Welcome!

Today's topic: Modular Operators

We use modular arithmetic every day when we talk about time.

Example: If it is 11am, what time will it be in 5 hours?
It will be 4pm rather than 16am.

4 is the remainder when we divide 16 by 12

Notation

$$x \pmod{n} = r$$

This means when x is divided by n, there is a remainder of r. We say: "x modulo n is equal to r."

Examples

$7 \pmod{4} =$

$15 \pmod{3} =$

$19 \pmod{4} =$

$21 \pmod{5} =$

Exercises

7 (mod 5) =

8 (mod 4) =

8 (mod 3) =

17 (mod 8) =

37 (mod 6) =

124 (mod 60) =

Word Problem!

Using a regular deck of 52 cards, I dealt all the cards in the deck to 3 people (including myself). Were the cards dealt evenly?

Modular Addition

We will look at **two methods** for modular addition!

Let's consider $(1824 + 326)(mod\ 2)$

Method 1

For method 1, we compute the addition first, then mod our answer

$1824 + 326 = 2150$. Since $2150 (mod\ 2) = 0$.

$(1824 + 326)(mod\ 2) = 0$

Method 2

For method 2, we compute each mod first, then add our answers.

We know $1824 (mod\ 2) = 326 (mod\ 2) = 0$.

Thus, $(1824 + 326)(mod\ 2) = 1824(mod\ 2) + 326(mod\ 2) = 0 + 0 = 0$

In general, $\boldsymbol{(x + y)\ (mod\ n) = x\ (mod\ n) + y\ (mod\ n)}$

Examples

(7+6) (mod 5) =

(19+28) (mod 5) =

Exercises

5 + 9(mod 4) =

43 +37 (mod 3) =

124 +199 (mod 5) =

34 + 121 (mod 11) =

Word Problem!

If my birthday was on Tuesday, March 5, 2019, what day of the week will my birthday be on in 2020? (Note the year 2020 is a leap year with 366 days.)

Modular Multiplication

Modular multiplication is very similar to modular addition.

We define it as: $(x \times y)\ (mod\ n) = [x\ (mod\ n) \times y\ (mod\ n)]\ (mod\ n)$

Exercises

(5 x 9) (mod 8) =

(7 x 15) (mod 7) =

(5782 x 2579) (mod 10) =

(603 x 123) (mod 60) =

(16 x 25) (mod 12) =

(34 x 122) (mod 11) =

Word Problem!

A liter of milk is 4 cups, and one cake recipe uses 3 cups. If I have 8 liters of milk, how many cakes can I make? And how many cups of milk will be leftover, if any?

Common Bases

Base	Application	Example
2	Even/odd numbers Binary codes	A number n is even if n (mod 2) = 0, and odd otherwise. We also use base 2 when converting from binary to decimal form, as we will see later.
4	Years between 2 consecutive leap years	If any given year n is either [n (mod 400) = 0] or [n (mod 4) = 0 and n (mod 100) $\neq$ 0] then it is a leap year, otherwise it is not.
7	Days in a week	If today is Sunday, then in 16 days it is Tuesday (since 16 (mod 7) = 2).
10	Metric measurements	We use base 10 when converting between metric measurements, such as meters to millimeters.
12	Hours on an analog clock	If it’s 7 pm now, it will be 2 am in 7 hours (since (7 + 7) (mod 12) = 14 (mod 12) = 2).
24	Hours in a day	If its 2 pm now, in 54 hours it will be 8 am (since (54 + 2) (mod 24) = 8).

Common Bases (continued)

Base	Application	Example
28, 29, 30, 31	Days in a month	If today is the 4th of April, then it will be the 8th of May in 34 days.
52	Weeks in a year	If today is the 6th week of the year, then it will be the 16th week of next year in 62 weeks.
60	Seconds in a minute and minutes in an hour	155 seconds is equivalent to 2 minutes and 35 seconds.
100	Years in a century	In 344 years, it will be the 60th year of that century, since (344 + 2016) (mod 100) = 60.
360	Degrees in a full circle	Rotating 420° is equivalent to rotating 60° since 420 (mod 360) = 60.
365	Days in a year	If today is the 65th day of the year, then in 750 days, it will be 85th day of that year.

Word Problem!

(a) Was the year 1900 a leap year?

(b) What about 2000?

Binary Numbers and Codes

- A binary code is any system that only uses 2 states: 1/0, on/off, true/false etc.
- A binary number is any number containing only 1's and 0's. Examples: 101, 000000, 1111111, 10001001010010, 10001111101010, 0101010101010
- Binary numbers have all sorts of applications, such as
  - Computers
  - Calculators
  - TV's
  - Barcodes
  - CD's and DVD's
  - Braille Binary codes

Binary Numbers and Codes

- There are multiple ways to express a binary code, the two most common forms of writing a binary code using numbers are ‘Decimal form’ and ‘Binary form’.

- For example: 1101 in binary form becomes 13 in decimal form. And 1001 becomes 9.

- We can do these conversions ourselves! Let's do a quick review

Review of Exponents

$$x^0 = 1$$
$$x^1 = x$$
$$x^2 = x \times x$$
$$x^3 = x \times x \times x$$
$$x^4 = x \times x \times x \times x$$

$$x^5 = x \times x \times x \times x \times x$$

and so on... (for any x). Also, fill out this table, it will be very useful for the rest of the lesson.

n	0	1	2	3
2^n	$2^0=$	$2^1=$	$2^2=$	$2^3=$
n	4	5	6	7
2^n	$2^4=$	$2^5=$	$2^6=$	$2^7=$

Converting Binary to Decimal

To convert a binary number to its decimal form, follow 3 simple steps.
Let's go through these steps and convert 1001 to decimal form.

Step 1:

Write out the number - but leave lots of space between your digits, like this: 1 0 0 1

Converting Binary to Decimal

Step 2:

Multiply each number by a 2, and starting with an exponent of 0 on the very last 2, and increase the exponent by 1 each time, like this:

$[1 \times (2^3)]$ $[0 \times (2^2)]$ $[0 \times (2^1)]$ $[1 \times (2^0)]$

Converting Binary to Decimal

Step 3:

Sum them up and calculate:

$$[1 \times (2^3)] + [0 \times (2^2)] + [0 \times (2^1)] + [1 \times (2^0)]$$

$$= [1 \times (8)] + [0 \times (4)] + [0 \times (2)] + [1 \times (1)]$$

$$= 8 + 0 + 0 + 1 = 9$$

Exercise 1

Convert each of the following binary numbers to decimal form.

110 →

101 →

00111 →

100001 →

Word Problem!

(a) How many different 5-digit binary numbers are there?

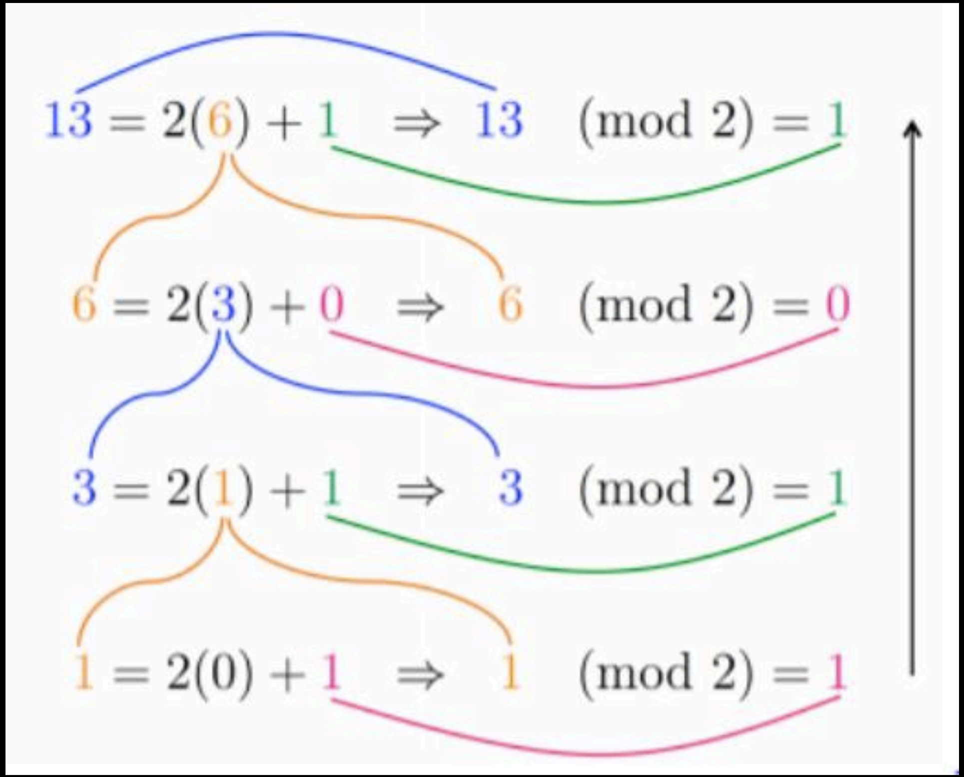
(b) How many of those end in a 1?

Converting Decimal to Binary

Now, this is the part where modular arithmetic comes in handy! We know that if we compute any number (mod 2), it will either be 0 or 1, and so that's exactly what we use for converting decimal numbers to binary. Here's what we need to do:

1. Compute our number (mod 2) and that will be our last digit.
2. Compute our quotient (mod 2) and place that as our 2nd last digit.
3. Repeat until our quotient is 0.

Example: Convert 13 to Binary



Now reading from the bottom up, 13 in decimal form is 1101 in binary form.

Exercise 2

Convert each of the following numbers to binary form.

a) 76

b) 193

Exercise 2

Convert each of the following numbers to binary form.

c) 97

d) 255