LET'S GET STARTED WITH NUMBERS

Xin Li
Math Circle, Spring 2018
University of Central Florida

WHAT IS A NUMBER?

Give me an example of a number

Give me an example of a natural number

Give me an example of an integer

Can you describe all the integers

0, 1, -1, 2, -2, 3, -3, …

…, -3, -2, -1, 0, 1, 2, 3, …

Let's focus on the properties of integers

That is the study of "number theory"

EVEN V.S. ODD

We can divide the set of all integers into two separate groups: even integers and odd integers

So, an integer is either even or odd but cannot be both

Give me an example of an even integer

Give me an example of an odd integer

How do you check if an integer is even or not? (What is an even integer?)

DEFINITION OF AN EVEN INTEGER

In words: An integer is call even if it is a multiple of 2

In symbol (representation):

n is even if $n = 2k$ for some integer k.

In symbol (easier, but harder to use):

n is even if n is divisible by 2

DEFINITION OF AN ~~EVEN~~ODD INTEGER

In words: An integer is call ~~even~~odd if it is NOT a multiple of 2

In symbol (representation):

n is ~~even~~odd if $n = 2k+1$ for some integer k.

In symbol (easier, but harder to use):

n is even if n is NOT divisible by 2

PROPERTIES OF EVEN NUMBERS

Squares

Pick an even integer

Square the even integer

Is the resulting integer even or odd?

Repeat

What is the pattern you observed?

The square of an even integer is again even.

"THE SQUARE OF AN EVEN INTEGER IS AGAIN EVEN"

Is this always true?

We have not (and could not) check it for every even integer.

How to convince others?

We need to use logic!

Let's first imagine the following conversation

AN IMAGINARY CONVERSATION

Us: Pick an even integer

Stranger: 12

Us: How do you know that 12 is even?

Stranger: $12 = 2 \times 6$

Us: Note that

$$
\begin{aligned}
12^2 &= 12 \times 12 \\
&= 2 \times 6 \times 2 \times 6 \quad \textit{(why not use 144?)} \\
&= 2 \times (6 \times 2 \times 6)
\end{aligned}
$$

Which is even (why?)

AN IMAGINARY CONVERSATION

Us: Pick an even integer

Stranger: -234500000000000000000000000068

Us: How do you know that -234500000000000000000000000068 is even?

Stranger: $-23450\ldots068 = 2 \times (-117250..034)$

Us: Note that
$(-23450\ldots068)^2 = 23450\ldots068 \times 23450\ldots068$

$= 2 \times 117250..034 \times 2 \times 117250..034$ *(why not use a calculator? Your calculator wont work!)*

$= 2 \times (117250..034 \times 2 \times 117250..034)$

Which is even (why?)

DO YOU SEE ANY PATTERN?

Is it important to multiply it out, like writing $12^2 = 144, \quad 234568^2 = \cdots$?

AN IMAGINARY CONVERSATION

Us: Pick an even integer

Stranger: n

Us: How do you know that n is even?

Stranger: $n = 2k$ for some integer k

Us: Note that
$n^2 = n \times n = 2k \times 2k = 2 \times (k \times 2k)$

(No need to multiply out $k \times 2k$: *It is an integer!)*

Which is even (why?)

WE GOT A PROOF!

"The square of an even integer is again even."

Proof.

Pick an even integer n.

Then $n = 2k$ for some integer k

Note that

$$n^2 = (2k) \times (2k) = 2 \times (k \times 2k),$$

which is even (by the definition of an even integer) since $k \times 2k$ is an integer.

This completes our proof.

EXERCISE

What can you say about the square of an odd integer?

Experiment with a few odd integers.

What is the pattern?

Can you verify (that is prove) your claim?

WHAT IS MATHEMATICS?

Is it about numbers?

Is it about geometry?

Is it about algebra?

???

It is about recognizing “patterns”!

All the rules you have learned so far in your mathematics classes are statement of patterns

1+1=2 is a pattern: 1 apple and 1 apple make 2 apple; 1 orange and 1 orange make 2 orange;…

RATIONAL AND IRRATIONAL NUMBERS

Give me a fraction

Rational numbers

Pick a rational number r. Then $r = \frac{m}{n}$ for some integers m, n.

Irrational numbers

Not rational !

Give me an example of an irrational number.

$\sqrt{2}$ IS AN IRRATIONAL NUMBER

How to prove this claim?

Proof by contridiction.

Assume that, on the contrary, $\sqrt{2}$ is a rational number.

Then $\sqrt{2} = \frac{m}{n}$

If both m and n are even, we cancel factor 2 from m and n to simplify the fraction. Repeat this until one of m and n is odd and the other is even.

Squaring both sides:

$2 = \frac{m^2}{n^2}$ or $2n^2 = m^2$

PROOF CONTINUED

So, m^2 is even.

Then m must be even (as we discussed before).

Thus, $m = 2k$ for some integer k.

Use this back to the equality $2n^2 = m^2$, we get

$$2n^2 = (2k)^2 = 4k^2.$$

So, $n^2 = 2k^2$.

PROOF CONTINUED

We see that n^2 is even and so n itself is even.

So far, we have show that both m and n are even, contradicting our simplification above.

This contradiction implies that our assumption that $\sqrt{2}$ is a rational number must be wrong.

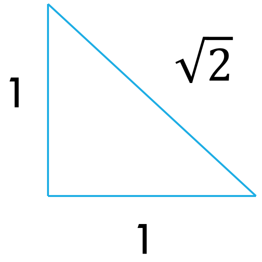
Done

A LITTLE BIT ABOUT HISTORY

Hippasus was credited for the discovery of the rationality of $\sqrt{2}$, the most famous irrational !

But he was drown to death due to his discovery by the Pythagoreans

It is an evil number for the Pythagoreans since they believe that all lengths (numbers) are constructible using ruler and a unit length.



EUCLID'S ELEMENT

Book X, Prop. 117 contains a proof by Aristotle

Aristotle, 384-324BC