega = [0, 1]$ and $h = 1/(n+1)$. In particular we have $\Delta_n : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ being a linear operator. Consider the eigenvalue problem

$$-\Delta_n \varphi = \mu\varphi,$$

with boundary conditions given in (7). It is well known that the eigenvalues for this problem are given by

$$\mu_j = \frac{2}{h^2}\left(1 - \cos\left(\frac{j\pi}{n+1}\right)\right), \quad j = 0, 1, \ldots, n, \tag{8}$$

(cf. e.g. [17]), with the corresponding normalized eigenvectors

$$\varphi_j = \left(\varphi_j^{(k)}\right)_{0 \le k \le n} \in \mathbb{R}^{n+1}, \qquad \varphi_j^{(k)} = \sqrt{\frac{2}{n+1}} \cos\left(\frac{j(k+1/2)\pi}{n+1}\right)$$

for $j = 1, 2, \ldots, n$, and $\varphi_0 = (1/\sqrt{n+1})\vec{1}$, where $\vec{1}$ is the $(n+1)$-dimensional vector of ones.

In order to understand the asymptotic of μ_j and φ_j, notice that

$$\cos\left(\frac{j\pi}{n+1}\right) = 1 - \frac{1}{2}\left(\frac{j\pi}{n+1}\right)^2 + \frac{1}{4!}\left(\frac{j\pi}{n+1}\right)^4 + O\left(n^{-6}\right),$$

which leads to

$$\left(\frac{j\pi}{n+1}\right)^2 n^2 - \frac{2}{4!}\left(\frac{j\pi}{n+1}\right)^4 n^2 \le \mu_j \le \left(\frac{j\pi}{n+1}\right)^2 n^2 \le (j\pi)^2 . \qquad (9)$$

Since in one dimension $\kappa_j = (j\pi)^2$, we have $\mu_j \to \kappa_j$ as $n \to \infty$.

Remark 2.3 *All this is of course very well known. Nevertheless, for the remaining we need good quantitative estimates of* $\mid \frac{\mu_j}{\kappa_j} - 1 \mid$ *as well as* $\mid \mu_j - \kappa_j \mid$ *and the corresponding analogue for higher dimensions (cf. Corollary 2.8). Notice that the eigenvalues of the discrete Laplacian in (8) are up to an affine transformation the zeros of a Chebyshev polynomial. Therefore, for any given mesh size, only the lower eigenvalues* μ_j *come close to* κ_j *(in the sense we need), whereas the upper half of the eigenvalues have completely different asymptotic.*

We note that the normalized eigenfunctions for the continuous case given in Lemma 2.1 can be written as

$$\psi_j(x) = \hat{c}\cos(j\pi x)$$

for some constant $\hat{c}$, which implies that we can write $\varphi_j^{(k)}$ in the form

$$\varphi_j^{(k)} = \frac{1}{\hat{c}}\sqrt{\frac{2}{n+1}} \cdot \psi_j\left(\frac{k+1/2}{n+1}\right). \qquad (10)$$

Similar results hold for the higher space dimensions $d = 2, 3$. Here the discrete Laplacian is the linear operator $\Delta_n : \mathbb{R}^{(n+1)^d} \to \mathbb{R}^{(n+1)^d}$ given by

$$\Delta_n u(x) = \frac{1}{h^2}\sum_{\ell=1}^{d}(u(x+he_\ell) - 2u(x) + u(x-he_\ell)). \qquad (11)$$

evaluated at an equidistant grid of points $x \in \Omega$, and

$$x = \left(x_{k_1}^{(1)}, \ldots, x_{k_d}^{(d)}\right), \quad x_{k_\ell}^{(\ell)} = (k_\ell + 1/2)\cdot h, \quad k_\ell = 0, 1, \ldots, n, \quad \ell = 1, \ldots, d.$$

Again $h = 1/(n+1)$ and $e_\ell \in \mathbb{R}^d$ is the unit vector whose ℓth component is one. Using this notation, the Neumann boundary conditions take the form

$$u(x - he_\ell) = u(x), \qquad k_\ell = 0, n+1 \text{ and } \ell = 1, \ldots, d.$$

The eigenvalues and eigenfunctions of $-\Delta_n$ are obtained from the eigenvalues and eigenfunctions for one dimension in the following way. Using $\tilde{j} = (j_1, ..., j_d) \in [0, ..., n]^d$, we get

$$\mu_{\tilde{j}} = \mu_{j_1} + \cdots + \mu_{j_d}, \qquad (12)$$

and

$$\varphi_{\tilde{j}}(x) = \varphi_{j_1}\left(x^{(1)}_{k_1}\right)\cdots\varphi_{j_d}\left(x^{(d)}_{k_d}\right) = c\cdot\psi_{\tilde{j}}\left(x^{(1)}_{k_1},\ldots,x^{(d)}_{k_d}\right) \tag{13}$$

where $\varphi_{\tilde{j}} \in \mathbb{R}^{(n+1)^d}$, $k_\ell = 0,\ldots,n$, and $\ell = 1,...,d$ (cf. e.g. [18], Section 4.4).

Now consider the following results which relate the eigenvalues of the discrete Laplacian to those of the continuous Laplacian.

Lemma 2.4 *Assume* **(A1)**. *Let* $0 < \hat{\rho} \ll 1$, *and suppose for some* $\tilde{j} \in [0,\ldots,n]^d$ *that* $0 \le \kappa_{\tilde{j}} \le 6\hat{\rho}n^2$ *and* $n \ge 6/\hat{\rho}$. *Then the eigenvalues* $\kappa_{\tilde{j}}$ *and* $\mu_{\tilde{j}}$ *of the continuous and discrete Laplace operators, given in Lemma* 2.1 *and (12), respectively, satisfy*

$$\left|\frac{\mu_{\tilde{j}}}{\kappa_{\tilde{j}}} - 1\right| \le \hat{\rho}.$$

Proof. From (9) and (12), we see that

$$\left|\mu_{\tilde{j}} - \kappa_{\tilde{j}}\left(\frac{n}{n+1}\right)^2\right| \le \frac{2n^2}{4!(n+1)^4}\pi^4\left(j_1^4+\cdots+j_d^4\right),$$

which implies

$$\left|\frac{\mu_{\tilde{j}}}{\kappa_{\tilde{j}}} - 1 + 1 - \left(\frac{n}{n+1}\right)^2\right| \le \frac{2n^2}{4!(n+1)^4}\pi^2\left(j_1^2+\cdots+j_d^2\right) = \frac{2n^2}{4!(n+1)^4}\kappa_{\tilde{j}}\ .$$

From this estimate it follows that

$$\left|1 - \left(\frac{n}{n+1}\right)^2\right| \le \frac{\hat{\rho}}{2}\ , \tag{14}$$

and

$$\frac{2n^2}{4!(n+1)^4}\kappa_{\tilde{j}} \le \frac{\hat{\rho}}{2} \tag{15}$$

imply $\left|\mu_{\tilde{j}}/\kappa_{\tilde{j}} - 1\right| \le \hat{\rho}$. It is easy to check that $n \ge 6/\hat{\rho}$ implies (14) and $\kappa_{\tilde{j}} \le 6\hat{\rho}n^2$ implies (15), hence, the proof is complete. □

Corollary 2.5 *Assume* $0 < \hat{\rho} \ll 1$, $\epsilon > 0$, **(A1)** *and* **(A2)** *hold. Then for*

$$n \ge \max\left[\sqrt{\frac{f'(m)}{2\hat{\rho}\epsilon^2}}, \frac{6}{\hat{\rho}}\right]. \tag{16}$$

and

$$\kappa_{\tilde{j}} \le (3f'(m))/\epsilon^2, \quad \textit{for some} \quad \tilde{j} \in [0,\ldots,n]^d, \tag{17}$$

we have $\left|\mu_{\tilde{j}}/\kappa_{\tilde{j}} - 1\right| \le \hat{\rho}$.

Proof. Inequality (16) implies

$$\frac{3f'(m)}{\epsilon^2} \leq 6\hat{\rho}n^2,$$

and by (17) $\kappa_{\tilde{j}} \leq 6\hat{\rho}n^2$. Hence, an application of Lemma 2.4 completes this proof. □

Unfortunately, in our application we can only bound $\mu_{\tilde{j}}$ instead of $\kappa_{\tilde{j}}$. In order to obtain the $\hat{\rho}$ accuracy of the eigenvalues for bounded $\mu_{\tilde{j}}$ as well in Corollary 2.8, we give two lemmas for the one dimensional eigenvalues beforehand.

Lemma 2.6 *Assume* $0 \ll c_0 < 1$ *is given,* **(A1)** *holds with* $d = 1$*, and*

$$n \geq N_{c_0} := (1 - \sqrt[4]{c_0})^{-1} - 1 \quad \textit{as well as} \quad j \leq \delta_{c_0} \cdot (n+1),$$

are satisfied, where we set $\delta_{c_0} := \frac{1}{\pi} \cdot \sqrt{12(1-\sqrt{c_0})}$*. Then* $\mu_j \geq c_0 \cdot \kappa_j$*.*

Proof. We know from (9) that

$$\mu_j \geq \left(\frac{n}{n+1}\right)^2 \left(1 - \frac{2}{4!}\frac{(j\pi)^2}{(n+1)^2}\right)\kappa_j \;, \tag{18}$$

since $\kappa_j = (j\pi)^2$. The assumptions on n and j are made in such a way that both factors in (18) in front of κ_j can be estimated by $\sqrt{c_0}$ from below. □

Lemma 2.7 *Assume* **(A1)** *and* **(A2)** *holds with* $d = 1$*. Then if* n *satisfies (16) for given* $0 < \hat{\rho} \ll 1$*,* $\epsilon > 0$*, and* $\mu_j \leq \frac{2f'(m)}{\epsilon^2}$ *holds, we have*

$$j \leq \frac{n+1}{\pi} \cdot \arccos(1 - 2\hat{\rho}).$$

Proof. The assumption on μ_j together with (8) and (16) imply the inequality $1 - \cos(\frac{j\pi}{n+1}) \leq 2\hat{\rho}$, from which the assertion follows. □

These lemmas for the one dimensional eigenvalues enable us to improve Corollary 2.5.

Corollary 2.8 *Assume* $0 < \hat{\rho} < 0.45$*,* $\epsilon > 0$*,* **(A1)** *and* **(A2)** *hold. Let* $\kappa_{\tilde{j}}$ *and* $\mu_{\tilde{j}}$ *be the eigenvalues of the continuous and discrete Laplace operators, given in Lemma* 2.1 *and (12), respectively. Then for*

$$n \geq \max\left[\sqrt{\frac{f'(m)}{2\hat{\rho}\epsilon^2}}, \frac{6}{\hat{\rho}}, 10\right] \tag{19}$$

and

$$\mu_{\tilde{j}} \leq (2f'(m))/\epsilon^2, \quad \textit{for some} \quad \tilde{j} \in [0, \ldots, n]^d, \tag{20}$$

we have $\left|\mu_{\tilde{j}}/\kappa_{\tilde{j}} - 1\right| \leq \hat{\rho}$ *and* $\left|\mu_{\tilde{j}} - \kappa_{\tilde{j}}\right| \leq \frac{3f'(m)}{\epsilon^2} \cdot \hat{\rho}$*.*

Proof. Given Corollary 2.5, it suffices to show that $\kappa_{\bar{j}} \leq 3f'(m)/\epsilon^2$, or, equally well we have to show $\kappa_{\bar{j}} \leq \frac{3}{2} \cdot \mu_{\bar{j}}$.

Since $\kappa_{\bar{j}} = \sum_{i=1}^{d} \kappa_{j_i}$ and similar for $\mu_{\bar{j}}$ (cf. (12)), it is enough to show $\kappa_{j_i} \leq \frac{3}{2} \cdot \mu_{j_i}$ for all $i = 1, \ldots, d$. Using Lemma 2.7 with $\mu_{j_i} \leq \frac{2f'(m)}{\epsilon^2}$ gives

$$j_i \leq \frac{n+1}{\pi} \arccos(1 - 2\hat{\rho}) \leq \delta_{c_0} \cdot (n+1)$$

for $c_0 = \frac{2}{3}$, since $\hat{\rho} \leq 0.45$. Applying Lemma 2.6 concludes the proof. □

Remark 2.9 *Note that these results do not rely on the specific finite difference discretization being considered. However, the analysis here does rely on having good approximations to the eigenvalues of the discrete Laplace operator. In fact, a similar analysis should follow, using the same procedure for other spatial discretizations, including non-uniform finite differences, collocation, and finite elements.*

A starting point, for instance, could be a generalization to more general domains. There are several papers in the literature on good estimates of eigenvalues of the discrete Laplacian versus the continuous Laplacian on more general domains than just cubes (cf. Hubbard [20] or Kuttler [21]). Indeed, the upper and lower bounds of Hubbard might be a possible starting point to obtain nontrivial generalizations of Lemma 2.4 and Corollary 2.8 to more general domains. However, since our main emphasis is to outline the basic procedure how spinodal decomposition could be treated in the discrete setting, we did restrict ourselves to the simplest case and have not yet pursued the general domain case.

The next step in our analysis is to consider the eigenvalues of the spatially discrete Cahn-Hilliard equation linearized about $u_0 = m$. Using (11), we can write (1) in spatially discrete form as

$$\dot{u} = \mathcal{F}(u) \equiv -\Delta_n \left(\epsilon^2 \Delta_n u + f(u)\right). \tag{21}$$

where $\dot{u} = du/dt$. Now, if we let $\hat{A}_\epsilon$ be the discrete linear operator

$$\hat{A}_\epsilon v = -\Delta_n \left(\epsilon^2 \Delta_n v + f'(m) v\right), \tag{22}$$

then the eigenvalues for $\hat{A}_\epsilon$ are given by $\lambda_\epsilon(\mu_{\bar{j}})$ with λ_ϵ defined in (5), and since $\kappa_{\bar{j}} \approx \mu_{\bar{j}}$ for all sufficiently small $\mu_{\bar{j}}$, we have $\lambda_\epsilon(\kappa_{\bar{j}}) \approx \lambda_\epsilon(\mu_{\bar{j}})$.

In order to better understand the linearized equation, we briefly discuss the solutions to the abstract equation for the discrete case

$$v_t = \hat{A}_\epsilon v, \qquad v(0) = \bar{v} \in \mathbb{R}^{(n+1)^d}. \tag{23}$$

We can write $\bar{v}$ as a Fourier series

$$\bar{v} = \sum_{\bar{j}\in[0,\ldots,n]^d\setminus\{0\}} \chi_{\bar{j}}\varphi_{\bar{j}}, \tag{24}$$

where $\chi_{\bar{j}} = \langle \bar{v}, \varphi_{\bar{j}}\rangle$, the $\varphi_{\bar{j}}$ form an orthonormal set of eigenvectors corresponding to $\lambda_\epsilon(\mu_{\bar{j}})$, and $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product in $\mathbb{R}^{(n+1)^d}$. Therefore, the solution of (23) is given by

$$v(t) = \sum_{\bar{j}\in[0,\ldots,n]^d\setminus\{0\}} e^{\lambda_\epsilon(\mu_{\bar{j}})t}\chi_{\bar{j}}\varphi_{\bar{j}}, \tag{25}$$

for all $t \geq 0$. Similar results hold for the continuous equation where we sum to infinity and we use the eigenvalues $\lambda_\epsilon(\kappa_{\bar{j}})$ and the eigenfunctions $\psi_{\bar{j}}$, cf. [22].

2.1.3 The Dominating Subspace

Let Y be the space for the discrete problem corresponding to X. Notice that $Y = \vec{1}^\perp \subset \mathbb{R}^{(n+1)^d}$, where now $\vec{1}$ denotes a vector with only ones as entries. We define the dominating subspace of Y in the following way.

Definition 2.10 *Fix constants $\gamma^{--} = -1 < 0 \ll \gamma^- < \gamma^+ < 1$. In what follows we consider for simplicity only those values of $\epsilon > 0$, such that the spectrum of $\hat{A}_\epsilon$ is disjoint from the set $\{\gamma^{--},\gamma^-,\gamma^+\}\cdot\lambda_\epsilon^{\max}$. Then the spectrum of $\hat{A}_\epsilon$ denoted by $\sigma(\hat{A}_\epsilon)$, can be divided into four disjoint sets, such that*

$$\sigma(\hat{A}_\epsilon) = \sigma_\epsilon^{--} \cup \sigma_\epsilon^- \cup \sigma_\epsilon^+ \cup \sigma_\epsilon^{++}, \tag{26}$$

with $\sigma_\epsilon^{--} \subset (-\infty,\gamma^{--})\cdot\lambda_\epsilon^{\max}$, $\sigma_\epsilon^- \subset (\gamma^{--},\gamma^-)\cdot\lambda_\epsilon^{\max}$, $\sigma_\epsilon^+ \subset (\gamma^-,\gamma^+)\cdot\lambda_\epsilon^{\max}$, and $\sigma_\epsilon^{++} \subset (\gamma^+,1]\cdot\lambda_\epsilon^{\max}$. Hence, if we let Y_ϵ^{--}, Y_ϵ^-, Y_ϵ^+, and Y_ϵ^{++} be the subspaces of Y that are generated by the eigenfunctions of $\hat{A}_\epsilon$ corresponding to the eigenvalues in σ_ϵ^{--}, σ_ϵ^-, σ_ϵ^+, and σ_ϵ^{++}, respectively, then we have the decomposition

$$Y = Y_\epsilon^{--} \oplus Y_\epsilon^- \oplus Y_\epsilon^+ \oplus Y_\epsilon^{++}. \tag{27}$$

Furthermore, if $v : \mathbb{R} \to Y$ is a full orbit of (23), then

$$v(t) = v^{--}(t) + v^-(t) + v^+(t) + v^{++}(t), \tag{28}$$

where the superscript on v corresponds to the appropriate subspace of Y. In addition, we can define a decomposition of X similar to (27) according to the eigenvalues of A_ϵ. Here X_ϵ^{--} is infinite dimensional. Notice that for ϵ fixed and $n \to \infty$, the subspace Y_ϵ^{--} is of considerably higher dimension than that of the other subspaces.

It is important to be careful here because it could happen that $\mu_{\bar{j}}$ contributes to Y_ϵ^{++}, but $\kappa_{\bar{j}}$ contributes to X_ϵ^+. Nevertheless, the difference between eigenvalues for the discrete and continuous operators can be controlled for a sufficiently fine discretization of Ω and bounded $\mu_{\bar{j}}$, as was shown in Corollary 2.8.

2.2 Dynamics of the Linear Problem

Since the equilibrium point $u_0 = m$ is unstable, we know that most orbits starting inside some neighbourhood of m will exit that neighbourhood after some time. In this section, we are interested in the behaviour of these orbits. It is shown that the strongly unstable subspace $Y_\epsilon^+ \oplus Y_\epsilon^{++}$ is dominating the behaviour near $u_0 = m$. In fact, an orbit starting close to the homogeneous equilibrium will exit a certain neighbourhood of that equilibrium close to the subspace $Y_\epsilon^+ \oplus Y_\epsilon^{++}$ with a high probability. When this happens, we observe spinodal decomposition. It is worthwhile mentioning that this happens although the space of stable and moderately unstable modes, $Y_\epsilon^{--} \oplus Y_\epsilon^-$, may be of considerably higher dimension than that of the strongly unstable subspace $Y_\epsilon^+ \oplus Y_\epsilon^{++}$, especially as $n \to \infty$.

2.2.1 Choosing a Spatial Mesh

Using (4) we find that the dimensions $\dim X_\epsilon^-$, $\dim X_\epsilon^+$, and $\dim X_\epsilon^{++}$ are all proportional to ϵ^{-d}. Hence, the same is true for the subspaces $Y_\epsilon^-, Y_\epsilon^+$, and Y_ϵ^{++}, since all contributing eigenvalues are $\hat\rho$ accurate compared with the eigenvalues of the continuous problem. To be more precise, there exist some $n^*_{\hat\rho} > 0$ such that for all

$$n \geq n^*_{\hat\rho} \cdot \frac{1}{\epsilon} \quad \text{and} \quad \epsilon > 0 \quad \text{sufficiently small,} \tag{29}$$

we have from Corollary 2.8 that $|\mu_{\tilde{j}} - \kappa_{\tilde{j}}| \leq \frac{3f'(m)\hat\rho}{\epsilon^2}$ for all $\mu_{\tilde{j}}$ contributing to Y_ϵ^-, Y_ϵ^+, and Y_ϵ^{++}. It is now easy to see that this gives the following estimates for the dimensions of the subspaces of Y

$$\dim Y_\epsilon^{++} \sim \dim X_\epsilon^{++} \sim \epsilon^{-d}$$

$$\dim Y_\epsilon^{+} \sim \dim X_\epsilon^{+} \sim \epsilon^{-d}$$

$$\dim Y_\epsilon^{-} \sim \dim X_\epsilon^{-} \sim \epsilon^{-d},$$

for small $\epsilon > 0$. It is clear that, due to the infinite dimension of the subspace X_ϵ^{--}, we loose the $\hat\rho$-accuracy of the eigenvalues corresponding to Y_ϵ^{--}.

The dimension of this strongly stable subspace can trivially be estimated by

$$\dim Y_\epsilon^{--} \leq (n+1)^d.$$

Since we only assume (29), $\dim Y_\epsilon^{--}$ can be much larger than ϵ^{-d}. However, we can ensure

$$\dim Y_\epsilon^{--} \sim \epsilon^{-d},$$

if we additionally to (29) assume $n \leq \hat n/\epsilon$ for some $\hat n > 1$. We refer to this case as $n \sim \frac{1}{\epsilon}$.

As we consider these two cases, we will need the following propositions and notation. Let v be a full orbit of (23), with $v(0) = \bar v$. In addition, let

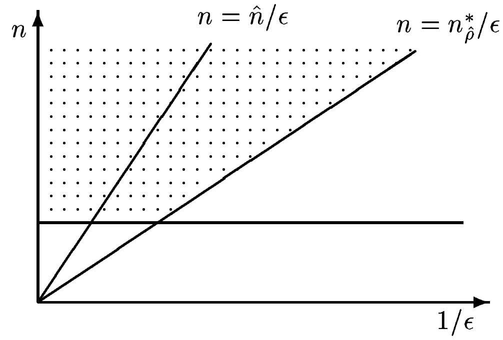

Figure 1: Spinodal decomposition occurs in the shaded region where the dimension of the subspace Y_ϵ^{--} is of a different order than the others only if $n \not\sim \frac{1}{\epsilon}$.

$B_R(0) \subset \mathbb{R}^{(n+1)^d}$ be a Euclidean ball centered at zero with radius R, and let $v^* = v(t^*)$ denote the point where $v(t)$ exits the ball $B_R(0)$ at time $t^* > 0$. Furthermore, $\bar{v}$ and v^* can be decomposed analogously to (28), and $\|\cdot\|$ is the standard Euclidean norm in $\mathbb{R}^{(n+1)^d}$.

Proposition 2.11 *Assume the decomposition of the phase space $Y = \vec{1}^\perp$ as provided by Definition 2.10. If $\overline{v} \in B_r(0) \subset \mathbb{R}^{(n+1)^d}$ and for some $\rho \in (r, R)$*

$$r < (\varrho R^{-\gamma^{--}})^{1/(1-\gamma^{--})}, \tag{30}$$

then $\|v^{*,--}\| < \varrho$. In other words, every orbit of (23) starting in $B_r(0)$ has a small Y_ϵ^{--}-component upon leaving $B_R(0)$, provided the orbit leaves $B_R(0)$ at all.

Proof. Similar to the proof of C1 in [22]. □

Proposition 2.12 *Let r be as in (30), and suppose that $\bar{v} \in B_r(0)$ is such that*

$$\|\bar{v}^{++}\| > R\varrho^{-\gamma}\|\bar{v}^{-}\|^{\gamma}, \tag{31}$$

where $\gamma = \gamma^+/\gamma^- > 1$, then the orbit through $\bar{v}$ exits $B_R(0)$ near the dominating subspace $Y_\epsilon^+ \oplus Y_\epsilon^{++}$. More precisely besides $\|v^{*,--}\| < \rho$ we also have $\|v^{*,-}\| < \varrho$.

Proof. Similar to the proof of C2 in [22]. □

Thus for small $\rho > 0$, the Y_ϵ^{--} and the Y_ϵ^{-} components of an orbit are small when exiting $B_R(0)$ if only (31) is satisfied for the initial condition $\overline{v} \in B_r(0)$. Therefore those orbits starting inside $B_r(0)$ but outside the parabola shaped region (31) will leave a larger neighbourhood $B_R(0)$ close to the subspace $Y_\epsilon^+ \oplus Y_\epsilon^{++}$.

2.2.2 Probability Estimates

Our next goal is to show that most $\overline{v} \in B_r(0)$ satisfy (31) and therefore the behaviour of the orbits near $u_0 = m$ is kind of dominated by $Y_\epsilon^{+} \oplus Y_\epsilon^{++}$.

The question is with what probability does an initial condition, originating inside $B_r = B_r(0)$, lead to an orbit that will exit $B_R(0)$ within a ϱ-neighbourhood of the subspace $Y_\epsilon^{+} \oplus Y_\epsilon^{++}$? We here assume $0 < \rho \ll R$ are given and ask for a range for r such that such a statement like this makes sense. Before we can address this question, we must first define some notation.

Definition 2.13 *Assume the spectral splitting (27) introduced in Definition 2.10. As a measure on the space* $Y = Y_\epsilon^{--} \oplus Y_\epsilon^{-} \oplus Y_\epsilon^{+} \oplus Y_\epsilon^{++}$ *we use the* N*-dimensional Lebesgue measure on* $\mathbb{R}^N$ *and denote it by* $\Upsilon^{(N)}(\cdot)$*, where we set* $N = \dim Y = (n+1)^d - 1$*. Furthermore, let* $G_r \subset B_r$ *be the set of all points satisfying (31).*

Assuming (30), Proposition 2.11 and 2.12 imply that orbits starting at $\overline{v} \in G_r$ leave the ball $B_R(0)$ ρ–close to the subspace $Y_\epsilon^{+} \oplus Y_\epsilon^{++}$. Therefore, we only have to show that the volume of G_r is large compared to the volume of B_r.

Theorem 2.14 *Assume the situation of Definition 2.13 and let* $0 < \rho \ll R$ *and some* $0 \ll p < 1$ *be given. Then, if* n *and* ϵ *satisfy (29), we have*

$$\frac{\Upsilon^{(N)}(G_r)}{\Upsilon^{(N)}(B_r)} \geq p, \quad \text{for all} \quad 0 \leq r \leq r_0. \tag{32}$$

Here $r_0 \in (0, \rho)$ *depends on* $p, \rho, R, \gamma^-, \gamma^+$ *and* γ^{--} *but is independent of* ϵ *and* n*.*

Proof. This is an application of Theorem 3.1 in [22]. We only have to guarantee that $\dim Y_\epsilon^{-}$ and $\dim Y_\epsilon^{++}$ are of the same order, which was done in the last subsection. □

Notice that the above theorem holds, even if Y_ϵ^{--} is of considerably larger dimension as the other subspaces of Y.

2.3 Wavelength Estimates

Now we are left to consider the wavelength of spinodally decomposed states. In general, when considering a cosine wave of the form $y = A\cos(kx)$, we can use a standard definition of a wavelength w given in general physics by $w = 2\pi/k$, where k is called the wave number. Similarly, we can write the wavelength for the eigenfunctions $\psi_{\tilde{j}}$ as

$$w = \frac{2\pi}{\sqrt{\kappa_{\tilde{j}}}}.$$

The goal of this subsection is to derive a notion of wavelength for the functions in the strongly unstable subspace $Y_\epsilon^+ \oplus Y_\epsilon^{++} \subset \mathbb{R}^{(n+1)^d}$. Compared to the above, the elements in $Y_\epsilon^+ \oplus Y_\epsilon^{++}$ are no longer eigenfunctions to one eigenvalue, but superpositions of eigenfunctions to different eigenvalues $\mu_{\tilde{j}}$ (such that $\lambda_\epsilon(\mu_{\tilde{j}}) \geq \gamma^- \lambda_\epsilon^{\max}$). Another difficulty besides that stems from the fact that elements in $Y_\epsilon^+ \oplus Y_\epsilon^{++}$ are only discrete.

But since the eigenfunctions for the discrete case are just restrictions of the continuous eigenfunctions (see (13)), we will make use of a notion of wavelength known in that case (cf. [22]). We start choosing a subset $I_\epsilon \subset \{0, \ldots, n\}^d$ such that

$$Y_\epsilon^+ \oplus Y_\epsilon^{++} = \operatorname{span}\{\varphi_{\tilde{j}} : \tilde{j} \in I_\epsilon\},$$

where $0 \ll \gamma^- < 1$ is assumed to be fixed. Furthermore, we choose $0 \ll \gamma_1 < 1$ and $0 \ll \frac{1}{\gamma_2} < 1$, such that

$$\gamma_1 \leq \frac{\mu_\epsilon^{\max}}{\mu_{\tilde{j}}} \leq \gamma_2, \quad \text{for all} \quad \tilde{j} \in I_\epsilon \tag{33}$$

holds with γ_1 as large as possible and γ_2 as small as possible. Here we use $\mu_\epsilon^{\max} = f'(m)/(2\epsilon^2)$ which is the point where λ_ϵ is maximal. It is clear that both γ_1 and γ_2 can be calculated explicitly from $\gamma^- < 1$. Also for $\eta := \frac{\gamma_1}{\gamma_2} < 1$ it is immediate that $\eta(\gamma^-) \to 1$ as $\gamma^- \to 1$.

We want to establish a notion of wavelength of order ϵ for functions related to elements in the dominating subspace $Y_\epsilon^+ \oplus Y_\epsilon^{++}$. This is made precise by showing that any ball $B_{\bar{r}}(x_0) \subset [0,1]^d$ which is contained in a nodal domain of these functions has radius $\bar{r} \leq K\epsilon$ for some $K \in \mathbb{R}^+$. In order to make this precise we need the following condition.

Definition 2.15 *Let $\delta > 0$ be given. We say that $x_0 \in [0,1]^d$ satisfies* **Condition $\mathbf{C}_\delta$** *with respect to the vector $\beta := (\beta_{\tilde{j}})_{\tilde{j} \in I_\epsilon}$, if either the angle between β and $\psi_{x_0} := (\psi_{\tilde{j}}(x_0))_{\tilde{j} \in I_\epsilon}$ satisfies*

$$\frac{|\langle \beta, \psi_{x_0} \rangle|}{\|\beta\| \cdot \|\psi_{x_0}\|} \geq \delta > 0, \tag{34}$$

or

$$\left| \sum_{\tilde{j} \in I_\epsilon} \beta_{\tilde{j}} \psi_{\tilde{j}}(x_0) \right| \geq \delta \cdot \min \left\{ - \sum_{\tilde{j} \in I_\epsilon^-} \beta_{\tilde{j}} \psi_{\tilde{j}}(x_0), \sum_{\tilde{j} \in I_\epsilon^+} \beta_{\tilde{j}} \psi_{\tilde{j}}(x_0) \right\}, \tag{35}$$

where I_ϵ^- and I_ϵ^+ are the subsets of I_ϵ with $\beta_{\tilde{j}} \psi_{\tilde{j}}(x_0) < 0$ and $\beta_{\tilde{j}} \psi_{\tilde{j}}(x_0) > 0$, respectively. Scalar product and norm above are Euclidean in $\mathbb{R}^{|I_\epsilon|}$.

Theorem 2.16 *For every $\varphi = \sum_{\tilde{j} \in I_\epsilon} \beta_{\tilde{j}} \varphi_{\tilde{j}} \in Y_\epsilon^+ \oplus Y_\epsilon^{++} \subset \mathbb{R}^{(n+1)^d}$ with $\beta_{\tilde{j}} \in \mathbb{R}$, $n \geq \frac{n^*(\gamma^-)}{\epsilon}$ ($n^*(\gamma^-)$ is to be determined in the proof) and $0 < \epsilon < 1$,* *we have that the corresponding superposition of continuous eigenfunctions* $\psi = \sum_{\tilde{j}\in I_\epsilon} \beta_{\tilde{j}}\psi_{\tilde{j}}$ *has a wavelength of the order* $O(\epsilon)$ *in the following sense:*

There exists some $\delta = \delta(\eta) > 0$ *depending on* $0 \ll \eta = \gamma_1/\gamma_2 < 1$ *(cf. (33)) which converges to 0 as* $\eta \to 1$*, such that at any* $x_0 \in [0,1]^d$ *satisfying Condition* $\mathcal{C}_\delta$ *with respect to* $\beta \in \mathbb{R}^{|I_\epsilon|}$*, we have that for any ball* $B_{\bar{r}}(x_0) \subset [0,1]^d$ *which is contained in a nodal domain of* ψ*, its radius* $\bar{r}$ *necessarily satisfies the inequality*

$$\bar{r} \le \frac{2\pi}{\sqrt{f'(m)}}\sqrt{2\gamma_2}\cdot\epsilon.$$

Proof. The claim follows from Theorem 4.8 of [22], once we have shown

$$0 < \tilde{\gamma}_1 = \frac{\mu_\epsilon^{\max}}{\kappa_{\tilde{j}}} \le \tilde{\gamma}_2, \quad \text{for all} \quad \tilde{j} \in I_\epsilon, \tag{36}$$

where we set $\tilde{\gamma}_1 := 1 - 2(1-\gamma_1)$ and $\tilde{\gamma}_2 = 1 + 2(\gamma_2 - 1) > 1$. Notice that still $\tilde{\eta} = \frac{\tilde{\gamma}_1}{\tilde{\gamma}_2} \to 1$ as $\gamma^- \to 1$.

To obtain (36) we need $-(1-\gamma_1) < \mu_\epsilon^{\max}\left(\frac{1}{\kappa_{\tilde{j}}} - \frac{1}{\mu_{\tilde{j}}}\right) < \gamma_2 - 1$. This is obtained from Corollary 2.8 applied with $\hat{\rho} := \min\{(1-\gamma_1)/\gamma_1, (\gamma_2-1)/\gamma_2\} > 0$ and also determines $n^*(\gamma^-)$. □

This theorem gives a bound of order ϵ for the wavelength of some functions corresponding to elements in the dominating subspace $Y_\epsilon^+ \oplus Y_\epsilon^{++}$, but only with the extra assumption Condition $\mathcal{C}_\delta$.

It is obtained in Subsection 4.3 in [22] that these points $x_0 \in [0,1]^d$ may be typical points.

3 Nonlinear Dynamics

In this section, we derive a similar probability estimate as that given in Theorem 2.14 for the nonlinear equation (21). Our approach is very much guided by [33].

For simplicity, we will treat here only the case $n \sim \frac{1}{\epsilon}$, although with some more effort it should be possible to give a similar result for $n \ge \frac{1}{\epsilon}$. The advantage of $n \sim \frac{1}{\epsilon}$ is that we do not have to distinguish between elements in Y_ϵ^{--} and Y_ϵ^- since their dimension is of the same order. Therefore, the following abstract part is dealing only with a decomposition into three different subspaces.

3.1 The Abstract Equation

The following result is a restatement of a part of Theorem 2.8 of Maier-Paape and Wanner [23] with the special focus that our problem is only finite dimensional.

Let Z denote a finite dimensional real Hilbert space with scalar product $(\cdot,\cdot)_0$ and induced norm $||\cdot||_0$. We consider evolution equations of the form

$$\dot{u} = Au + F(u) \; , \quad u(0) = u_0 \; , \tag{37}$$

and assume that the following hypotheses hold.

(H1) *The operator A is a symmetric linear operator on the Hilbert space Z.*

Thus, A generates a linear analytic group $S(t) = \exp(At) : Z \to Z$, $t \in \mathbb{R}$.

(H2) *There exists a decomposition $Z = Z^- \oplus Z^+ \oplus Z^{++}$ such that these spaces are pairwise orthogonal with respect to $(\cdot,\cdot)_0$. Each of these three spaces is invariant with respect to both A and $S(t)$. We denote the restrictions of $S(t)$ and A to these subspaces by corresponding superscripts. Furthermore, we assume that there exist real constants*

$$0 < a^- < b^- < a^+ < b^+$$

such that for every $u^{++} \in Z^{++}$, $u^+ \in Z^+$, $u^- \in Z^-$ the estimates

$$\begin{aligned} ||S^{++}(t)u^{++}||_0 &\le e^{b^+ t} \cdot ||u^{++}||_0 && \textit{for} \quad t \le 0 \; , \\ ||S^{+}(t)u^{+}||_0 &\le e^{a^+ t} \cdot ||u^{+}||_0 && \textit{for} \quad t \ge 0 \; , \\ ||S^{+}(t)u^{+}||_0 &\le e^{b^- t} \cdot ||u^{+}||_0 && \textit{for} \quad t \le 0 \; , \\ ||S^{-}(t)u^{-}||_0 &\le e^{a^- t} \cdot ||u^{-}||_0 && \textit{for} \quad t \ge 0 \; , \end{aligned} \tag{38}$$

hold. Finally define

$$\hat{\gamma}^- := \frac{a^- + b^-}{2} \; , \qquad \hat{\gamma}^+ := \frac{a^+ + b^+}{2} \; . \tag{39}$$

According to this splitting we rewrite (37) in the form

$$\begin{aligned} \dot{u}^- &= A^- u^- + F^-(u^- + u^+ + u^{++}) \\ \dot{u}^+ &= A^+ u^+ + F^+(u^- + u^+ + u^{++}) \\ \dot{u}^{++} &= A^{++} u^{++} + F^{++}(u^- + u^+ + u^{++}) \; . \end{aligned} \tag{40}$$

We have to impose the following condition on the nonlinearity $F := F^- \oplus F^+ \oplus F^{++} : Z^- \oplus Z^+ \oplus Z^{++} \to Z^- \oplus Z^+ \oplus Z^{++}$.

(H3) *The mapping F satisfied $F(0) = 0$ and $F'(0) = 0$. It is globally Lipschitz continuous with Lipschitz constant L_F.*

The following theorem states that upon leaving a ball $B_R(0) \subset Z^- \oplus Z^+ \oplus Z^{++}$ most orbits of (40) originating "sufficiently close" to the equilibrium 0 stay close to the subspace $Z^+ \oplus Z^{++}$. Before that, however, we have to introduce a measure on the space $Z^- \oplus Z^+ \oplus Z^{++}$ which will be used for our probability assertion.

As in Subsection 2.2 for $N = \dim Z$ we denote by $\Upsilon^{(Z)}$ the standard Lebesgue measure on $Z = Z^- \oplus Z^+ \oplus Z^{++} \cong \mathbb{R}^N$.

Theorem 3.1 *Consider the equation (40) and assume that* **(H1)**, **(H2)** *are satisfied. Choose constants* $C_* > 0$, $R > 0$, $\rho \in (0, R)$, *define* $\sigma > 0$ *by the identity* $\sigma^2 + 1 = R^2/\rho^2$, *and let*

$$M_* := \min \left\{ \frac{1}{1 + 2C_*} \, , \, \left(\int_0^1 \left(1 - s^2\right)^{C_*/2} \, ds \right)^2 \right\} < 1 \, .$$

Furthermore, suppose that

$$\dim Z^- \leq C_* \cdot \dim Z^{++} \, ,$$

and that **(H3)** *holds with*

$$0 \leq L_F \leq \frac{\min\{b^- - a^-, b^+ - a^+\}}{6 + \sigma + 1/\sigma} \, . \tag{41}$$

Define $\hat{\gamma} := \hat{\gamma}^+/\hat{\gamma}^- > 1$, *let* $0 < p \ll 1$ *be arbitrary and assume that* r *satisfies*

$$0 < \frac{r}{R} \leq \left(\frac{\min\left\{p^2 \, , \, M_*\right\}}{6} \right)^{1/(\hat{\gamma}-1)} \cdot \left(\frac{\rho}{R} \right)^{\hat{\gamma}/(\hat{\gamma}-1)} \, . \tag{42}$$

Let M_r *denote the set of all initial conditions* $u_0 \in Z$ *of (40) satisfying* $||u_0||_0 < r$ *such that for the corresponding solution* $u(\cdot)$ *one of the following two properties holds.*

(i) The solution remains in $B_R(0) = \{u \in Z : ||u||_0 < R\}$ *for arbitrary* $t \geq 0$.

(ii) There exists a time $t^* > 0$ *such that* $||u(t^*)||_0 = R$ *and* $||u^-(t^*)||_0 \geq \rho$, *i.e., the forward orbit leaves* $B_R(0)$ *and upon leaving its* Z^-*-component has norm at least* ρ.

Then the estimate

$$\left(\frac{\Upsilon^{(Z)}(M_r)}{\Upsilon^{(Z)}(B_r(0))} \right) \leq p \tag{43}$$

holds.

Proof. Theorem 2.8 and Corollary 2.10 of [23]. □

The set M_r in the above theorem contains all "bad" initial conditions, i.e., those initial conditions which do not lead to orbits leaving the neighbourhood $B_R(0)$ close to $Z^+ \oplus Z^{++}$. On the other hand, the initial conditions in $B_r(0) \setminus M_r$ will lead to orbits which do leave $B_R(0)$ close to $Z^+ \oplus Z^{++}$, i.e., the behaviour of orbits originating in $B_r(0) \setminus M_r$ is dominated by the subspace $Z^+ \oplus Z^{++}$. The estimate (43) gives that in fact most initial values in $B_r(0)$ are dominated by $Z^+ \oplus Z^{++}$.

3.2 Setup for Discrete Cahn-Hilliard

Rather than looking at the spatially discrete version for $u = u(t) \in \mathbb{R}^{(n+1)^d}$

$$u_t = -\Delta_n(\epsilon^2 \Delta_n u + f(u)), \tag{44}$$

of the original equation (1) with mass constraint $\frac{1}{(n+1)^d} \sum_{i_j \in \{0,\dots,n\}^d} u_{i_j} = m$, we will rewrite the equation as was done in [16]. Consider a change of variables such that $w = u - m$. Then, after we replace w by u again, we can write (44) as

$$u_t = -\Delta_n(\epsilon^2 \Delta_n u + f(m+u)) \tag{45}$$

with mass constraint $\frac{1}{(n+1)^d} \sum_{i_j \in \{0,\dots,n\}^d} u_{i_j} = 0$. Now, define the nonlinear function $\tilde{F} : \mathbb{R}^{(n+1)^d} \to \mathbb{R}^{(n+1)^d}$ by

$$\tilde{F}(u) = -\Delta_n \tilde{f}(u), \tag{46}$$

where

$$\tilde{f}(u) := f(m+u) - f'(m)u - f(m). \tag{47}$$

Obviously $\tilde{f}(0) = \tilde{f}'(0) = 0$ holds. Using (22), this allows us to write (44) after the change of variables in the form

$$\dot{u} = \hat{A}_\epsilon u + \tilde{F}(u). \tag{48}$$

We wish to apply the results of Subsection 3.1 to this finite-dimensional problem. In order to apply these results we have to satisfy the assumptions **(H1)**, **(H2)**, and **(H3)**. The main restriction is the global Lipschitz condition on the nonlinearity, which will be achieved by considering a suitable small neighbourhood of the equilibrium 0. The maximal possible size of this neighbourhood will depend on the size of certain spectral gaps.

Lemma 3.2 *Assume that* **(A1)** *and* **(A2)** *of Section 2 are satisfied, so that in particular we have* $d \in \{1,2,3\}$. *Furthermore, fix two constants* $-1 < c_* < c^* < 1$ *and let* $\lambda_\epsilon^{\max}$ *be defined as in (6).*

Then there exist constants $\epsilon_0, \alpha_0 > 0$ *depending only on* c_*, c^*, d, *and* $f'(m)$ *such that for arbitrary* $0 < \epsilon \le \epsilon_0$ *and* $n \ge n^*_{\hat{\rho}}/\epsilon$ *as in (29) with* $0 < \hat{\rho} < \min\{\frac{1}{60 f'(m)} \cdot \delta_c, 0.45\}$ *and* $\delta_c := \frac{c^* - c_*}{4}$ *the following holds. The linear operator* $\hat{A}_\epsilon$ *has eigenvalues* $\lambda_*(\epsilon)$ *and* $\lambda^*(\epsilon)$ *satisfying both*

$$c_* \cdot \lambda_\epsilon^{\max} \le \lambda_*(\epsilon) < \lambda^*(\epsilon) \le c^* \cdot \lambda_\epsilon^{\max}$$

and

$$\lambda^*(\epsilon) - \lambda_*(\epsilon) \ge \alpha_0 \cdot \epsilon^{d-2} .$$

Moreover, the whole interval $(\lambda_*(\epsilon), \lambda^*(\epsilon))$ *is part of the resolvent set of* $\hat{A}_\epsilon$.

Proof. Using (4) and Corollary 2.8 we obtain immediately

$$\#\{\tilde{j} \in \{0,...,n\}^d \setminus \{0\} \mid (c_* + \delta_c)\lambda_\epsilon^{\max} \le \lambda_\epsilon(\mu_{\tilde{j}}) \le (c^* - \delta_c)\lambda_\epsilon^{\max}\}$$
$$\le \#\{\tilde{j} \in \mathbb{N}_0^d \setminus \{0\} \mid c_* \cdot \lambda_\epsilon^{\max} \le \lambda_\epsilon(\kappa_{\tilde{j}}) \le c^* \lambda_\epsilon^{\max}\} \le C \cdot \epsilon^{-d}.$$

Hence,

- the number of eigenvalues of $\hat{A}_\epsilon$ in the interval $(c_* + \delta_c, c^* - \delta_c) \cdot \lambda_\epsilon^{\max}$ is bounded above by $C \cdot \epsilon^{-d} - 1$, where C depends only on c_*, c^*, d, and $f'(m)$,

and similarly,

- both in the interval $[c^* - \delta_c, c^*] \cdot \lambda_\epsilon^{\max}$ and in $[c_*, c_* + \delta_c] \cdot \lambda_\epsilon^{\max}$ there is at least one eigenvalue of $\hat{A}_\epsilon$.

Therefore the conclusion can be drawn along the lines of the proof of Lemma 3.2 in [23]. □

The above lemma states that it is possible to find gaps in the spectrum of $\hat{A}_\epsilon$ whose size can be controlled as $\epsilon \to 0$. We will use this fact to obtain the spectral gaps needed for assumption **(H2)** and **(H3)** in Subsection 3.1 (cf. (41)). To this end, choose constants

$$0 \ll \underline{c}^- < \overline{c}^- < \underline{c}^+ < \overline{c}^+ < 1 \,, \tag{49}$$

where typically the differences $\overline{c}^- - \underline{c}^-$ and $\overline{c}^+ - \underline{c}^+$ will be small. The following results are an immediate consequence of Lemma 3.2. They give the appropriate choices of the spectral gaps for the operators $\hat{A}_\epsilon$ needed for applying the results of the last subsection to the Cahn-Hilliard equation.

Corollary 3.3 *Assume that the assumptions of Lemma 3.2 are satisfied. Then with the constants from (49) there exist intervals*

$$\begin{array}{rcccccl} J_\epsilon^- & := & [a_\epsilon^-, b_\epsilon^-] & \subset & [\underline{c}^-, \overline{c}^-] & \cdot & \lambda_\epsilon^{\max} \,, \\ J_\epsilon^+ & := & [a_\epsilon^+, b_\epsilon^+] & \subset & [\underline{c}^+, \overline{c}^+] & \cdot & \lambda_\epsilon^{\max} \end{array}$$

such that for sufficiently small $\epsilon > 0$ the following holds.

(a) Each of the intervals J_ϵ^-, J_ϵ^+ is contained in the resolvent set of $\hat{A}_\epsilon$.

(b) There is an ϵ-independent constant $C > 0$ such that the length of each of the intervals J_ϵ^- and J_ϵ^+ is at least $C \cdot \epsilon^{d-2}$. The constant C depends only on d, $f'(m)$, and the constants in (49).

As a consequence of this corollary, we can now define the subspace decomposition of $Y = \vec{1}^\perp \subset \mathbb{R}^{(n+1)^d}$ needed for applying the results of Subsection 3.1.

Definition 3.4 *Using the constants introduced in Corollary 3.3, define the intervals $\tilde{I}_\epsilon^- := (-\infty, a_\epsilon^-)$, $\tilde{I}_\epsilon^+ := (b_\epsilon^-, a_\epsilon^+)$, and $\tilde{I}_\epsilon^{++} := (b_\epsilon^+, \lambda_\epsilon^{\max}]$. Furthermore, let Z_ϵ^-, Z_ϵ^+, and Z_ϵ^{++} denote the sum of all eigenvectors of the operator $\hat{A}_\epsilon$ corresponding to eigenvalues in $\tilde{I}_\epsilon^-$, $\tilde{I}_\epsilon^+$, and $\tilde{I}_\epsilon^{++}$, respectively.*

As in Subsection 2.1.3 we denote the restrictions of $\hat{A}_\epsilon$ or the corresponding (linear) analytic semigroup $\hat{S}_\epsilon(t)$ to each of the subspaces defined above by the appropriate superscript. Notice that this general setting is simpler than that of the continuous case because the equations define a flow rather than a semiflow. This is true because the subspaces of Y are finite dimensional. With these definitions we can proceed to verifying the two hypotheses **(H1)** and **(H2)** from Subsection 3.1 for the linearization of the Cahn-Hilliard equation.

Lemma 3.5 *Assume that* **(A1)** *and* **(A2)** *hold. Let* $\hat{A}_\epsilon : Y \to Y$ *denote the discrete linear operator from (22), let* $\hat{S}_\epsilon(t) : Y \to Y$, $t \in \mathbb{R}$, *denote the corresponding analytic group. Furthermore, consider the constants and intervals introduced in Corollary 3.3 and Definition 3.4 with* n *further restricted to* $n \le \hat{n}/\epsilon$ *for some given* $\hat{n} > n^*_{\hat{\rho}}$. *Then the following assertions hold for arbitrary* $0 < \epsilon \le \epsilon_0$, *where* ϵ_0 *depends only on* d, *the derivative* $f'(m)$, *and the constants in (49).*

(a) The subspaces Z_ϵ^-, Z_ϵ^+, *and* Z_ϵ^{++} *have dimensions proportional to* ϵ^{-d}. *Furthermore, the spaces* Z_ϵ^-, Z_ϵ^+, *and* Z_ϵ^{++} *are orthogonal with respect to the Euclidean scalar product* $< \cdot, \cdot >$ *in* $Y \subset \mathbb{R}^{(n+1)^d}$.

(b) For every $u^{++} \in Z_\epsilon^{++}$, $u^+ \in Z_\epsilon^+$, $u^- \in Z_\epsilon^-$, *the estimates corresponding to (38) such as for instance*

$$||S_\epsilon^{++}(t)u^{++}|| \le e^{b_\epsilon^+ t} \cdot ||u^{++}|| \quad \textit{for} \quad t \le 0 \, ,$$

hold. Here $||\cdot||$ *is the Euclidean norm in* $Y \subset \mathbb{R}^{(n+1)^d}$.

Proof. The assertions of *(a)* follow easily from Corollary 2.8, (4), Corollary 3.3, and Definition 3.4. The expansion (25) can be used to show *(b)*. □

According to the above lemma the linear part of the discrete Cahn-Hilliard equation (48) satisfies both **(H1)** and **(H2)** from Subsection 3.1 on the finite dimensional space $Y = Z_\epsilon^- \oplus Z_\epsilon^+ \oplus Z_\epsilon^{++}$. Moreover, the asymptotic behaviour for $\epsilon \to 0$ of certain spectral gaps in the spectrum of $\hat{A}_\epsilon$ leads to the following remark.

Remark 3.6 Using the results of Corollary 3.3 and Lemma 3.5 we can deduce that the constant introduced in (41) takes the form

$$C_\epsilon^+ = \frac{\min\{b_\epsilon^- - a_\epsilon^-, b_\epsilon^+ - a_\epsilon^+\}}{6 + \sigma + 1/\sigma} \, ,$$

and from Corollary 3.3 we deduce $C_\epsilon^+ \ge C \cdot \epsilon^{d-2}$ for $\epsilon \to 0$. Hence, **(H3)** with (41) is satisfied if $0 \le L_F \le C \cdot \epsilon^{d-2}$. The above constant C depends only on d, $f'(m)$, the constants in (49), and on σ.

For the discrete Cahn-Hilliard equation (48) the nonlinearity $\tilde{F}$ is continuously differentiable, but it does not satisfy the strong global Lipschitz condition of hypothesis **(H3)**. However, this can be achieved by the following standard cut-off result.

Lemma 3.7 *Assume that* **(A2)** *holds and consider* $\tilde{f}$ *from (47). Then there exists a constant* $\alpha = \alpha_{\tilde{f}} > 0$ *such that the following is true. For every Lipschitz constant* $0 < L < 1$ *there exists a continuously differentiable mapping* $\hat{f} : \mathbb{R} \to \mathbb{R}$ *satisfying*

$$|\hat{f}(u) - \hat{f}(v)| \leq L|u - v| \quad \textit{for all} \quad u, v \in \mathbb{R},$$

as well as

$$\hat{f}(u) = \tilde{f}(u) \quad \textit{for all} \quad u \in \mathbb{R} \quad \textit{with} \quad |u| \leq \alpha \cdot L\ .$$

It is easy to apply this lemma to our situation.

Lemma 3.8 *Assume that* **(A1)** *and* **(A2)** *hold. Consider* $\tilde{F} : \mathbb{R}^{(n+1)^d} \to \mathbb{R}^{(n+1)^d}$, $u \mapsto -\Delta_n \tilde{f}(u)$ *from (46). Then there exists a constant* $\alpha = \alpha_{\tilde{f}} > 0$ *such that for all* $0 < L \leq 1$ *there exists a mapping* $\hat{F} : \mathbb{R}^{(n+1)^d} \to \mathbb{R}^{(n+1)^d}$ *with*

$$||\hat{F}(u) - \hat{F}(v)|| \leq d(n+1)^2 L||u - v|| \quad \textit{for all} \quad u, v \in \mathbb{R}^{(n+1)^d}$$

and

$$\hat{F}(u) = \tilde{F}(u) \quad \textit{for all} \quad u \in \mathbb{R}^{(n+1)^d} \quad \textit{with} \quad ||u|| \leq \alpha \cdot L. \tag{50}$$

Proof. Let $\hat{F}(u) := -\Delta_n(\hat{f}(u))$ with $\hat{f}$ from Lemma 3.7. Then (50) is obviously satisfied. We have

$$\begin{aligned} ||\hat{F}(u) - \hat{F}(v)|| &\leq ||\Delta_n||_{\mathcal{L}(\mathbb{R}^{(n+1)^d})} \cdot ||\hat{f}(u) - \hat{f}(v)|| \\ &\leq \max_{\tilde{j} \in \{0,\ldots,n\}^d} \{\mu_{\tilde{j}}\} \cdot L||u - v|| \quad \text{for all} \quad u, v \in \mathbb{R}^{(n+1)^d}, \end{aligned}$$

and the assertion follows. □

Thus it is clear that hypothesis **(H3)** with (41) in our situation will be valid for that function $\hat{F} : \mathbb{R}^{(n+1)^d} \to \mathbb{R}^{(n+1)^d}$ which coincides with the function $\tilde{F}$ defined in (46) on a certain neighbourhood of the origin, and, assuming $n \sim 1/\epsilon$, the size of this neighbourhood will be proportional to ϵ^d. This fact will be used in the next subsection to explain the phenomenon of spinodal decomposition in the discrete case.

3.3 Spinodal Decomposition

In the previous subsection we have established all the properties we need for applying the abstract results from Subsection 3.1 to the discrete Cahn-Hilliard equation. Our standing hypothesis is as follows.

Assumption 3.9 *We discuss solutions of a discretized Cahn-Hilliard equation (44) for $\epsilon > 0$, where the domain $\Omega \subset \mathbb{R}^n$ and the nonlinearity f satisfy the standing assumptions of Section 2,* **(A1)** *and* **(A2)***, respectively. The discretization is chosen in such a way that $n \sim 1/\epsilon$, i.e., $n \in [n^*_{\hat{\rho}}/\epsilon, \hat{n}/\epsilon]$, for some fixed $\hat{n} \in \mathbb{N}$. $n^*_{\hat{\rho}}$ is chosen as in Lemma 3.2, (19) and (29).*

Remark 3.10 *The above assumption $n \geq n^*_{\hat{\rho}}/\epsilon$ with accuracy $\hat{\rho}$ as needed in Lemma 3.2 gives the condition*

$$n \geq \frac{1}{\epsilon} \cdot \sqrt{\frac{120 f'(m)^2}{\hat{\delta}_c}}, \quad \textit{where} \quad \hat{\delta}_c := \min\{\overline{c}^+ - \underline{c}^+, \overline{c}^- - \underline{c}^-\}.$$

Therefore the proportionality constant between n and $1/\epsilon$ increases for γ^- closer to 1, i.e., when the strongly dominant subspace is smaller.

According to Subsection 3.2 the discrete Cahn-Hilliard equation may be reformulated as an abstract evolution equation

$$u_t = \hat{A}_\epsilon u + \tilde{F}(u) \tag{51}$$

with equilibrium 0, whose solutions correspond to solutions of the nontranslated discrete Cahn-Hilliard equation (44) via the translation $u \to u + m$. After choosing a set of constants $\underline{c}^-$, $\overline{c}^-$, $\underline{c}^+$, and $\overline{c}^+$ as in (49) we may apply the foregoing results:

According to Remark 3.6 and Lemma 3.8 there exists a constant $\epsilon_0 > 0$ (depending only on d, $f'(m)$, and the constants in (49)) such that for every $0 < \epsilon \leq \epsilon_0$ there is a neighbourhood

$$U_\epsilon = m + B_{R_\epsilon}(0) \subset m + Y \subset m + \mathbb{R}^{(n+1)^d}$$

of the homogeneous equilibrium $\bar{u}_0 \equiv m$ of (44) with the following properties. The radius R_ϵ is proportional to ϵ^d as $\epsilon \to 0$, and on $U_\epsilon - m$ the function $\tilde{F}$ satisfies a Lipschitz condition with Lipschitz constant of the order $O(\epsilon^{d-2})$. Next we modify $\tilde{F}$ outside the neighbourhood $U_\epsilon - m$ using Lemma 3.8. This furnishes a new function $\hat{F}$ which satisfies a global Lipschitz condition with Lipschitz constant $0 \leq L_{\hat{F}} \leq C \cdot \epsilon^{d-2}$ as required in Remark 3.6. Inside of $U_\epsilon - m$, both $\tilde{F}$ and $\hat{F}$ coincide. Notice that the proportionality constant of $R_\epsilon \sim \epsilon^d$ depends on d, $f'(m)$, $\hat{n}$, σ, and the constants in (49).

Using the spectral gaps and dichotomy estimates of Subsection 3.2 we may now apply all the results of Section 3.1 to the family of evolution equations

$$u_t = \hat{A}_\epsilon u + \hat{F}(u) , \quad 0 < \epsilon \leq \epsilon_0 . \tag{52}$$

The global results for (52) will furnish local results for (51) when restricted to the neighbourhood $U_\epsilon - m$.

Next let us define the analogues of the sets M_r introduced in Subsection 3.1. Recall that we called the initial conditions from these sets "bad", because the behaviour of the corresponding orbits was not dominated by the strongly unstable subspace $Z^+ \oplus Z^{++}$. We use the notation

$$B_{r,\epsilon}(0) = \{v \in Z_\epsilon^- \oplus Z_\epsilon^+ \oplus Z_\epsilon^{++} \;:\; ||v|| < r\} \tag{53}$$

for balls in the finite-dimensional subspace $Z_\epsilon^- \oplus Z_\epsilon^+ \oplus Z_\epsilon^{++}$ of $Y \subset \mathbb{R}^{(n+1)^d}$. For arbitrary, but fixed, $0 < c_1 \ll 1$ we set

$$\rho_\epsilon := c_1 \cdot R_\epsilon \;, \tag{54}$$

so that ρ_ϵ is also proportional to ϵ^d as $\epsilon \to 0$. We define for $r \in (0, \rho_\epsilon)$ the set

$$\begin{aligned} M_{r,\epsilon} = \{ & v_0 \in Z_\epsilon^- \oplus Z_\epsilon^+ \oplus Z_\epsilon^{++} \;:\; ||v_0|| < r \text{ and the corresponding} \\ & \text{solution } u(\cdot) = u(\cdot; v_0) \text{ of (45) to this initial condition} \\ & \text{satisfies either (i) or (ii) below } \} \;, \end{aligned} \tag{55}$$

where (i) and (ii) are defined as follows :

(i) $u(t)$ remains in the ball $B_{R_\epsilon,\epsilon}(0)$ for arbitrary nonnegative times $t \geq 0$.

(ii) There exists an exit time $t^* > 0$ such that both

$$||u(t^*)|| = R_\epsilon \quad \text{ and } \quad ||P_\epsilon^-(u(t^*))|| \geq \rho_\epsilon$$

are satisfied.

Here $P_\epsilon^- : \mathbb{R}^{(n+1)^d} \to Z_\epsilon^-$ denotes the orthogonal projection. The ratio of the volumes of $M_{r,\epsilon}$ and $B_{r,\epsilon}(0)$ can now finally be estimated using Theorem 3.1.

Theorem 3.11 *We consider solutions of the discrete Cahn-Hilliard equation (44) and assume that Assumption 3.9 holds. Then there exists a constant* $\epsilon_0 > 0$ *(depending only on d,* $f'(m)$*, and the constants in (49)) such that for every* $0 < \epsilon \leq \epsilon_0$ *the above definitions of constants and sets are well-defined for an arbitrary, but fixed, constant* $0 < c_1 \ll 1$*, cf. (53), (54) and (55).*

Then for every $0 < p \ll 1$ *there exists a radius* $r_0 = r_0(\epsilon)$ *(depending on* c_1*, p,* $f'(m)$*, d,* $\hat{n}$*, and the constants in (49)) which is proportional to* ϵ^d *as* $\epsilon \to 0$ *such that*

$$\frac{\text{vol}(M_{r,\epsilon})}{\text{vol}(B_{r,\epsilon}(0))} \;\leq\; p \tag{56}$$

for all $0 < r \leq r_0(\epsilon)$ *and every* $0 < \epsilon \leq \epsilon_0$*.*

Proof. The proof is along the lines of the proof of Theorem 3.11 in [23]. Since $\tilde{F}$ and $\hat{F}$ coincide on $U_\epsilon - m = B_{R_\epsilon}(0)$ we can argue for the solutions of (52). The ratio $\dim Z_\epsilon^- / \dim Z_\epsilon^{++}$ is bounded by an ϵ-independent constant $C_* > 0$ according to Lemma 3.5*(a)*. Therefore Theorem 3.1 yields the estimate (56) for all $r \le r_0(\epsilon)$, where due to (42) we have

$$r_0(\epsilon) = R_\epsilon \cdot \left(\frac{\min\{p^2, M_*\}}{6} \right)^{1/(\gamma_\epsilon - 1)} \cdot \left(\frac{\rho_\epsilon}{R_\epsilon} \right)^{\gamma_\epsilon/(\gamma_\epsilon - 1)} \sim \epsilon^d$$

with

$$\gamma_\epsilon^- = (a_\epsilon^- + b_\epsilon^-)/2, \quad \gamma_\epsilon^+ = (a_\epsilon^+ + b_\epsilon^+)/2,$$

and

$$\gamma_\epsilon := \frac{\gamma_\epsilon^+}{\gamma_\epsilon^-} \in \left[\underline{c}^+/\overline{c}^-, \overline{c}^+/\underline{c}^- \right] \subset (1, \infty) \, ,$$

since $\rho_\epsilon = c_1 \cdot R_\epsilon$ and R_ϵ is proportional to ϵ^d as $\epsilon \to 0$. □

4 Conclusions

We have studied spinodal decomposition for the spatially discrete Cahn-Hilliard equation. In this analysis, the equation was divided into a linear part and a nonlinear part. The analysis for the nonlinear part followed similarly to that for the Cahn-Hilliard PDE. For the linear part, it was shown that if the linear operator for the discrete problem was a good approximation to the linear operator for the continuous problem, then spinodal decomposition occurs in the semidiscretization. Spinodal decomposition takes place when n is close to, or greater than $1/\epsilon$, and when the initial condition starts close enough to m.

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References

[1] N.D. Alikakos, P.W. Bates and X. Chen, Convergence of the Cahn-Hilliard equation to the Hele-Shaw model, *Arch. Ration. Mech. An.*, **128**, (1994) 165-205.

[2] N.D. Alikakos, L. Bronsard and G. Fusco, Slow motion in the gradient theory of phase transitions via energy and spectrum, *Calc. Var. Partial Dif.*, **6**, (1998) 39-66.

[3] P.W. Bates and J. Xun, Metastable patterns for the Cahn-Hilliard equation. I, *J. Differ. Equations*, **111**, (1994) 421-457.

[4] P.W. Bates and J. Xun, Metastable patterns for the Cahn-Hilliard equation. I. Layer dynamics and slow invariant manifold, *J. Differ. Equations*, **117**, (1995) 165-216.

[5] L. Bronsard and D. Hilhorst, On the slow dynamics for the Cahn-Hilliard equation in one space dimension, *Proc. R. Soc. London, Ser. A*, **439**, (1992) 669-682.

[6] J.W. Cahn, On spinodal decomposition, *Acta Metall.*, **9**, (1961) 795-801.

[7] J.W. Cahn, Phase separation by spinodal decomposition in isotropic systems, *J. Chem. Phys.*, **42**, (1965) 93-99.

[8] J.W. Cahn, Spinodal decomposition, *T. Metall. Soc. AIME*, **242**, (1968) 166-180.

[9] J.W. Cahn, S.N. Chow and E.S. Van Vleck, Spatially discrete nonlinear diffusion equations, *Rocky Mt. J. Math.*, **25**, (1995) 87-117.

[10] J.W. Cahn and J.E. Hilliard, Free energy of a nonuniform system I. Interfacial free energy, *J. Chem. Phys.*, **28**, (1958) 258-267.

[11] R. Courant and D. Hilbert, Methods of Mathematical Physics. Vol. I, Interscience Publishers, Inc., New York, 1953.

[12] C.M. Elliott, The Cahn-Hilliard model for the kinetics of phase separation, *Inter. Ser. Numer. Math.*, **88**, (1989) 35-73.

[13] C.M. Elliott and D.A. French, Numerical studies of the Cahn-Hilliard equation for phase separation, *IMA J. Appl. Math.*, **38**, (1987) 97-128.

[14] C.M. Elliott and S. Zheng, On the Cahn-Hilliard equation, *Arch. Ration. Mech. An.*, **96**, (1986) 339-357.

[15] P.C. Fife, Models for phase separation and their mathematics, *Electron J. Differ. Equ.*, **48**, (2000) 1-26.

[16] C.P. Grant, Spinodal decomposition for the Cahn-Hilliard equation, *Commun. Part. Diff. Eq.*, **18**, (1993) 453-490.

[17] C.P. Grant and E.S. Van Vleck, Slowly-migrating transition layers for the discrete Allen-Cahn and Cahn-Hilliard equations, *Nonlinearity*, **8**, (1995) 861-876.

[18] W. Hackbusch, Theorie und Numerik elliptischer Differentialgleichungen, B.G. Teubner, Stuttgart, 1996.

[19] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Springer-Verlag, Berlin, 1981.

[20] B. Hubbard, Bounds for eigenvalues of the free and fixed membrane by finite difference methods, *Pacific J. Math.*, **11**, (1961) 559–590.

[21] J.R. Kuttler, Estimating Eigenvalues with a posteriori/a priori Inequalities, *Research Notes in Mathematics*, Pitman, **135**, (1985).

[22] S. Maier-Paape and T. Wanner, Spinodal Decomposition for the Cahn-Hilliard Equation in Higher Dimensions. Part I: Probability and Wavelength Estimate, *Commun. Math. Phys.*, **195**, (1998) 435-464.

[23] S. Maier-Paape and T. Wanner, Spinodal Decomposition for the Cahn-Hilliard Equation in Higher Dimensions. Nonlinear Dynamics, *Arch. Ration. Mech. An.*, **151**, (2000) 187-219.

[24] L. Modica, The gradient theory of phase transitions and the minimal interface criterion, *Arch. Ration. Mech. An.*, **98**, (1987) 123-142.

[25] E.-M. Nash, Finite-Elemente und Spektral-Galerkin Verfahren zur numerischen Lösung der Cahn-Hilliard Gleichung und verwandter nichtlinearer Evolutionsgleichungen, Institut für Mathematik, Universität Augsburg, Shaker, Aachen, 2000.

[26] B. Nicolaenko and B. Scheurer, Low-dimensional behavior of the pattern formation Cahn-Hilliard equation, *Trends in the theory and practice of nonlinear analysis (Arlington, Tex., 1984)*, North-Holland, Amsterdam, (1985) 323–336.

[27] A. Novick-Cohen and L.A. Segel, Nonlinear aspects of the Cahn-Hilliard equation, *Physica D*, **10**, (1984) 277-298.

[28] F. Pockels, Über die partielle Differentialgleichung $\Delta u + k^2 u = 0$ und deren Auftreten in der mathematischen Physik, Teubner, Leipzig, 1891.

[29] R.L. Pego, Front migration in the nonlinear Cahn-Hilliard equation, *Proc. R. Soc. London, Ser. A*, **422**, (1989) 261-278.

[30] S.M. Rankin III, Semilinear evolution equations in Banach spaces with application to parabolic partial differential equations, *T. Am. Math. Soc.*, **336**, (1993) 523-535.

[31] T.M. Rogers, K.R. Elder and R.C. Desai, Numerical study of the late stages of spinodal decomposition, *Physics. Rev. B*, **37**, (1988) 9638-9649.

[32] E. Sander and T. Wanner, Monte Carlo simulations for spinodal decomposition, *J. Stat. Phys.*, **95**, (1999) 925-948.

[33] E. Sander and T. Wanner, Unexpectedly linear behavior for the Cahn-Hilliard equation, *SIAM J. Appl. Math.*, **60**, (2000) 2182-2202 (electronic).

[34] B.E.E. Stoth, Convergence of the Cahn-Hilliard equation to the Mullins-Sekerka problem in spherical symmetry, *J. Differ. Equations*, **125**, (1996) 154-183.

[35] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Second ed., Springer-Verlag, New York, 1997.

[36] J.D. van der Waals, The thermodynamic theory of capillarity flow under the hypothesis of a continuous variation in density, *Verhandelingen der Koninklijke Nederlandsche Akademie van Wetenschappen te Amsterdam*, **1**, (1893) 1-56.

[37] S. Zheng, Asymptotic behavior of solutions to the Cahn-Hilliard equation, *Applied Analysis*, **23**, (1986) 165-184.

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