The Extremal Function for $K_{10}$ Minors

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Roadmap

1. The Four Color Theorem and Hadwiger’s Conjecture
2. The Extremal Function for $K_t$ Minors
3. Proof Outline of Our Conjecture
4. Future Work
For $t \in \mathbb{Z}^+$, a graph $G$ is $t$-colorable if there exists a mapping $c : V(G) \rightarrow \{1, 2, ..., t\}$ such that $c(u) \neq c(v)$ for every edge $uv \in E(G)$. 
Preliminaries

For $t \in \mathbb{Z}^+$, a graph $G$ is $t$-colorable if there exists a mapping $c : V(G) \rightarrow \{1, 2, \ldots, t\}$ such that $c(u) \neq c(v)$ for every edge $uv \in E(G)$.

For graphs $H$ and $G$, say $G$ has an $H$-minor if a graph isomorphic to $H$ can be obtained from a subgraph of $G$ by contracting edges, denoted as $G > H$. 
The Four Color Theorem (FCT)

The Four Color Theorem (Appel and Haken’76)

Every planar graph is 4-colorable.
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Every planar graph is 4-colorable.

Theorem (Kuratowski’30; Wagner’37)
A graph is planar if and only if it has no $K_5$ or $K_{3,3}$ minor.
### The Four Color Theorem (FCT)

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Restatement: Every graph with no $K_5$ or $K_{3,3}$ minor is 4-colorable.

Is every graph with no $K_5$ minor 4-colorable?
For every integer \( t \geq 0 \), every graph with no \( K_{t+1} \) minor is \( t \)-colorable.
Hadwiger’s Conjecture

For every integer $t \geq 0$, every graph with no $K_{t+1}$ minor is $t$-colorable.

- $t \leq 3$: Easy
- $t = 4$: HC ($\iff$ FCT) Wagner (1937)
- $t = 5$: HC ($\iff$ FCT) Robertson, Seymour, and Thomas (1993)
- $t \geq 6$: OPEN

Dirac (1952); Hadwiger (1943)
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The Extremal Function for $K_t$ minors

**Theorem (Mader’68)**

For every integer $t = 1, 2, ..., 7$, a graph on $n \geq t$ vertices and at least $(t - 2)n - \binom{t-1}{2} + 1$ edges has a $K_t$ minor.
The Extremal Function for $K_t$ minors

Theorem (Mader’68)

For every integer $t = 1, 2, \ldots, 7$, a graph on $n \geq t$ vertices and at least $(t - 2)n - \binom{t-1}{2} + 1$ edges has a $K_t$ minor.

- Counter-example for $t = 8$: $K_{2,2,2,2,2}$
The Extremal Function for $K_t$ minors

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- **Counter-example for** $t = 8$: $K_{2,2,2,2,2}$

- **More counter-examples:** $(K_{2,2,2,2,2,5})$-cockades! - graphs obtained from disjoint copies of $K_{2,2,2,2,2}$ by identifying cliques of size 5
The Extremal Function for $K_t$ minors

For positive integers $t$ and $n$, let

$$M(t, n) = (t - 2)n - \binom{t - 1}{2} + 1.$$
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**Theorem for $K_8$ Minors (Jørgensen’94)**

Every graph on $n \geq 8$ vertices and at least $M(8, n) = 6n - 20$ edges either has a $K_8$ minor or is a $(K_{2,2,2,2,2,5})$-cockade.

**Theorem for $K_9$ Minors (Song and Thomas’06)**

Every graph on $n \geq 9$ vertices and at least $M(9, n) = 7n - 27$ edges either has a $K_9$ minor or is a $(K_{1,2,2,2,2,2,6})$-cockade, or is isomorphic to $K_{2,2,2,3,3}$.

Can we prove a similar statement for $K_{10}$ minors?
The Extremal Function for $K_t$ minors

For positive integers $t$ and $n$, let

$$M(t, n) = (t - 2)n - \left(\frac{t - 1}{2}\right) + 1.$$ 

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The Extremal Function for $K_t$ minors

**Our Conjecture for $K_{10}$ Minors**

Every graph on $n \geq 8$ vertices and at least $M(10, n) = 8n - 35$ edges either has a $K_{10}$ minor or is isomorphic to one of the following: a $(K_{1,1,2,2,2,2,2,7})$-cockade, $K_{1,2,2,2,3,3}$, $K_{2,2,2,2} + C_5$, $K_{2,2,3,3,4}$, $K_{3,3,3} + C_5$, $K_{2,2,2,2,2,3}$, $K_{2,3,3,3,3}$, and $J - e$ where $J \in \{K_{2,2,2,2,2,3}, K_{2,3,3,3,3}\}$ and $e \in E(J)$.

Notation: $H + G$ denotes the graph obtained from $H \cup G$ by adding edges $xy$ for all $x \in V(H)$ and $y \in V(G)$. 
Current Status of Related Works

HC for $t = 5$ (Roberson, Seymour, and Thomas'93): Every graph with no $K_6$ minor is 5-colorable.

HC for $t = 6$ is open: Every graph with no $K_7$ minor is 6-colorable.
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- **Weaker Versions of HC when** $t \geq 6$:
  - Kawarabayashi and Toft'05: Every graph with no $K_7$ minor is either 6-colorable or has a $K_4, 4$ minor.
  - Albar and Gonçalves'13; Rolek and Song'17: For $t = 7, 8, 9$, every graph with no $K_t$ minor is $(2t - 6)$-colorable.
  - Rolek and Song'18: For $t \geq 6$, if every graph on $n \geq t$ vertices and at least $M(t, n)$ edges either contains a $K_t$ minor or is $(t - 1)$-colorable, then every graph with no $K_t$ minor is $(2t - 6)$-colorable.
  - Rolek and Song'18: For $t \leq 9$, every doubly-critical $t$-chromatic graph contains a $K_t$ minor.
  - Thomas and Yoo'18: For $t = 2, 3, \ldots, 9$, a triangle-free graph $G$ on $n \geq 2t - 5$ vertices and at least $(t - 2)n - (t - 2)^2 + 1$ edges has a $K_t$ minor.
  - Jakobsen'73, Jakobsen'83, Song'05: The extremal function for $K_t$ minors for $t \leq 8$. 
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- **Jakobsen’73, Jakobsen’83, Song’05**: The extremal function for $K_t^-$ minors for $t \leq 8$. 
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Outline of the proof of the $K_{10}$ Minor Conjecture

Conjecture (Thomas, Z.)

Every graph on $n \geq 8$ vertices and at least $8n - 35$ edges either has a $K_{10}$ minor or is isomorphic to one of the following: a $(K_{1,1,2,2,2,2,2,7})$-cockades, $K_{1,2,2,2,3,3}$, $K_{2,2,2,2} + C_5$, $K_{2,2,3,3,4}$, $K_{3,3,3} + C_5$, $K_{2,2,2,2,2,3}$, $K_{2,3,3,3,3}$, and $J - e$ where $J \in \{K_{2,2,2,2,2,3}, K_{2,3,3,3,3}\}$ and $e \in E(J)$.

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Definition

A graph $H$ be called an exceptional graph if $H$ is isomorphic one of the $K_{10}$ minor-free graphs in the above conjecture.
Minimal Counter-Example to the Conjecture

- **Exceptional graphs:** $(K_{1,1,2,2,2,2,2}, 7)$-cockade, $K_{1,2,2,2,3,3}$, $K_{2,2,2,2} + C_5$, $K_{2,2,3,3,4}$, $K_{3,3,3} + C_5$, $K_{2,2,2,2,2,3}$, $K_{2,3,3,3,3}$, and $J - e$ where $J \in \{K_{2,2,2,2,2,3}, K_{2,3,3,3,3}\}$ and $e \in E(J)$. 
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- Let \(G\) denote a **minimal counter-example** to the conjecture, i.e.
  
  1. \(e(G) \geq 8|G| - 35\)
  2. \(G \not\supseteq K_{10}\)
  3. \(G\) is not an exceptional graph
  4. subject to (1)-(3), \(|G|\) is minimum
  5. subject to (1)-(4), \(e(G)\) is minimum
Mineral Counter-Example to the Conjecture

**Notation:** For a graph $G$ and a vertex $x \in V(G)$, use $N(x)$ to denote the set of vertices adjacent to $x$ in $G$ as well as the induced subgraph of $G$ on the set $N(x)$.

**Lemma 1**

$$e(G) = 8n - 35 \geq 11/\delta(G) \geq 8$$ for every $x \in V(G)$

4. Every proper minor $G'$ of $G$ satisfies that $e(G') \leq 8|G'|-35$.

$G$ is 7-connected $e(G) = 8|G|-35, \delta(G) \geq 11 \Rightarrow \exists x \in V(G)$ such that $11 \leq d(x) \leq 15$

Case 1:

$\exists$ a component $K$ of $G - N[x]$ such that $N(K) = N(x)$

Case 2:

$\forall x \in V(G)$ with $11 \leq d(x) \leq 15, \forall$ component $K$ of $G - N[x], N(K) \subset N(x)$. 
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**Lemma**

1. $e(G) = 8n - 35$
2. $\delta(G) \geq 11$
3. $\delta(N(x)) \geq 8$ for every $x \in V(G)$
4. every proper minor $G'$ of $G$ satisfies that $e(G') \leq 8|G'| - 35$
5. $G$ is 7-connected
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\[ e(G) = 8|G| - 35, \quad \delta(G) \geq 11 \]
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\[e(G) = 8|G| - 35, \ \delta(G) \geq 11 \Rightarrow \exists x \in V(G) \text{ such that } 11 \leq d(x) \leq 15\]
**Minimal Counter-Example to the Conjecture**

**Notation:** For a graph $G$ and a vertex $x \in V(G)$, use $N(x)$ to denote the set of vertices adjacent to $x$ in $G$ as well as the induced subgraph of $G$ on the set $N(x)$.

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$$e(G) = 8|G| - 35, \quad \delta(G) \geq 11 \Rightarrow \exists x \in V(G) \text{ such that } 11 \leq d(x) \leq 15$$

- **Case 1:** $\exists$ a component $K$ of $G - N[x]$ such that $N(K) = N(x)$
- **Case 2:** $\forall x \in V(G)$ with $11 \leq d(x) \leq 15$, $\forall$ component $K$ of $G - N[x]$, $N(K) \subset N(x)$. 
Case 1: \( \exists x \in V(G) \) with \( 11 \leq d(x) \leq 15 \) and a component \( K \) of \( G - N[x] \) such that \( N(K) = N(x) \)

- If \( N(x) > K_8 \cup K_1 \Rightarrow G > K_{10}, \) a contradiction.
Case 1: \( \exists x \in V(G) \) with \( 11 \leq d(x) \leq 15 \) and a component \( K \) of \( G - N[x] \) such that \( N(K) = N(x) \)

- If \( N(x) > K_8 \cup K_1 \Rightarrow G > K_{10} \), a contradiction.

- The induced subgraph \( N(x) \) has the following properties:
  
  (i) \( 11 \leq |N(x)| \leq 15 \)  
  (ii) \( \delta(N(x)) \geq 8 \)  
  (iii) \( N(x) \not\sim K_8 \cup K_1 \)
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  (ii) \( \delta(N(x)) \geq 8 \)  
  (iii) \( N(x) \not\supseteq K_8 \cup K_1 \)
- There are only finitely many graphs satisfying (i)-(iii)!
Case 1: \( \exists x \in V(G) \) with \( 11 \leq d(x) \leq 15 \) and a component \( K \) of \( G - N[x] \) such that \( N(K) = N(x) \)

**Lemma (computer-assisted)**

Up to isomorphism, there are precisely 101 graphs \( H \) satisfying that (i) \( 11 \leq |H| \leq 15 \), (ii) \( \delta(H) \geq 8 \), (iii) \( H \not> K_8 \cup K_1 \), and (iv) every \( e \in E(H) \) has an end of degree 8.
Case 1: \( \exists x \in V(G) \) with \( 11 \leq d(x) \leq 15 \) and a component \( K \) of \( G - N[x] \) such that \( N(K) = N(x) \)

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- If \( N(x) \) is isomorphic to one of the 101 graphs
  \( \Rightarrow \) can always find some \( L \supseteq N(x) \) such that \( G - x \) has a rooted \( L \)-minor on \( N(x) \) and \( L > K_9 \)
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Up to isomorphism, there are precisely 101 graphs \( H \) satisfying that (i) \( 11 \leq |H| \leq 15 \), (ii) \( \delta(H) \geq 8 \), (iii) \( H \not\cong K_8 \cup K_1 \), and (iv) every \( e \in E(H) \) has an end of degree 8.

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- \( \Rightarrow G > K_{10} \), a contradiction
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Up to isomorphism, there are precisely 101 graphs \( H \) satisfying that (i) \( 11 \leq |H| \leq 15 \), (ii) \( \delta(H) \geq 8 \), (iii) \( H \not\supset K_8 \cup K_1 \), and (iv) every \( e \in E(H) \) has an end of degree 8.

- If \( N(x) \) is isomorphic to one of the 101 graphs
  \( \Rightarrow \) can always find some \( L \supseteq N(x) \) such that \( G - x \) has a rooted \( L \)-minor on \( N(x) \) and \( L > K_9 \)
  \( \Rightarrow G > K_{10} \), a contradiction

- Examples of \( L - N(x) \): a perfect matching of size 2 or 3, \( K_3 \cup P_2 \), \( P_4 \cup P_2 \), etc
Case 2: $\forall x \in V(G)$ with $11 \leq d(x) \leq 15$ and $\forall$ component $K$ of $G - N[x]$, $N(K) \subset N(x)$

- $\delta(G) \geq 11$, $e(G) = 8|G| - 35$, and every proper minor $G'$ of $G$ satisfies $e(G') \leq 8|G'| - 35$
- Can apply a similar argument used by Jørgensen, Song, and Thomas to show $G > K_{10}$, a contradiction
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Future Work

- Finish checking the argument in the current proof

- **The Extremal Function for $K_9^-$ Minors Conjecture:**
  Every graph on $n \geq 9$ vertices and at least $\frac{13}{2} n - 24$ edges either has a $K_9^-$ minor or falls into a few families of exceptional graphs.

- **Conjecture by Albar and Gonçalves:**
  Every graph that has every edge belonging to at least 7 triangles either has a $K_9$ minor or contains an induced $K_{1,2,2,2,2,2,2}$.

- Construction of graphs with average degree of order $t \sqrt{\log t}$ that lack a $K_t$ minor for large $t$