

Magic of numbers and Fibonacci Sequence

March 10, 2018

Mathematics Circle

Natural Numbers

- Kronecker, the German Mathematician said the following:
- "GOD CREATED THE NATURAL NUMBERS, THE REST ARE MAN'S HANDIWORK"
- Natural numbers are the counting numbers:
- 1, 2, 3, 4,.....

Nothing Becomes a Number

- Most of us think of zero by itself as nothing.
- It's story began in Mesopotamia, the cradle of civilization.
- The Babylonians had developed a place-value system for written numbers using two wedge-shaped symbols and base 60.
- We count a minute as sixty seconds and an hour as 60 minutes.

Story of Zero (contd.)

- Sometimes between 700 and 300 BC they introduced a . (dot) as a place holder. Some math. historians say that zero began as a place holder.
- The present decimal representation of number uses digits 1 to 9 and 0 was the contribution of Indian mathematicians and astronomers.
- The Arab merchants learned it and shared it with others.

Imagine life without Zero

- We write

$$18 = 1 \times 10 + 8$$

$$128 = 1 \times 100 + 2 \times 10 + 8$$

This has made life much simpler because of the place values being powers of 10 which essentially adds zeros to 1.

Also fractions could be represented as decimals.

Imagine writing numbers in base 60.

An activity with numbers

- Pick any three digit number with different digits in ones and hundreds place.
- Reverse the order of the digits
- (For example 182 becomes 281)
- Subtract the smaller number from the larger number ($281 - 182 = 099$)
- Reverse the order of digits of the difference: (It becomes 990)
- Always the sum is 1089: ($099 + 990 = 1089$)

Activity 2

- Take the year when you were born.
- To this add the year of an important event in your life.
- To this sum add the age you will be at the end of 2018.
- Finally add to this sum the number of years ago that important event took place.
- **The answer will always be.....**

Figurate Numbers

- Numbers can get their connotations from different geometric figures like triangles, squares, pentagons etc..
- These numbers have many interesting properties.
- We begin with a point and then the three vertices of a triangle.

Prime Numbers

- A prime number is a natural number greater than 1 whose only divisors (or factors) are 1 and itself.
- A natural number which is not a prime is a composite number.
- 2, 3, 5, 7, 11, 13, 17, 19, 23 are examples of prime numbers.
- 4, 6, 10, 12, 21, 25 are examples of composite numbers.

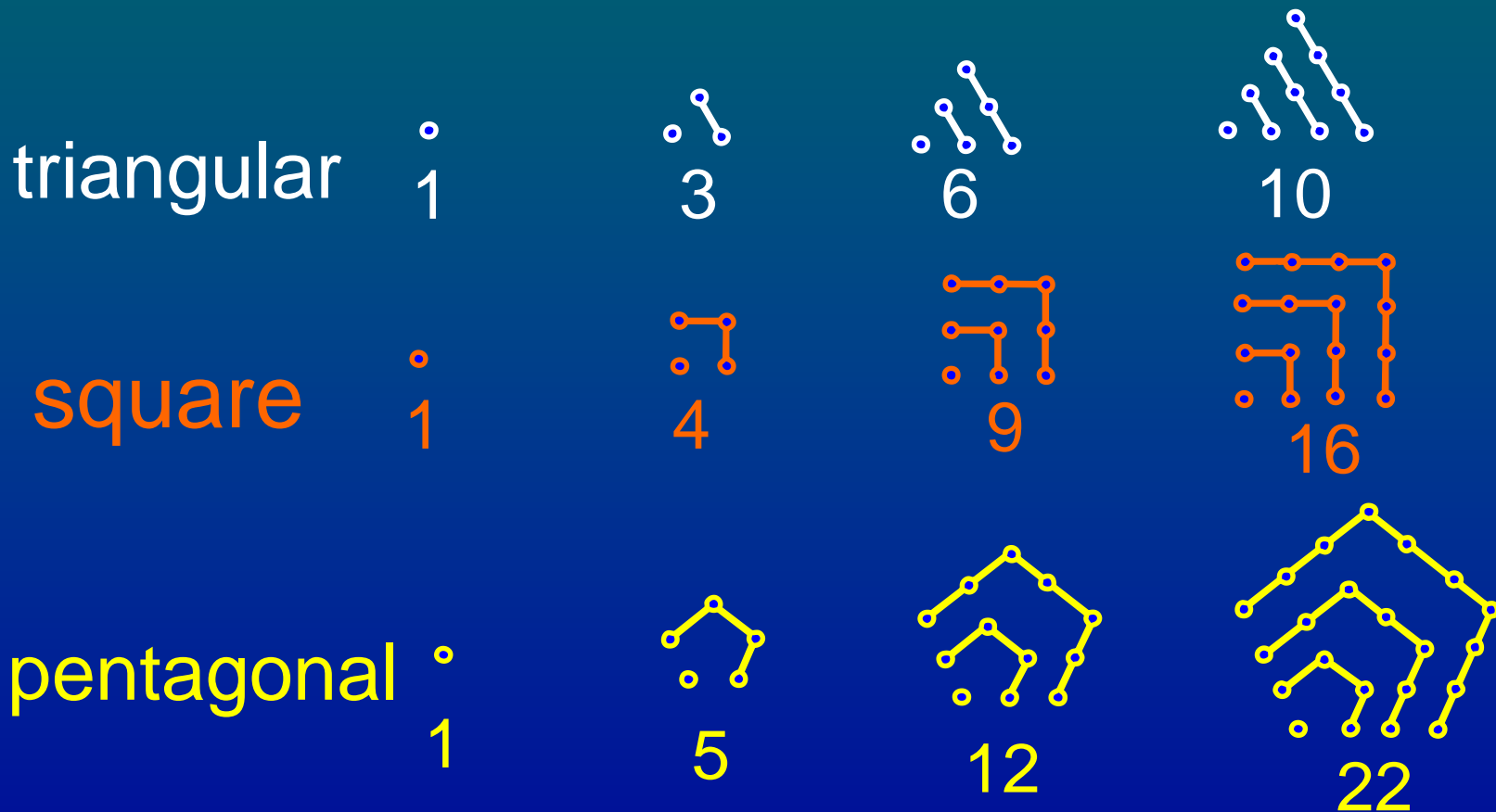
The Fibonacci Sequence



- The sequence begins with one. Each subsequent number is the sum of the two preceding numbers.
- $\text{Fib}(n) = \text{Fib}(n-1) + \text{Fib}(n-2)$
- Thus the sequence begins as follows:
- 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144....

Polygonal, Prime and Perfect Numbers

- Greeks tried to transfer geometric ideas to number theory. One of such attempts led to the appearance of polygonal numbers



Results about polygonal numbers

- General formula:

Let $X_{n,m}$ denote m^{th} n -agonal number. Then

$$X_{n,m} = m[1 + (n-2)(m-1)/2]$$

- Every positive integer is the sum of four integer squares (Lagrange's Four-Square Theorem, 1770)
- Generalization (conjectured by Fermat in 1670): every positive integer is the sum of n n -agonal numbers (proved by Cauchy in 1813)
- Euler's pentagonal theorem (1750):

$$\prod_{n=1}^{\infty} (1 - x^n) = 1 + \sum_{k=1}^{\infty} (-1)^k \left(x^{(3k^2-k)/2} + x^{(3k^2+k)/2} \right)$$

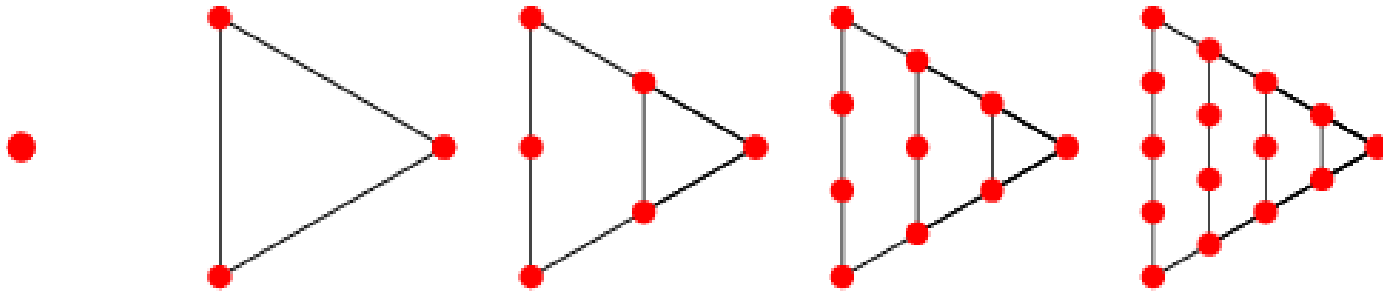
Prime numbers

- An (integer) number is called prime if it has no rectangular representation
- Equivalently, a number p is called prime if it has no divisors distinct from 1 and itself
- There are infinitely many primes. Proof (Euclid, “Elements”)

Perfect numbers

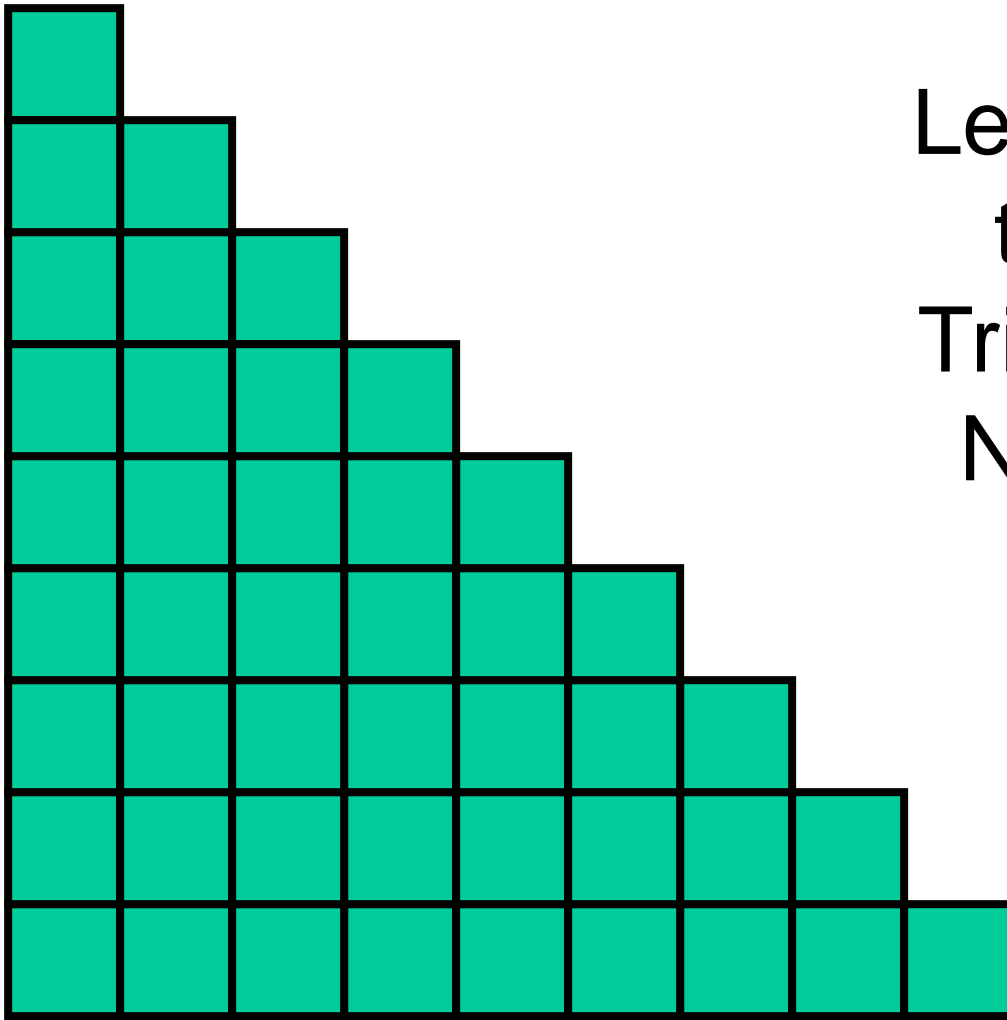
- Definition (Pythagoreans): A number is called perfect if it is equal to the sum of its divisors (including 1 but not including itself)
- Examples: $6=1+2+3$,
 $28=1+2+4+7+14$

Triangular numbers



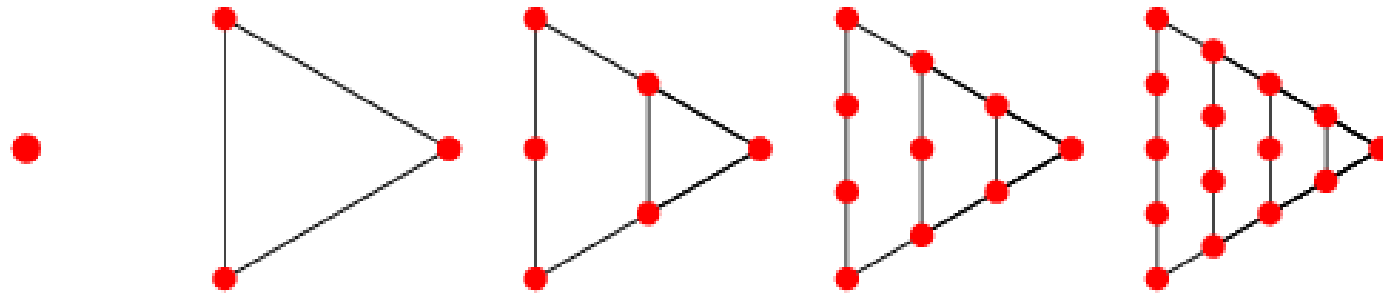
$$T_1 = 1 \quad T_2 = 3 \quad T_3 = 6 \quad T_4 = 10 \quad T_5 = 15$$

$$1; \quad 1+2=3 \quad 1+2+3=6 \quad 1+2+3+4=10 \quad 1+2+3+4+5=15$$



Let's Build
the 9th
Triangular
Number

$$1+2+3+4+5+6+7+8+9$$

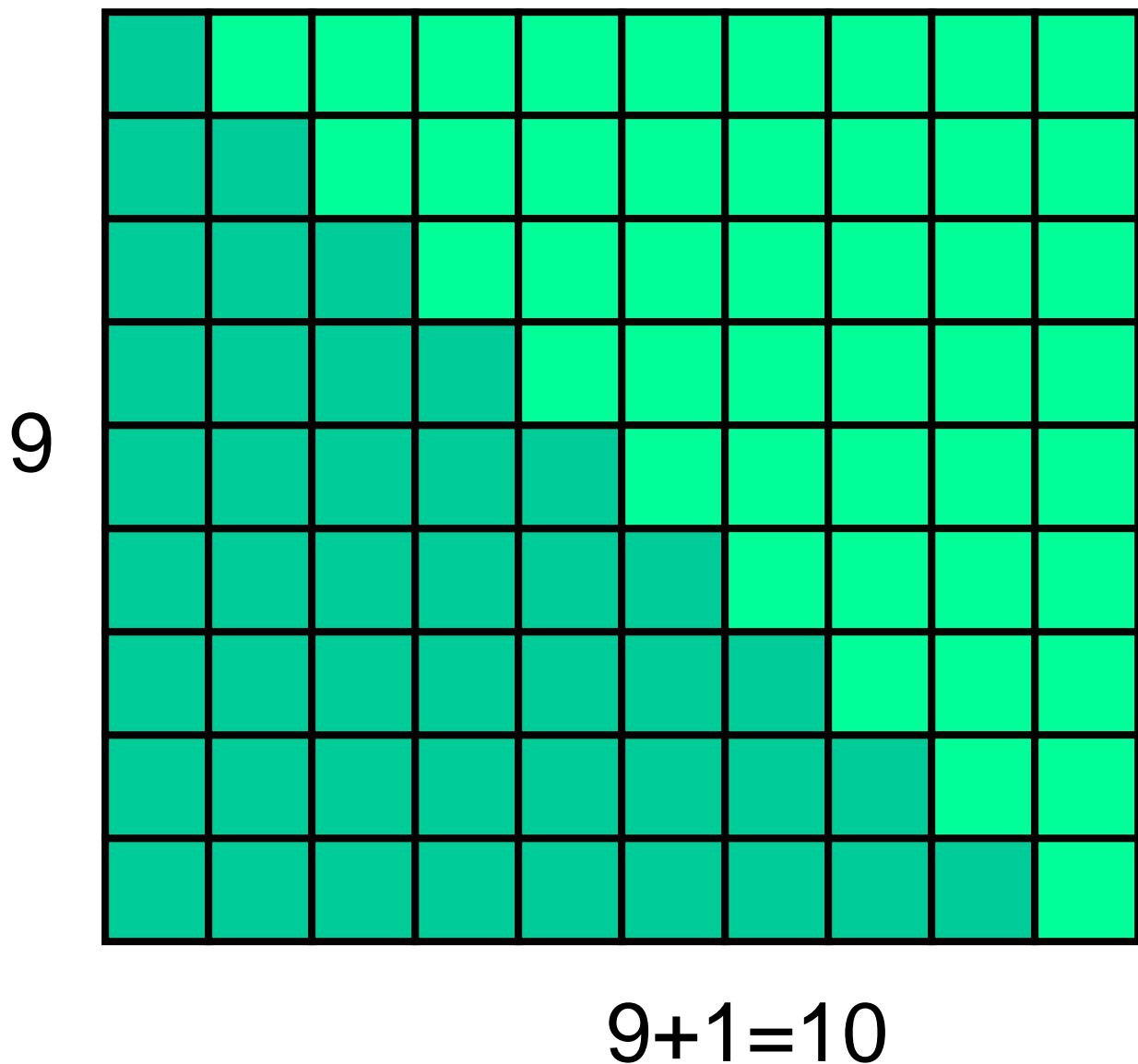


$$T_1 = 1 \quad T_2 = 3 \quad T_3 = 6 \quad T_4 = 10 \quad T_5 = 15$$

$$T_9 = 45$$

Q: Is there some easy way to get these numbers?

A: Yes, **take two copies of any triangular number and put them together**.....with multi-link cubes.

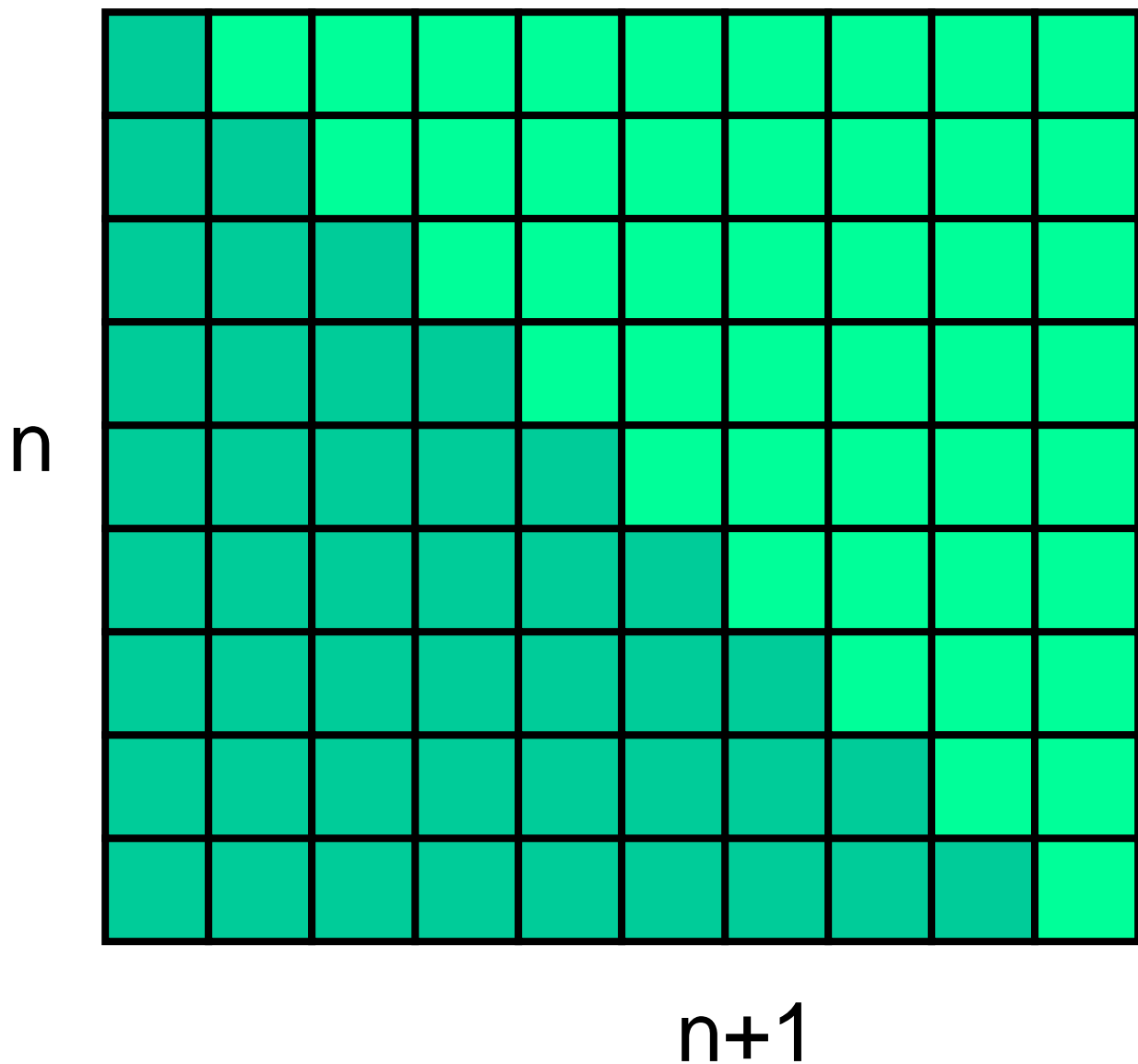


$$9 \times 10 = 90$$

Take half.

Each
Triangle
has 45.

$$T_9 = 45$$

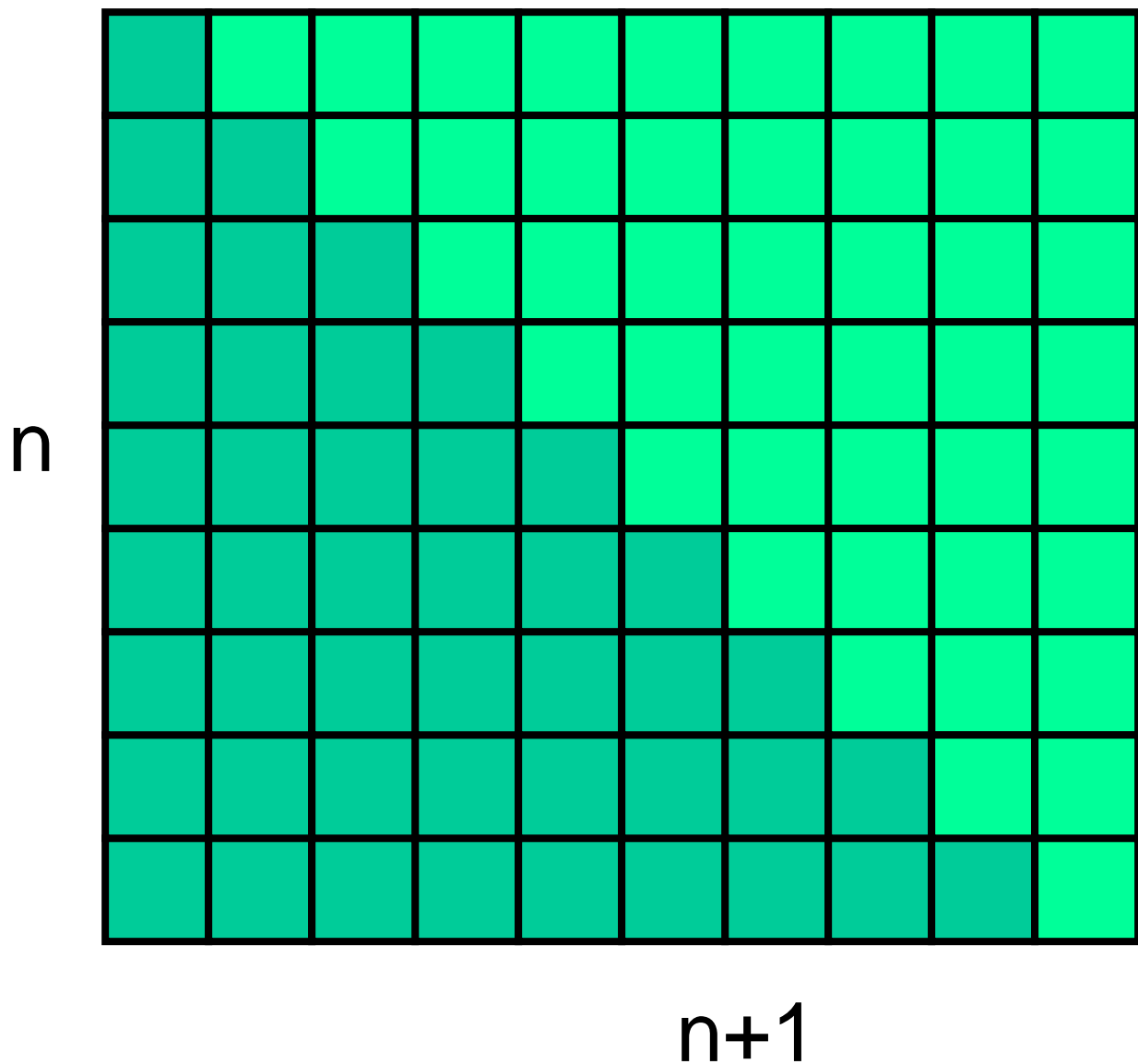


$$n(n+1)$$

Take half.

Each
Triangle
has
 $n(n+1)/2$

$$T_n = \frac{n(n+1)}{2}$$



$$n(n+1)$$

Take half.

Each
Triangle
has
 $n(n+1)/2$

$$T_n = \frac{n(n+1)}{2}$$

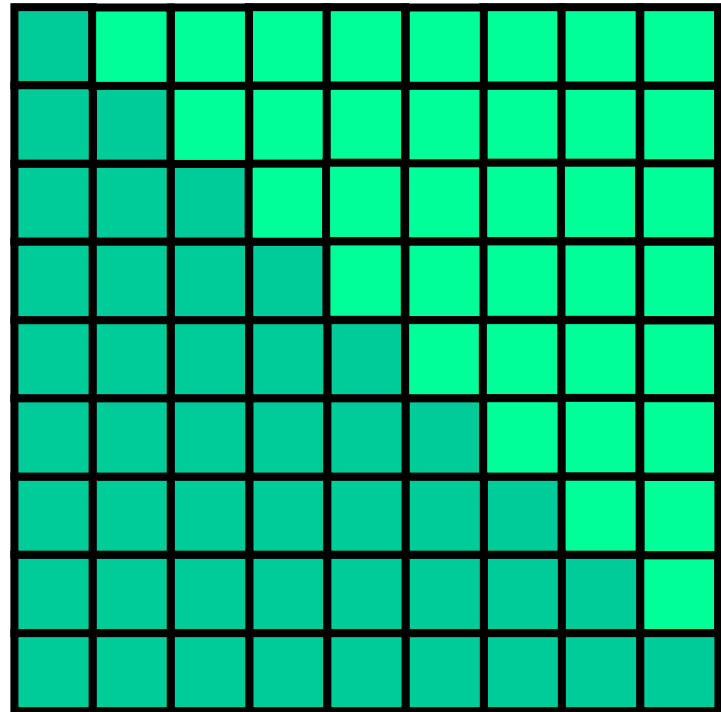
Another Cool Thing about Triangular Numbers

Put any triangular number together with the
next bigger (or next smaller).

And you get a Square!

$$T_8 + T_9 = 9^2 = 81$$

$$T_{n-1} + T_n = n^2$$



Another Cool Thing about Triangular Numbers

- First + **Second** $\rightarrow 1+3=4=2^2$
- Second + **Third** $\rightarrow 3+6=9=3^2$
- Third + **Forth** $\rightarrow 6+10=16=4^2$

$$T_{n-1} + T_n = n^2$$

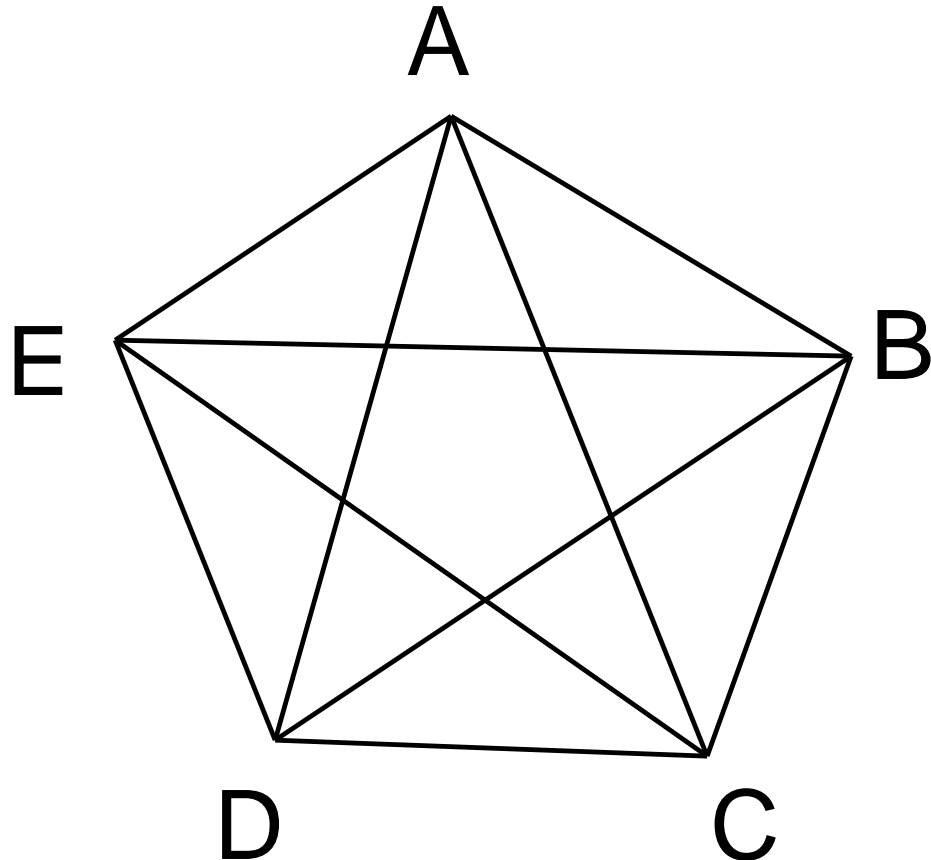
Interesting facts about Triangular Numbers

- The Triangular Numbers are the *Handshake Numbers*
- Which are the number of sides and diagonals of an n-gon.

Number of People in the Room	Number of Handshakes
2	1
3	3
4	6
5	10
6	15

Number of Handshakes

= Number of sides and diagonals of an n-gon.



Why are the handshake numbers Triangular?

Let's say we have 5 people: A, B, C, D, E.

Here are the handshakes:

A-B A-C A-D A-E

B-C B-D B-E

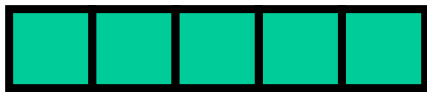
C-D C-E

D-E

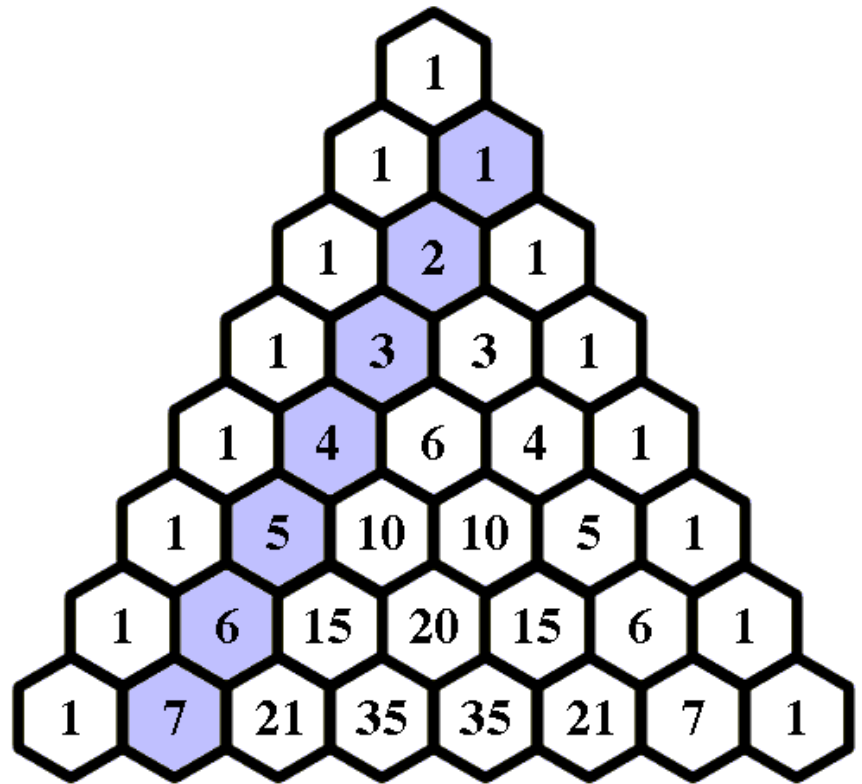
It's a Triangle !

PASCAL TRIANGLE

The first diagonal are
the “stick” numbers.

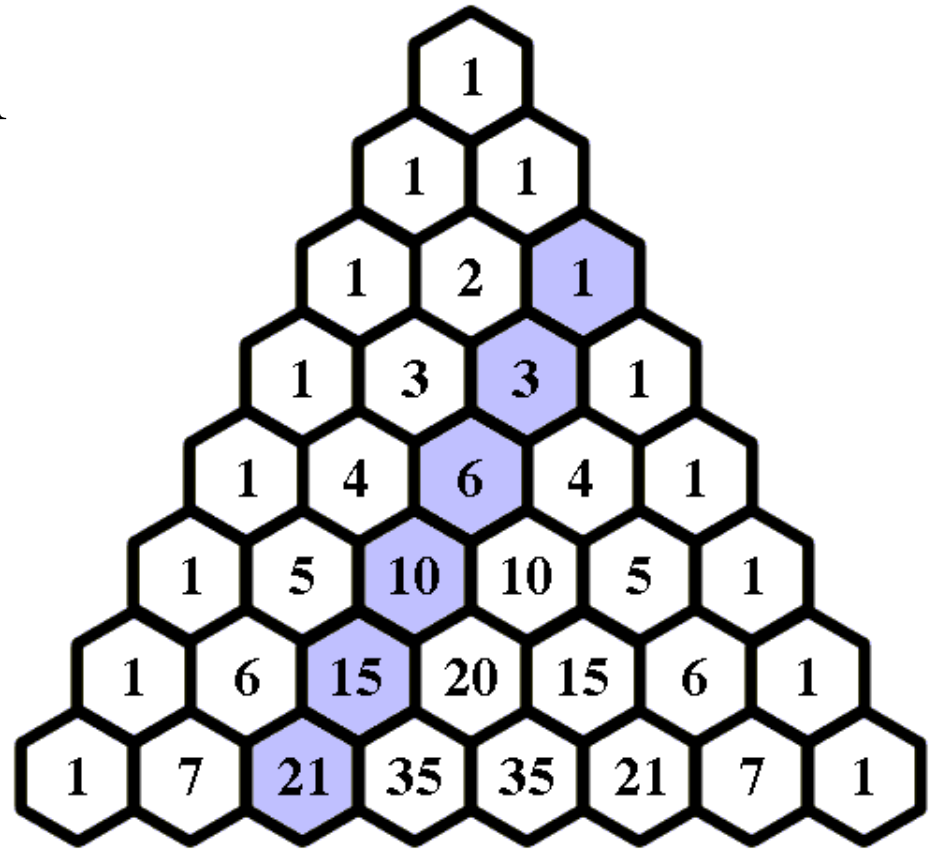


...boring, but a lead-in
to...



Triangle Numbers in Pascal Triangle

The second diagonal
are the triangular
numbers.

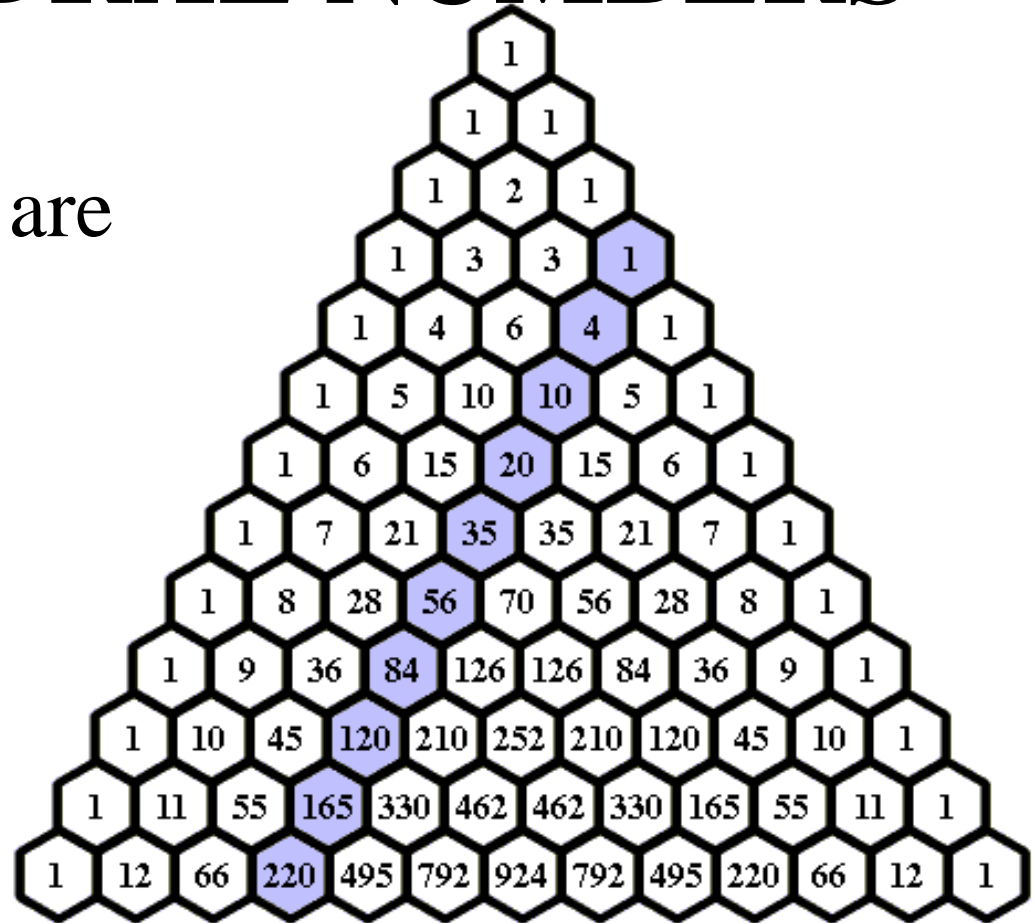


Why?

Because we use the Hockey Stick Principle
to sum up stick numbers.

TETRAHEDRAL NUMBERS

The third diagonal are
the tetrahedral
numbers.



Why?

Because we use the Hockey Stick Principle
to sum up triangular numbers.

Triangular and Hexagonal Numbers

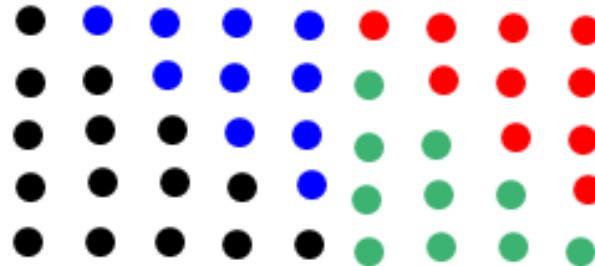
Relationships between **Triangular and Hexagonal Numbers....**decompose a hexagonal number into 4 triangular numbers.

Notation

T_n = nth Triangular number

H_n = nth Hexagonal number

Decompose a hexagonal number into 4 triangular numbers.



$$H_n = 1 + 5 + \dots + (4n - 3)$$

$$T_n = 1 + 2 + \dots + n$$

$$H_n = T_n + 3T_{n-1}$$

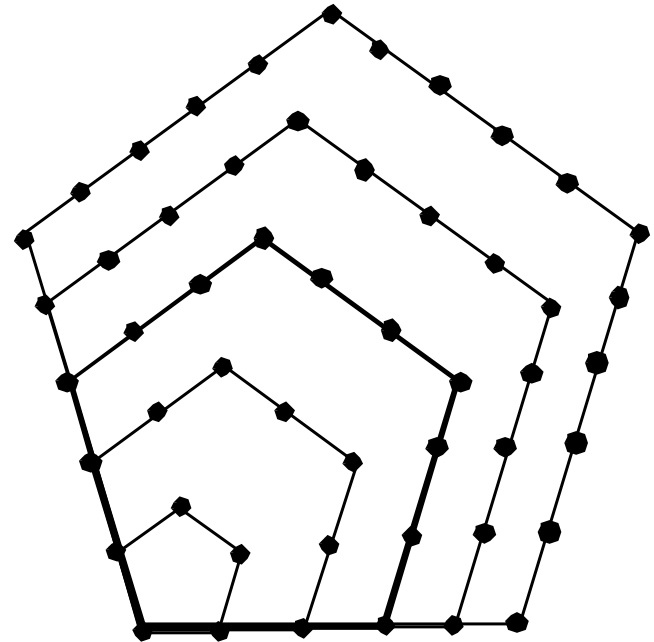
$$H_n = T_{2n-1}$$

$$H_n = n(2n - 1)$$

A Neat Method to Find Any Figurate Number

Number example:

Let's find the 6th pentagonal number.

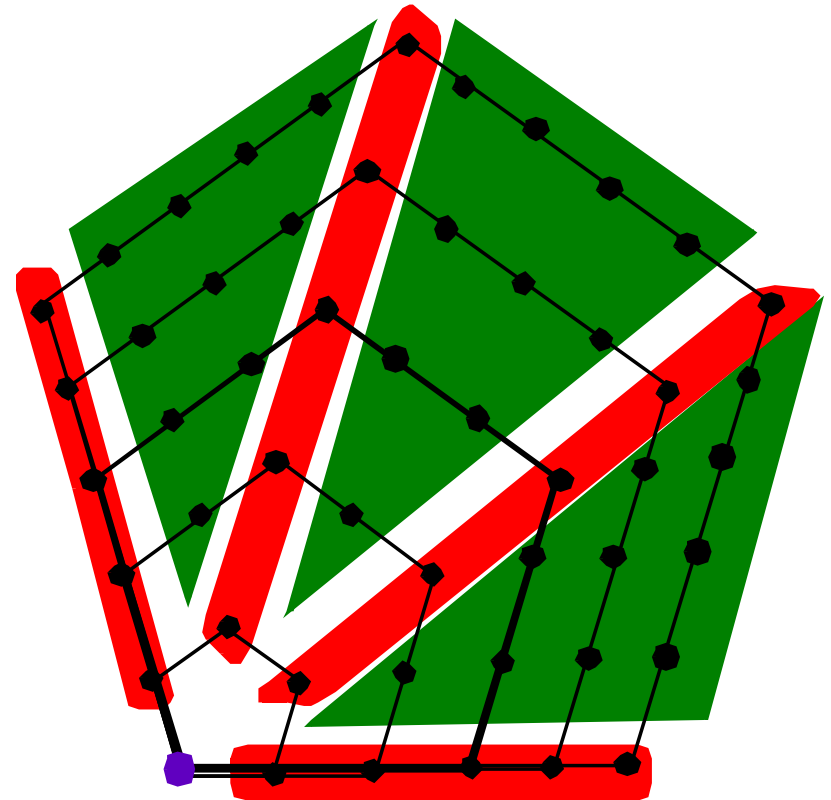


The 6th Pentagonal Number is:

- Polygonal numbers always begin with 1.
- Now look at the “Sticks.”
 - There are 4 sticks
 - and they are 5 long.
- Now look at the triangles!
 - There are 3 triangles.
 - and they are 4 high.

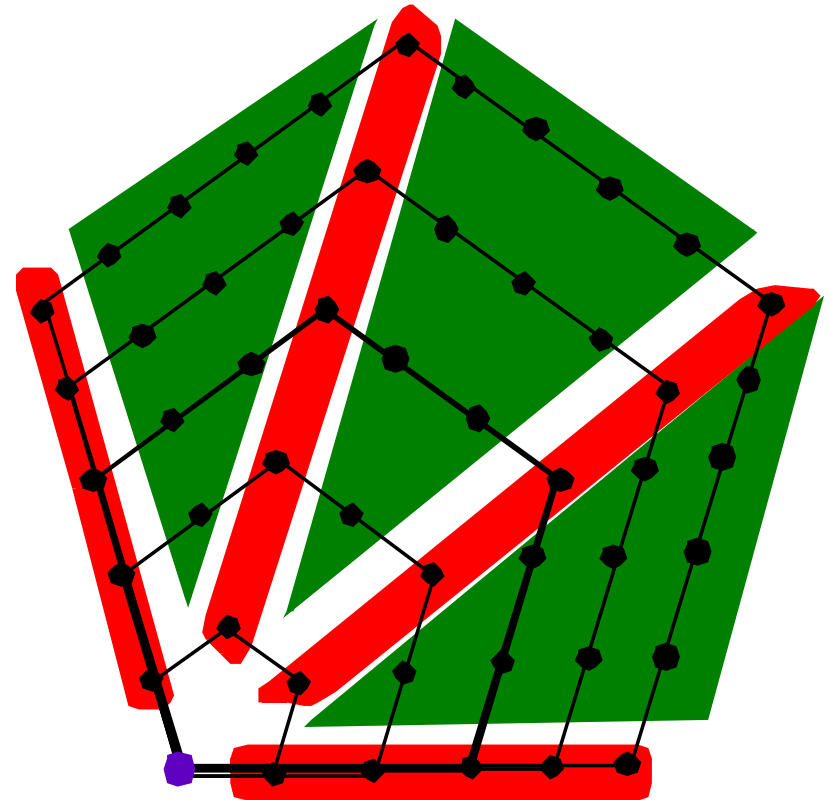
$$1 + 5 \times 4 + T_4 \times 3$$

$$1 + 20 + 30 = 51$$



The k^{th} n -gonal Number is:

- Polygonal numbers always begin with 1.
- Now look at the “Sticks.”
 - There are $n-1$ sticks
 - and they are $k-1$ long.
- Now look at the triangles!
 - There are $n-2$ triangles.
 - and they are $k-2$ high.



$$1 + (k-1)x(n-1) + T_{k-2}x(n-2)$$

Who Was Fibonacci?



- ~ Born in Pisa, Italy in 1175 AD
- ~ Full name was Leonardo Pisano
- ~ Grew up with a North African education under the Moors
- ~ Traveled extensively around the Mediterranean coast
- ~ Met with many merchants and learned their systems of arithmetic
- ~ Realized the advantages of the Hindu-Arabic system

Fibonacci's Mathematical Contributions

- ~ Introduced the Hindu-Arabic number system into Europe
- ~ Based on ten digits and a decimal point
- ~ Europe previously used the Roman number system
- ~ Consisted of Roman numerals
- ~ Persuaded mathematicians to use the Hindu-Arabic number system

1 2 3 4 5 6 7 8 9 0 .

The Fibonacci Numbers

- ~ Were introduced in the book *Liber Abaci*
- ~ Series begins with 1 and 1
- ~ Next number is found by adding the two preceding numbers together
- ~ Number obtained as the sum is the next number in the sequence
- ~ Pattern is repeated over and over

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, ...

$$F(n + 2) = F(n + 1) + F(n)$$

Fibonacci & the Rabbits

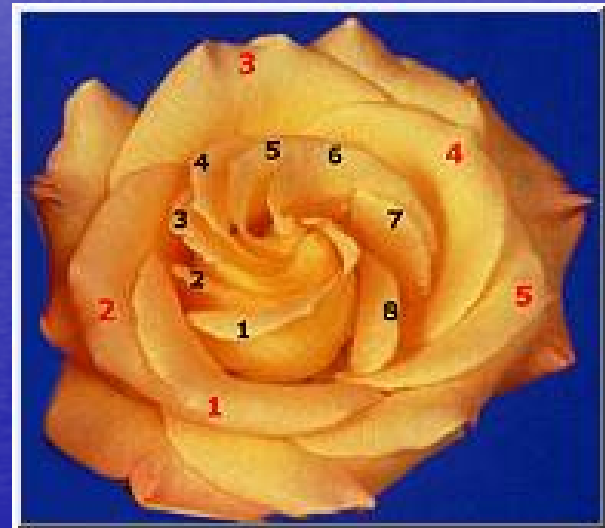
- Fibonacci applied his sequence to a problem involving the breeding of rabbits.
- Given certain starting conditions, he mapped out the family tree of a group of rabbits that initially started with only two members.
- The number of rabbits at any given time was always a Fibonacci number.
- Unfortunately, his application had little practical bearing to nature, since incest and immortality was required among the rabbits to complete his problem.

Applications of Fibonacci Sequence

- The Fibonacci sequence has far more applications than immortal rabbits.
- Fibonacci numbers have numerous naturally-occurring applications, ranging from the very basic to the complex geometric.

Fibonacci Numbers in Nature

- Many aspects of nature are grouped in bunches equaling Fibonacci numbers.
- For example, the number of petals on a flower tend to be a Fibonacci number.





White Calla Lily



euphorbia



Trillium



Columbine

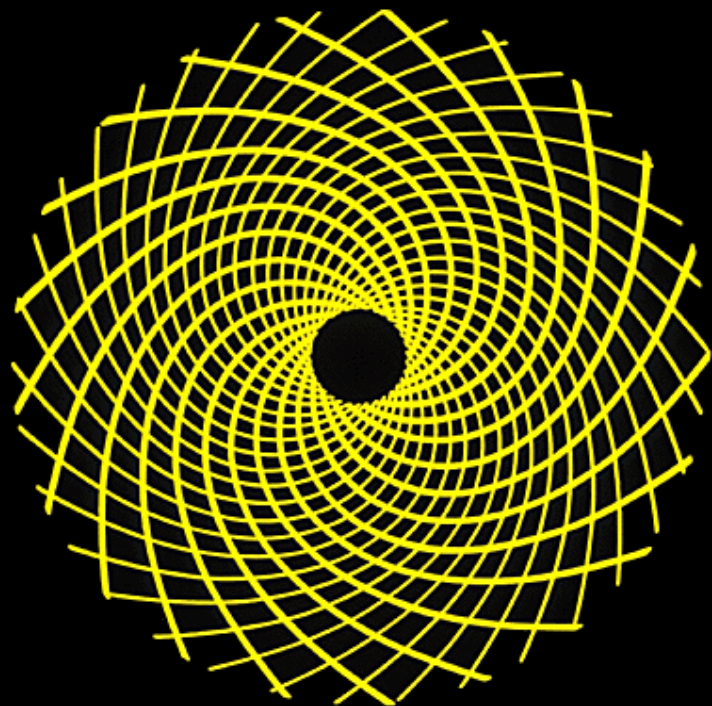


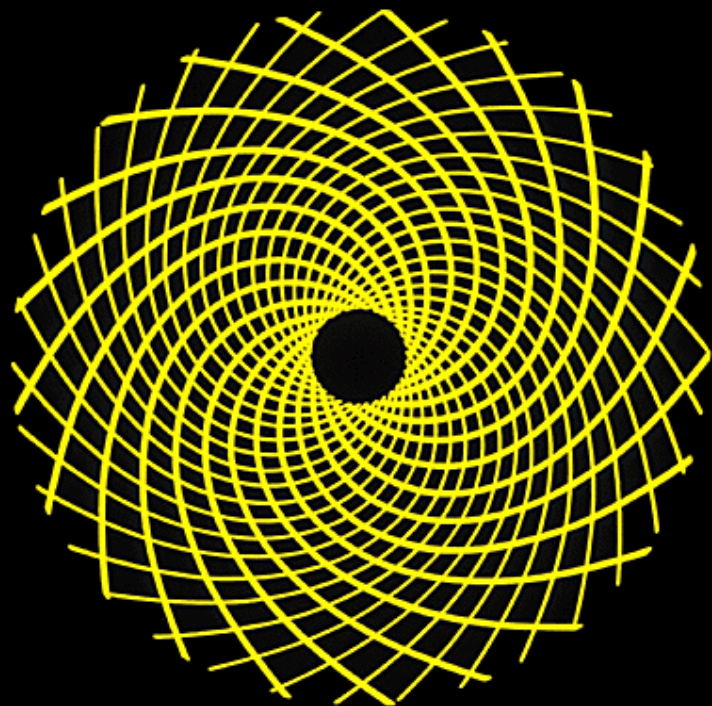
Bloodroot



Black-eyed Susan

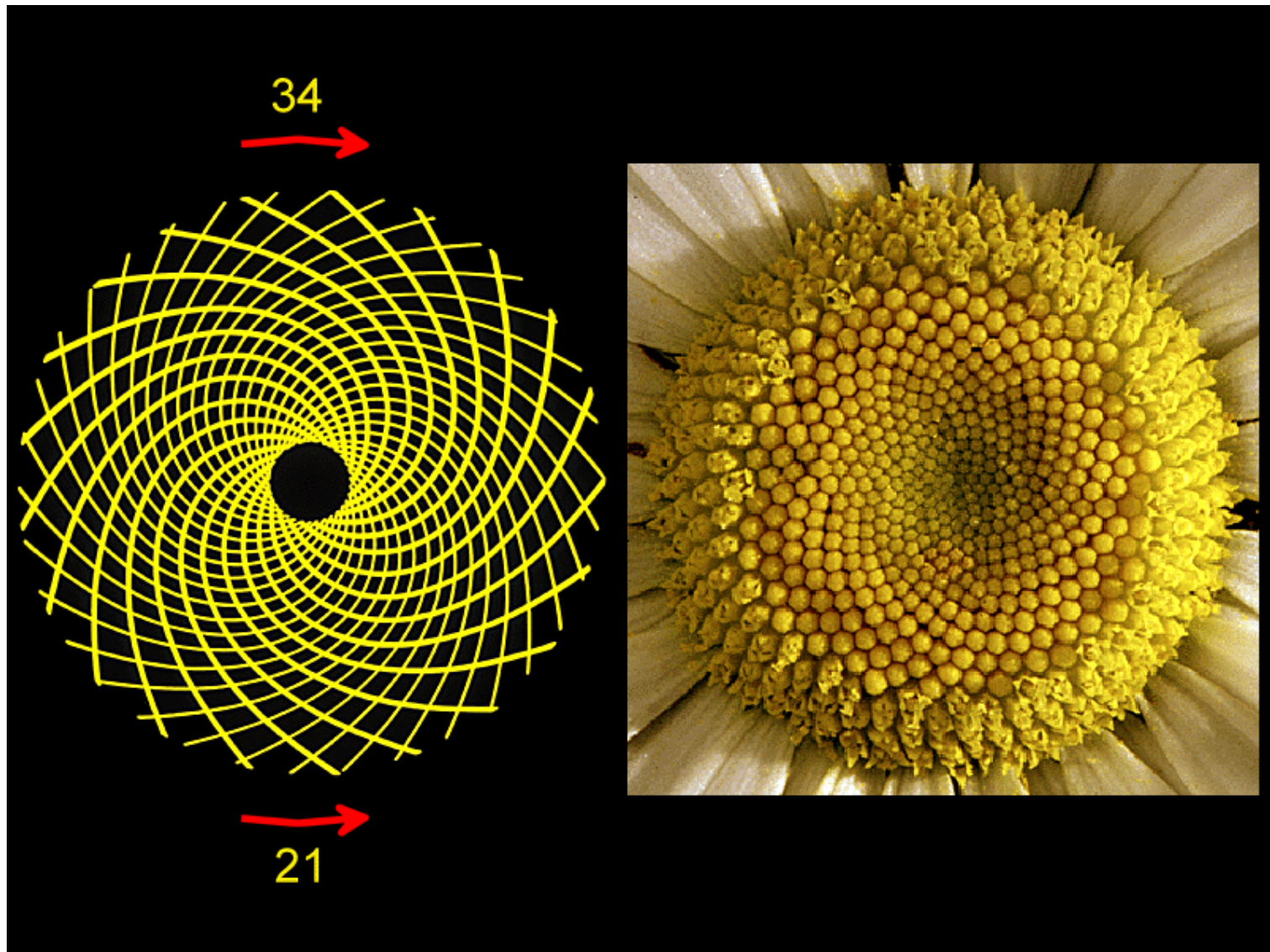






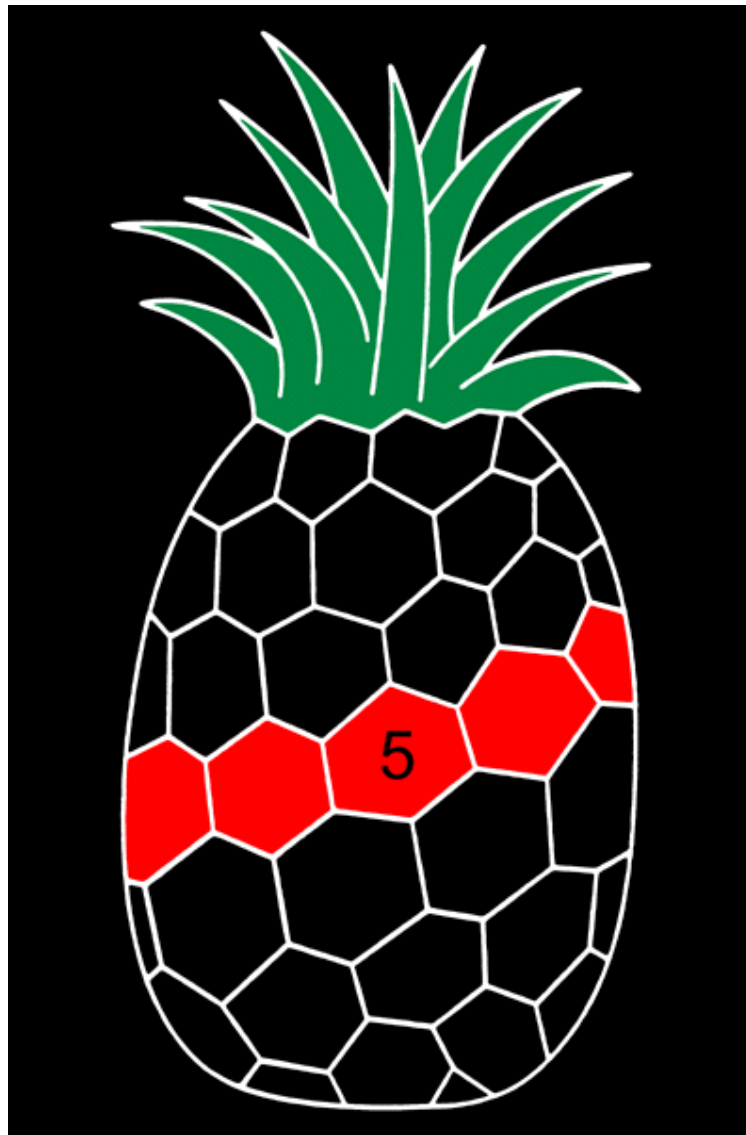
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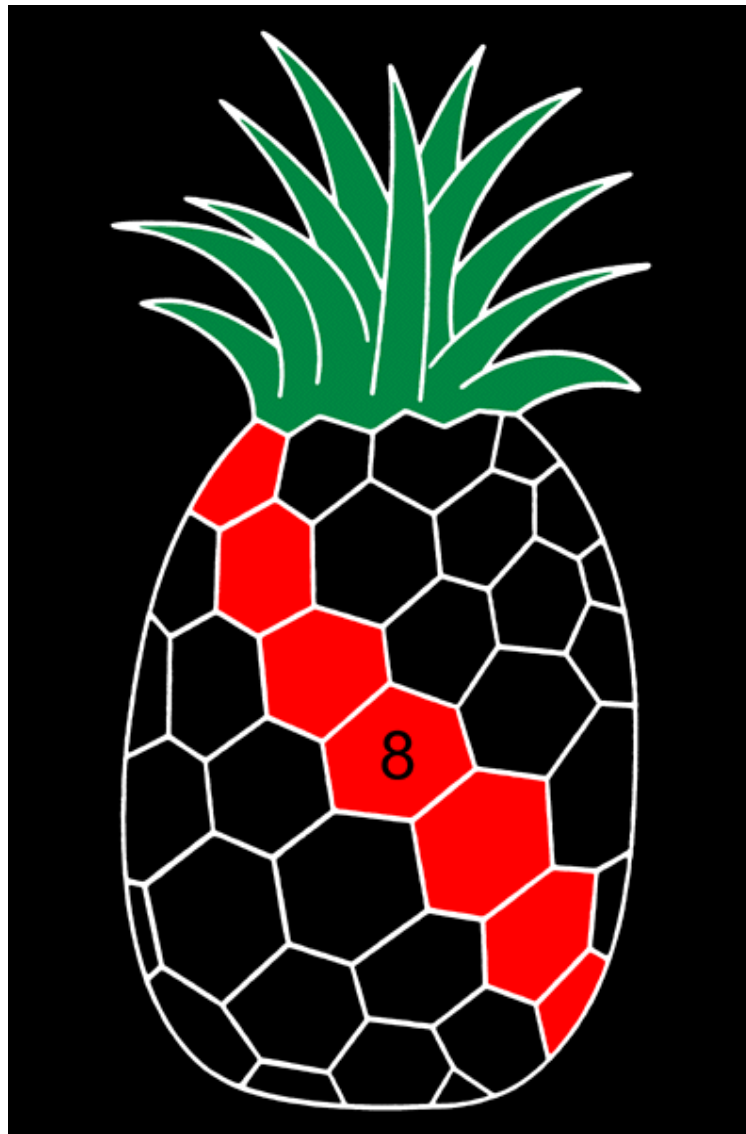


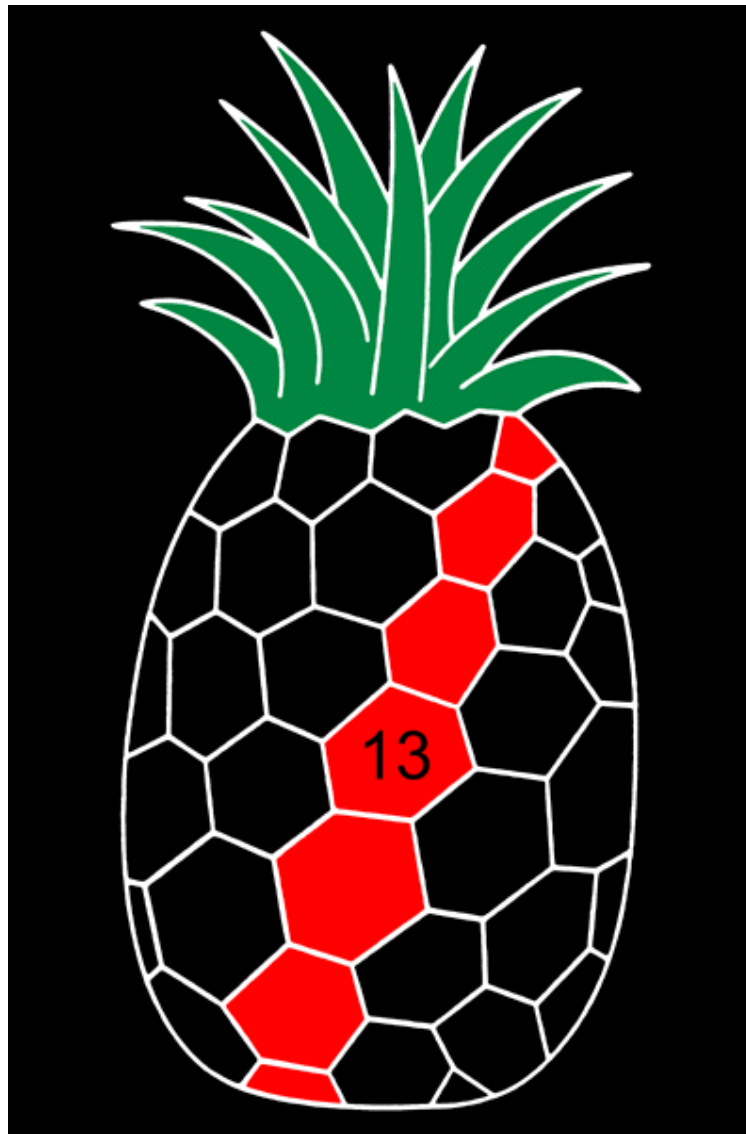


Pineapple







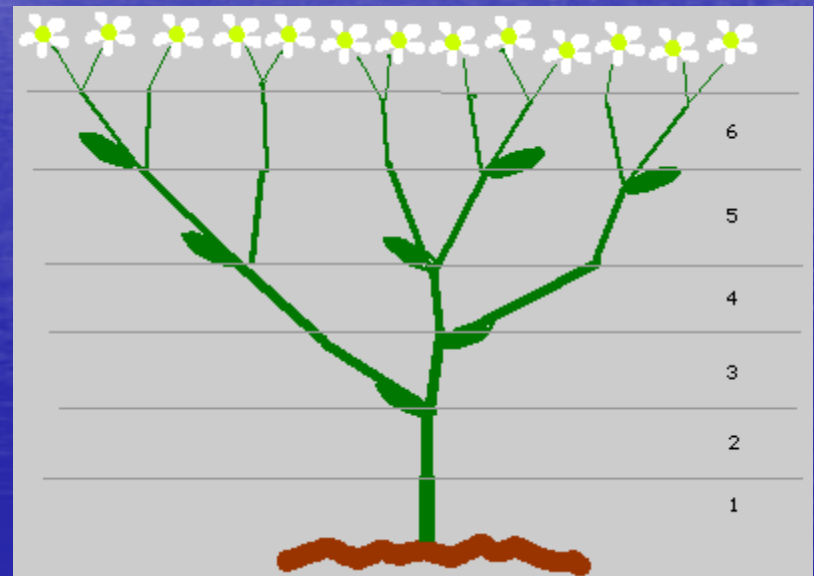


Examples of Flower Petals

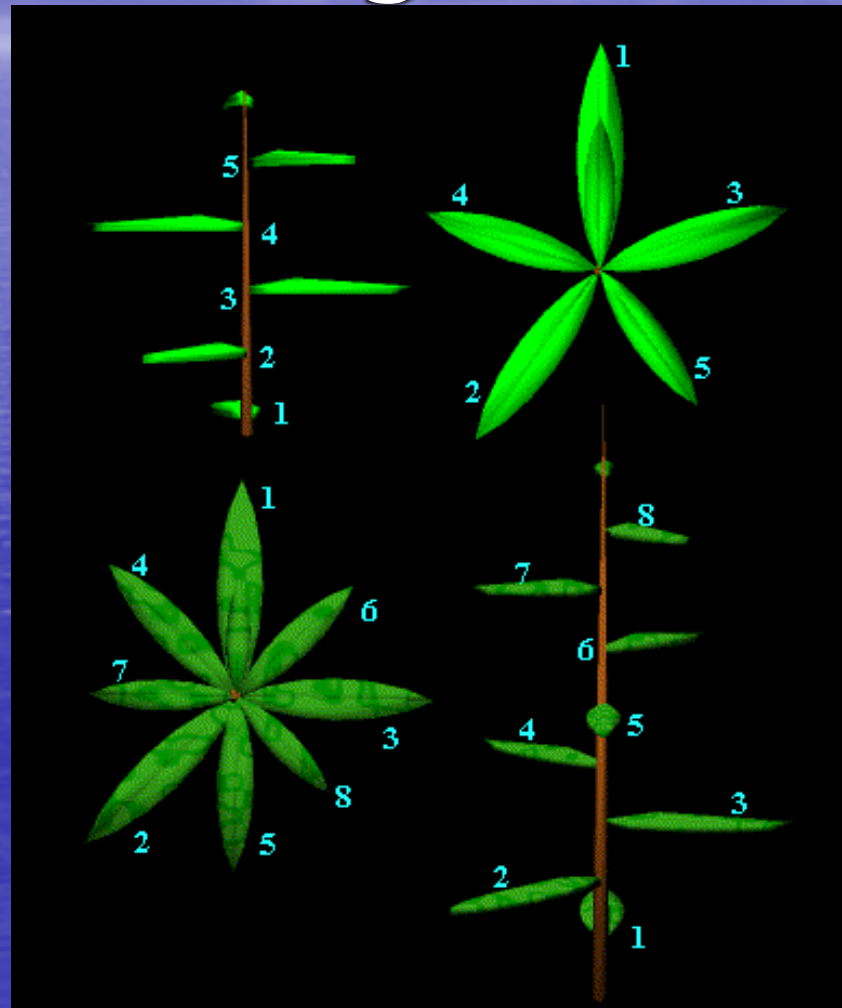
- 3 petals: lilies
- 5 petals: buttercups, roses
- 8 petals: delphinium
- 13 petals: marigolds
- 21 petals: black-eyed susans
- 34 petals: pyrethrum
- 55/89 petals: daisies

Leaves and Plants

- Leaves are also found in groups of Fibonacci numbers.
- Branching plants always branch off into groups of Fibonacci numbers.



Diagrams of Leaf Arrangements

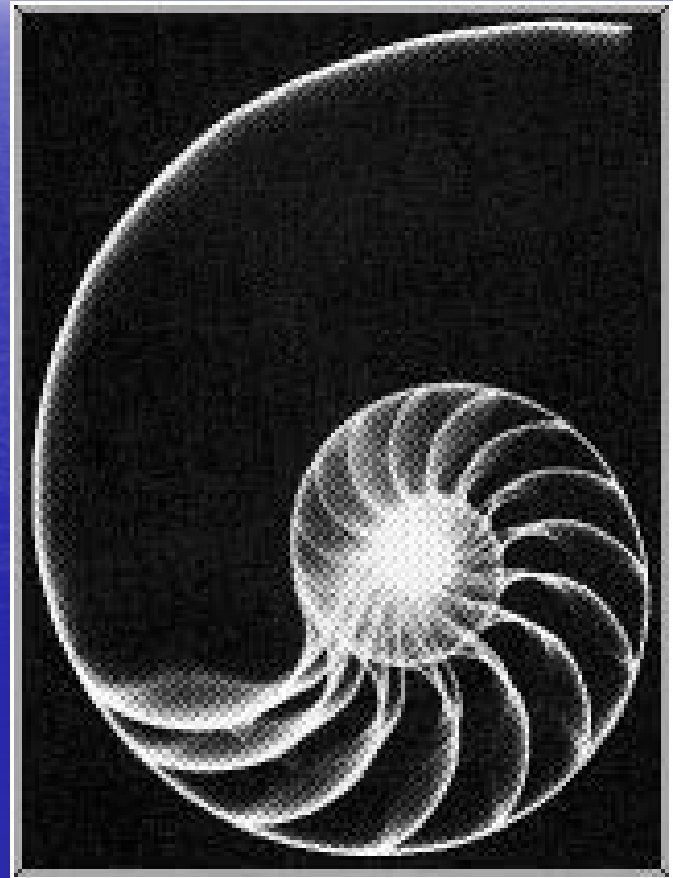


Think about Yourself

- Think about yourself.
You *should* have:
- 5 fingers on each hand
- 5 toes on each foot
- 2 arms
- 2 legs
- 2 eyes
- 2 ears
- 2 sections per leg
- 2 sections per arm

Geometric Applications

- Fibonacci numbers have geometric applications in nature as well.
- The most prominent of these is the Fibonacci spiral.

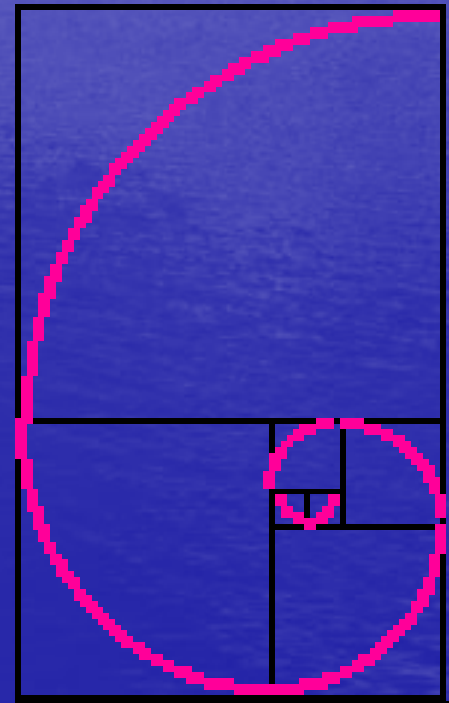


Construction of the Fibonacci Spiral

- The Fibonacci spiral is constructed by placing together rectangles of relative side lengths equaling Fibonacci numbers.

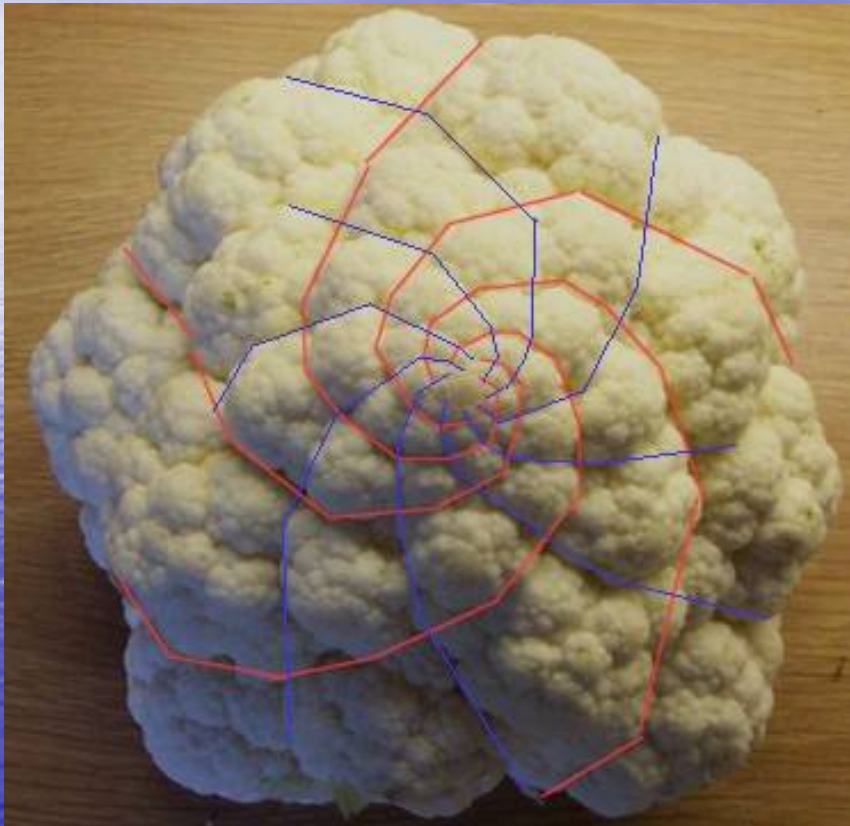


- A spiral can then be drawn starting from the corner of the first rectangle of side length 1, all the way to the corner of the rectangle of side length 13.

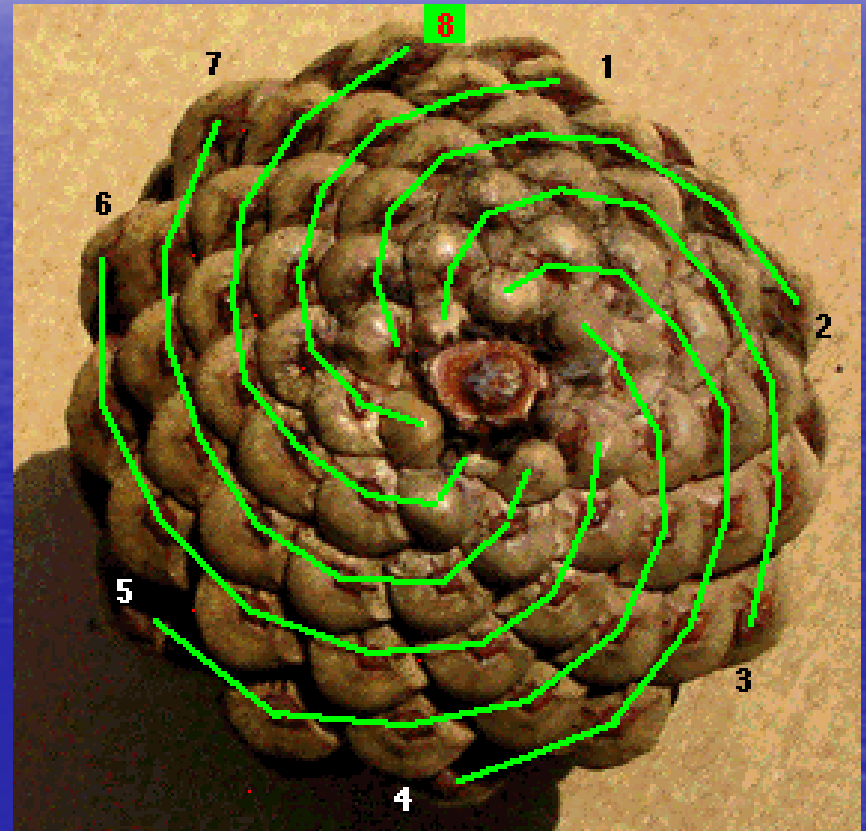


Other Applications of the Fibonacci Spiral

Cauliflower



Pine Cone



Applications to Music

- Music involves several applications of Fibonacci numbers.
- A full octave is composed of 13 total musical tones, 8 of which make up the actual musical octave.



More Applications

Fibonacci Ratio	Calculated Frequency	Tempered Frequency	Note in Scale	Musical Relationship
1/1	440	440.00	A	Root
2/1	880	880.00	A	Octave
2/3	293.33	293.66	D	Fourth
2/5	176	174.62	F	Aug Fifth
3/2	660	659.26	E	Fifth
3/5	264	261.63	C	Minor Third
3/8	165	164.82	E	Fifth
5/2	1,100.00	1108.72	C#	Third
5/3	733.33	740	F#	Sixth
5/8	275	277.18	C#	Third
8/3	1173.33	1174.64	D	Fourth
8/5	704	698.46	F	Aug. Fifth



More about Phi

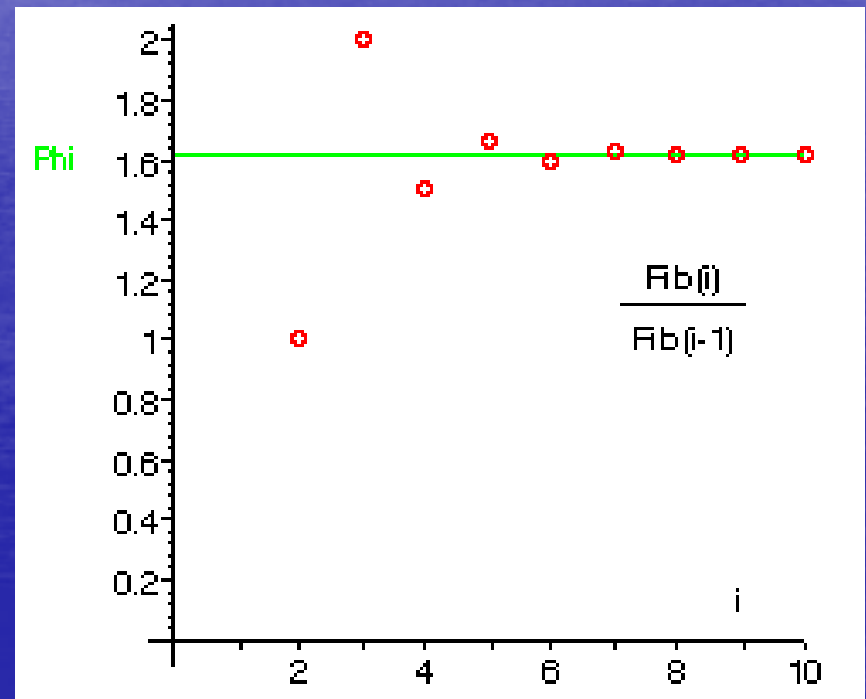
- One of the most significant applications of the Fibonacci sequence is a number that mathematicians refer to as Phi (Φ).
- Φ refers to a very important number that is known as the **golden ratio**.

What is Phi ?

- Phi is defined as the limit of the ratio of a Fibonacci number i and its predecessor, $\text{Fib}(i-1)$.
- Mathematically, this number is equal to:

$$\frac{1+\sqrt{5}}{2}$$

or approximately
1.618034.



Phi

- Phi can be derived by the equation:

$$x^2 - x - 1 = 0$$

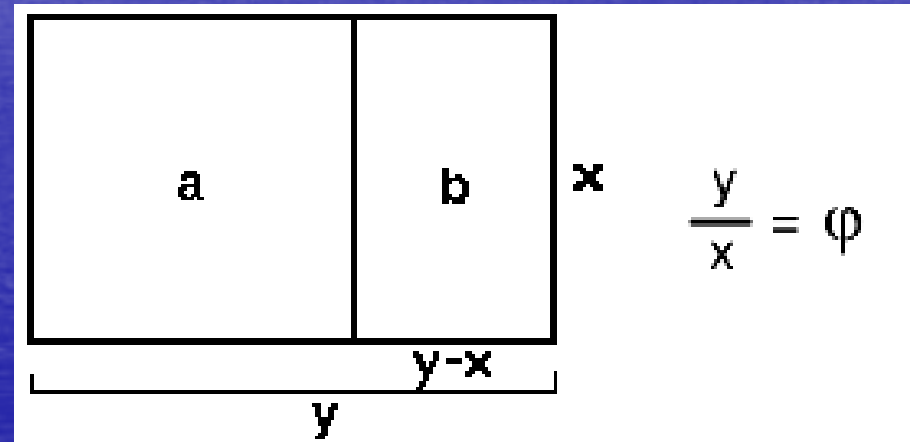
- With some fancy factoring and division, you get:

$$x - 1 = \frac{1}{x}$$

- This implies that Phi's reciprocal is smaller than 1. It is .618034, also known as phi (ϕ).

PHI (contd.)

- Is there anything mathematically definitive about Φ when used in geometry? You bet there is.
- A rectangle whose sides are in the **golden ratio** is referred to as a **golden rectangle**.
- When a **golden rectangle** is squared, the remaining area forms another **golden rectangle**!



Mathematical Appl.

- Without Φ , in order to find any Fibonacci number, you would need to know its two preceding Fibonacci numbers.
- But with Φ at your service, you can find any Fibonacci number knowing only its place in the sequence!

nth Fibonacci Number & Phi Binet Equation

$$Fib(n) = \frac{\Phi^n - ((-\Phi)^{-n})}{\sqrt{5}}$$

$$Fib(n) = \frac{\Phi^n - ((-\phi)^n)}{\sqrt{5}}$$

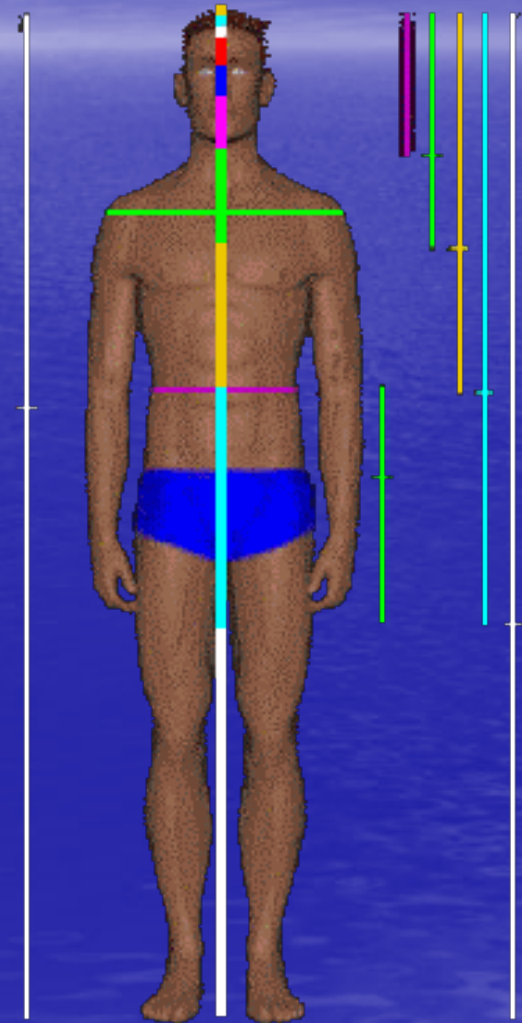
$$Fib(n) = \frac{\Phi^n - \frac{-1^n}{\Phi^n}}{\sqrt{5}}$$

Natural Applications of Phi

- Remember how flowers have leaves and petals arranged in sets of Fibonacci numbers?
- This ensures that there are Φ leaves and petals per turn of the stem, which allows for maximum exposure to sunlight, rain, and insects.

Phi and Human body

- How about your body?
- You have NO IDEA how many segments of the human body are related in size to each other by Φ !

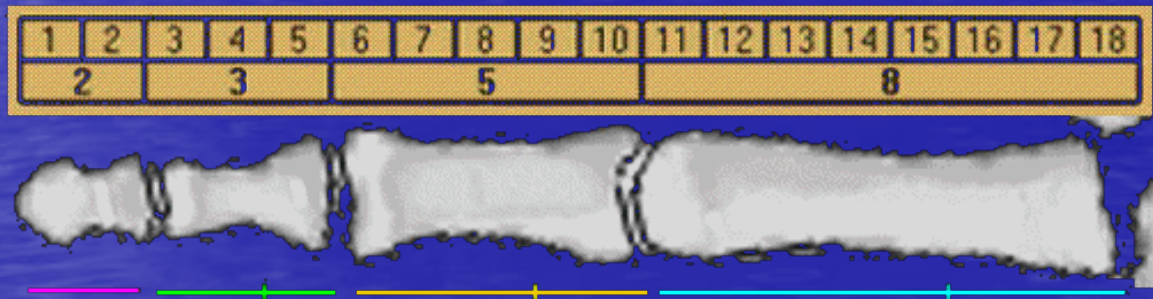


Relation with human body

- The human arm:



- The human finger:



Phi and human body (contd.)

- When used in dimensioning objects, it has always been thought that Φ produces the most visually appealing results.
- Many marketers have used Φ in their products over the years to make them more attractive to you.
- An extremely basic example: 3 x 5 greeting cards.

There are numerous other applications of the Fibonacci sequence, Fibonacci numbers, and Φ that were not covered in this presentation—simply because there are far too many to list. Feel free to research on your own if you found any of this interesting.

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