Mathematics of Infectious Diseases

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Happy St. Patrick’s Day!
Who Am I?

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Core Research

- New models, new approaches, new results, and new applications to complex biological systems
- Developed new graph-theoretic approaches to investigate dynamics of coupled systems on networks
- Established sharp threshold results for many heterogeneous infectious disease models
- Formulated and analyzed new mathematical models for waterborne diseases such as cholera
- Defined a new concept target reproduction number to mathematically measure intervention and control strategies in order to eradicate infectious diseases in heterogeneous host populations
Paper Passing Game

Once you receive the paper, you have choices to
▶ hold the paper, or
▶ pass the paper to one person with whom you shake hands, or
▶ rip it into two or more pieces and pass them to different persons with whom you shake hands

Repeat whenever hearing “NEXT”

Now consider the paper that you have passed after handshaking (effective contact) is an infectious disease

How many persons have been infected by the “paper” disease?

Zhisheng Shuai (U Central Florida)
Mathematics of Infectious Diseases
Central Florida Math Circle
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- Infectious diseases require a mode of transmission (direct or indirect transmission, waterborne, airborne, vector-borne, food-borne, etc.) to be transmitted to other individuals (infectious).
How to Model Infectious Diseases

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All models are wrong,
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- spread among heterogeneous host groups (heterogeneous model)
Some Success Stories of Mathematical Modeling in Disease Control

Malaria

A parasitic disease carried by mosquitoes (vector)
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  - developed a mosquito-human model and showed that malaria could be controlled without total destruction of the mosquito population
  - idea of a “threshold”
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- MacDonald [1957]
  - included adult and larval mosquitoes
  - showed that control of adult mosquitoes is more effective than control of larvae
Gonorrhea

A sexually transmitted bacterial disease
Gonorrhea

A sexually transmitted bacterial disease

- Hethcote and Yorke [1984]
  - found that a core group of very highly active individuals maintains the disease at an epidemic level
  - contact tracing is more efficient for control than routine screening
Measles

A viral childhood disease
Measles

A viral childhood disease

- Anderson and May [1992]
  *Infectious Diseases of Humans: Dynamics and Control*

  - guided design of vaccination programs
    (two dose strategy in UK)
  - two dose vaccination strategy started in US in 1989
Measles cases in the United States, 1944-2007

- Vaccine licensed
- Second dose recommended
- Second dose recommended

Cases (thousands)

Why Should We Do Mathematical Modeling of Infectious Diseases?

- Mathematical models and computer simulations can be used as experimental tools for testing control measures and determining sensitivities to changes in parameter values e.g.,
  - design of a vaccination strategy for rubella
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- Mathematical modeling of epidemics can lead to and motivate new results in mathematics e.g.,
  - ruling out periodic orbits in higher dimensional ODE systems
Disease Modeling Approach

Mathematical modeling is a trade off between

- simple models: highlight qualitative behavior
- detailed models: specific quantitative predictions
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In 1902 Sir Ronald Ross was awarded the Noble prize for medicine for proving that malaria is transmitted by mosquitoes
The Kermack-McKendrick Epidemic Model [1927]

Constant population divided into 3 compartments:

- $S(t) = \text{number of individuals susceptible to disease}$
- $I(t) = \text{number of individuals infected by disease}$
- $R(t) = \text{number of individuals recovered from disease}$

$N = S + I + R$ is total population number
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\[
\begin{align*}
S' &= -\beta SI \\
I' &= \beta SI - \gamma I \\
R' &= \gamma I
\end{align*}
\]

- \( \beta \) = effective contact coefficient
- \( 1/\gamma \) = average time of infection (depends on disease)

Length of the infective period is exponentially distributed
Dynamics of Kermack-McKendrick Model

- Asymptotic analysis for \( I' = \beta SI - \gamma I \)
  - If \( \beta S(0) < \gamma \) then \( I(t) \) decreases monotonically to 0
  - If \( \beta S(0) > \gamma \) then \( I(t) \to I_{\text{max}} \to 0 \) as \( t \to \infty \)
    leaving a positive number of susceptibles \( S(\infty) > 0 \)

\[
I_{\text{max}} = I(0) + S(0) - \frac{\gamma}{\beta} \left( 1 + \ln \frac{\beta S(0)}{\gamma} \right)
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- Basic reproduction number (threshold): \( R_0 = \frac{\beta S(0)}{\gamma} \)
  - \( R_0 < 1 \): disease dies out
  - \( R_0 > 1 \): there is a disease epidemic

The final size relation

\[
1 - R_0(\infty) = \exp(-R_0(\infty))
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Simulations with $R_0 = 2.7$
Simulations with Other $R_0$ Values

<table>
<thead>
<tr>
<th>$R_0$</th>
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<tbody>
<tr>
<td>0.9</td>
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<tr>
<td>1.8</td>
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<td>2.7</td>
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<td>3.6</td>
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Disease Control

- To control the disease (no epidemic) need to decrease $\mathcal{R}_0$ by decreasing $\beta$

  - For smallpox $\mathcal{R}_0 \approx 5$ vaccination of about 80% provided herd immunity (now eradicated !)
  - For measles in urban areas $\mathcal{R}_0 \approx 10^{-16}$ need to vaccinate over 90%
  - For ebola $\mathcal{R}_0 \approx 2$ need to vaccinate over only 50% (hypothetically)
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- To control the disease (no epidemic) need to decrease $R_0$ by decreasing $\beta$.

- For some diseases this can be done by vaccination.
  Vaccinate a fraction $p$ of population so that

  $$\beta(1 - p)S(0)\frac{1}{\gamma} < 1 \iff p > 1 - \frac{1}{R_0}$$
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