Gallai-Ramsey numbers of cycles

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Basic definitions

Let $G, H_1, \ldots, H_k$ be graphs.

- $G \rightarrow (H_1, \ldots, H_k)$ if every $k$-coloring of $E(G)$ contains a monochromatic copy of $H_i$ in color $i$ for some $i \in [k]$, where $[k] := \{1, 2, \ldots, k\}$.

- $R(H_1, \ldots, H_k) := \min\{n : K_n \rightarrow (H_1, \ldots, H_k)\}$.

- If $H = H_1 = \cdots = H_k$, write $R_k(H) := R(H_1, \ldots, H_k)$.

the classical **Ramsey number**

\[ K_5 \rightarrow (K_3, K_3) \]
# Known Ramsey numbers of odd cycles

<table>
<thead>
<tr>
<th>Ramsey Number</th>
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<tbody>
<tr>
<td>$R_2(C_{2n+1}) = 4n + 1$ for $n \geq 2$</td>
<td>Faudree, Schelp, 1974; Rosta, 1973</td>
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<tr>
<td>$R_3(C_3) = 17$</td>
<td>Greenwood, Gleason, 1955</td>
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<td>$R_3(C_5) = 17$</td>
<td>Yang, Rowlinson, 1992</td>
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<tr>
<td>$R_3(C_7) = 25$</td>
<td>Faudree, Schelten, Schiermeyer, 2003</td>
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</table>
For all $k \geq 1$ and $n \geq 2$: $R_k(C_{2n+1}) \geq n \cdot 2^k + 1$. 

(Bondy and Erdős 1973)
Lower bound for $R_k(C_{2n+1})$

For all $k \geq 1$ and $n \geq 2$: $R_k(C_{2n+1}) \geq n \cdot 2^k + 1$. 

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For all \( k \geq 1 \) and \( n \geq 2 \): \( R_k(C_{2n+1}) \geq n \cdot 2^k + 1 \).

(Bondy and Erdős 1973)
Conjecture (Bondy, Erdős, 1973)

For all $k \geq 3$ and $n \geq 2$, $R_k(C_{2n+1}) = n \cdot 2^k + 1.$

- When $k = 3$, it’s also known as **Triple Odd Cycle Conjecture**.
Ramsey numbers of odd cycles

**Conjecture (Bondy, Erdős, 1973)**  
*For all* $k \geq 3$ *and* $n \geq 2$,  
$$R_k(C_{2n+1}) = n \cdot 2^k + 1.$$  

- When $k = 3$, it’s also known as **Triple Odd Cycle Conjecture**.

**THM [Łuczak, 1999]**  
$R_3(C_{2n+1}) = 8n + o(n)$ as $n \to \infty$.

- This implies the Triple Odd Cycle Conjecture is asymptotically **true**.

**THM [Jenssen, Skokan, 2016]**  
For fixed $k$ and $n$ sufficiently large,  
$$R_k(C_{2n+1}) = n \cdot 2^k + 1.$$  

- **Bondy-Erdős** Conjecture is **true** for fixed $k$ and $n$ sufficiently large.

**THM [Day, Johnson, 2017]**  
For fixed $n$ and $k$ sufficiently large, there exists a constant $\epsilon(n) > 0$ such that  
$$R_k(C_{2n+1}) > 2n \cdot (2 + \epsilon)^{k-1}.$$  

- **Bondy-Erdős** Conjecture is **false** for fixed $n$ and $k$ sufficiently large.
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<thead>
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<tr>
<td>$R_2(C_4) = 6$</td>
<td>Chartrand, Schuster, 1971</td>
</tr>
<tr>
<td>$R_2(C_6) = 8$</td>
<td></td>
</tr>
<tr>
<td>$R_2(C_{2n}) = 3n - 1$, $n \geq 4$</td>
<td>Faudree, Schelp, 1974; Rosta, 1973</td>
</tr>
<tr>
<td>$R_3(C_4) = 11$</td>
<td>Bialostocki, Schönheim, 1984</td>
</tr>
<tr>
<td>$R_3(C_6) = 12$</td>
<td>Yang, Rowlinson, 1993</td>
</tr>
<tr>
<td>$R_3(C_8) = 16$</td>
<td>Sun, Yang, 2011</td>
</tr>
<tr>
<td>$R_4(C_4) = 18$</td>
<td>Exoo, 1987; Sun, Yang, Lin, Zheng, 2007</td>
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Lower bound for $R_k(C_{2n})$

For all $k \geq 1$ and $n \geq 2$,

$$R_k(C_{2n}) \geq \begin{cases} (k + 1)n & \text{if } k \text{ is odd} \\ (k + 1)n - 1 & \text{if } k \text{ is even.} \end{cases}$$

(Dzido, Nowik, Szuca, 2005)

$k = 3$:
Conjecture (Dzido, 2005)

For all $n \geq 3$, $R_3(C_{2n}) = 4n$. 
Conjecture (Dzido, 2005)

For all \( n \geq 3 \), \( R_3(C_{2n}) = 4n \).

**THM** [Figai, Łuczak, 2007]

\[ R_3(C_{2n}) = 4n + o(n) \text{ as } n \to \infty. \]

- This implies the Triple Even Cycle Conjecture is asymptotically true.

**THM** [Benevides, Skokan, 2009]

There exists \( n_1 \) such that for every \( n \geq n_1 \), \( R_3(C_{2n}) = 4n \).
Question: under what conditions will the Bondy-Erdős Conjecture be true for all colors and all odd cycles?
Gallai coloring

Rainbow Triangle:
A Gallai coloring is an edge-coloring of a complete graph with no rainbow triangle.

A Gallai $k$-coloring is a Gallai coloring that uses at most $k$ colors.
Theorem (Gallai, 1967)

For any Gallai coloring \( \tau \) of a complete graph \( G \) with \( |G| \geq 2 \), \( V(G) \) can be partitioned into nonempty sets \( V_1, V_2, \ldots, V_p \) with \( p > 1 \) so that at most two colors are used on the edges in \( E(G) \setminus (E(V_1) \cup \cdots \cup E(V_p)) \) and only one color is used on the edges between any fixed pair \( (V_i, V_j) \) under \( \tau \), where \( E(V_i) \) denotes the set of edges in \( G[V_i] \) for all \( i \in [p] \).
Gallai-Ramsey numbers

- $G \xrightarrow{\text{Gallai}} (H_1, \ldots, H_k)$ if every Gallai $k$-coloring of $E(G)$ contains a monochromatic copy of $H_i$ in color $i$ for some $i \in [k]$.

- $GR(H_1, \ldots, H_k) := \min\{n : K_n \xrightarrow{\text{Gallai}} (H_1, \ldots, H_k)\}$.

If $H = H_1 = \cdots = H_k$, write $GR_k(H) := GR(H_1, H_2, \ldots, H_k)$.

The Gallai-Ramsey number

- $GR_k(H) \leq R_k(H)$ for all $k \geq 1$.

- $GR_2(H) = R_2(H)$. 
$GR_k(P_3) = ?$

Lower Bound:

∴ $K_2 \not\rightarrow (P_3, \ldots, P_3)$.

Upper Bound:

∴ $K_3 \rightarrow (P_3, \ldots, P_3)$.

Hence, $GR_k(P_3) = 3$. 
$GR_k(P_3)$ =?

**Lower Bound:**

$\therefore K_2 \xrightarrow{\text{Gallai}} (P_3, \ldots, P_3)$.
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**Upper Bound:**

Hence, $GR_k(P_3) = 3$. 

Fangfang Zhang

Gallai-Ramsey numbers of cycles
$GR_k(P_3) =$?

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**Upper Bound:**

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Gallai-Ramsey numbers of cycles
$GR_k(P_3) =$?

Lower Bound:

$\therefore K_2 \xrightarrow{\text{Gallai}} (P_3, \ldots, P_3)$.

Upper Bound:

$\therefore K_3 \xrightarrow{\text{Gallai}} (P_3, \ldots, P_3)$.

Hence, $GR_k(P_3) = 3$. 
$GR_k(C_3) = ?$

For all $k \geq 2$, construct $G_k$ as follows: (Chung, Graham, 1983)

$G_{2k}$ ($k$ is even)
$G_{2k+1}$ ($k$ is odd)

∴ $GR_k(C_3) \geq \begin{cases} \frac{5k}{2} + 1 & \text{if } k \text{ is even} \\ 2 \cdot \frac{5(k-1)}{2} + 1 & \text{if } k \text{ is odd} \end{cases}$
$GR_k(C_3) =$?

For all $k \geq 2$, construct $G_k$ as follows:

(Chung, Graham, 1983)

$G_2$

$G_k(k \text{ is even})$

$G_k(k \text{ is odd})$

$\therefore GR_k(C_3) \geq \begin{cases} 5^{k/2} + 1 & \text{if } k \text{ is even} \\ 2 \cdot 5^{(k-1)/2} + 1 & \text{if } k \text{ is odd} \end{cases}$
Theorem (Chung, Graham, 1983)

For all \( k \geq 1 \),

\[
GR_k(C_3) = \begin{cases} 
5^{k/2} + 1 & \text{if } k \text{ is even} \\
2 \cdot 5^{(k-1)/2} + 1 & \text{if } k \text{ is odd}
\end{cases}
\]
General behavior of $GR_k(H)$

Theorem (Gyárfás, Sárközy, Sebő, Selkow, 2010)

Let $H$ be a fixed graph with no isolated vertices and let $k \geq 1$ be an integer. Then $GR_k(H)$ is:

(i) exponential in $k$ if $H$ is not bipartite,

(ii) linear in $k$ if $H$ is bipartite but not a star, and

(iii) constant (does not depend on $k$) when $H$ is a star.
$R_2(H)$ is closely related to $GR_k(H)$
$R_2(H)$ is closely related to $GR_k(H)$

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Gallai-Ramsey numbers of cycles
$R_2(H)$ is closely related to $GR_k(H)$

If the reduced graph $\mathcal{R}$ of a complete graph $G$ contains a monochromatic copy of a graph $H$, then so does $G$. 
A Conjecture of $GR_k(K_t)$

Conjecture (Fox, Grinshpun, Pach, 2015)

For all $k \geq 1$ and $t \geq 3$,

$$GR_k(K_t) = \begin{cases} (R_2(K_t) - 1)^{k/2} + 1, & \text{if } k \text{ is even} \\ (t - 1)(R_2(K_t) - 1)^{(k-1)/2} + 1, & \text{if } k \text{ is odd.} \end{cases}$$
A Conjecture of $GR_k(K_t)$

**Conjecture (Fox, Grinshpun, Pach, 2015)**

For all $k \geq 1$ and $t \geq 3$,

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(R_2(K_t) - 1)^{k/2} + 1, & \text{if } k \text{ is even} \\
(t - 1)(R_2(K_t) - 1)^{(k-1)/2} + 1, & \text{if } k \text{ is odd.} 
\end{cases}$$

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<thead>
<tr>
<th>$GR_k(K_t)$</th>
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<tbody>
<tr>
<td>$t = 3$</td>
<td>Chung, Graham, 1983</td>
</tr>
<tr>
<td>$t = 4$</td>
<td>Liu, Magnant, Saito, Schiermeyer, Shi, 2017+</td>
</tr>
<tr>
<td>$t \geq 5$</td>
<td>Open</td>
</tr>
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</table>

- **Schiermeyer** 2018: There is a Gallai 3-colored $K_{169}$ with no m.c. $K_5$.
- **Conjecture** [McKay, Radziszowski, 1997]: $R_2(K_5) = 43$. 

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Gallai-Ramsey numbers of cycles
What about $GR_k(C_{2n+1})$?

$GR_k(C_{2n+1}) \geq n \cdot 2^k + 1$.

(Bondy, Erdős, 1973)
What about $GR_k(C_{2n+1})$?

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<tr>
<td>$GR_k(C_5) = 2 \cdot 2^k + 1$</td>
<td>Fujita, Magnant, 2011</td>
</tr>
<tr>
<td>$GR_k(C_7) = 3 \cdot 2^k + 1$</td>
<td>Bruce, Song, 2017</td>
</tr>
<tr>
<td>$GR_k(C_9) = 4 \cdot 2^k + 1$</td>
<td>Bosse, Song, 2018+</td>
</tr>
<tr>
<td>$GR_k(C_{11}) = 5 \cdot 2^k + 1$</td>
<td>Bosse, Song, Zhang, 2018+</td>
</tr>
<tr>
<td>$GR_k(C_{13}) = 6 \cdot 2^k + 1$</td>
<td></td>
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<tr>
<td>$GR_k(C_{15}) = 7 \cdot 2^k + 1$</td>
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What about $GR_k(C_{2n+1})$?

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<td>$GR_k(C_{2n+1}) \leq k(n - 1) + n(4n + 1)3^{k-3}$</td>
<td>Fujita, Magnant, 2011</td>
</tr>
<tr>
<td>$GR_k(C_{2n+1}) \leq (2^{k+3} - 3)n \ln n$</td>
<td>Hall, Magnant, Ozeki, Tsugaki, 2014</td>
</tr>
<tr>
<td>$GR_k(C_{2n+1}) \leq (4n + n \log_2 n) \cdot 2^k$</td>
<td>Chen, Li, Pei, 2017</td>
</tr>
<tr>
<td>$GR_k(C_{2n+1}) \leq (n \log n) \cdot 2^k - (k + 1)n + 1$</td>
<td>Bosse, Song, Zhang, 2018+</td>
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</table>
Our result

**THM [Chen, Song, FFZ, 2018+]**

For all $k \geq 1$ and $n \geq 3$,

$$GR_k(C_{2n+1}) = n \cdot 2^k + 1.$$ 

- **Bondy-Erdős Conjecture** is true under Gallai-coloring.
Question: what about Gallai-Ramsey numbers of even cycles?
What about $GR_k(C_{2n})$?

$GR_k(C_{2n}) \geq (n - 1)k + n + 1.$

(Erdős, Faudree, Rousseau, Schelp, 1974)
What about \( GR_k(C_{2n}) \)?

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<tr>
<td>( GR_k(C_4) = k + 4 )</td>
<td>Faudree, Gould, Jacobson, Magnant, 2010</td>
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<td>( GR_k(C_6) = 2k + 4 )</td>
<td>Fujita, Magnant, 2011</td>
</tr>
<tr>
<td>( GR_k(C_8) = 3k + 5 )</td>
<td>Song, Zhang, 2018+</td>
</tr>
<tr>
<td>( GR_k(C_{10}) = 4k + 6 )</td>
<td>Lei, Shi, Song, Zhang, 2018+</td>
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<td>( GR_k(C_{12}) = 5k + 7 )</td>
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What about $GR_k(C_{2n})$?

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<td>$GR_k(C_{2n}) \leq (n - 1)k + O(n \log n)$</td>
<td>Fujita, Magnant, 2011</td>
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<td>$GR_k(C_{2n}) \leq (n - 1)k + 3n$</td>
<td>Hall, Magnant, Ozeki, Tsugaki, 2014</td>
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<td>$GR_k(C_{2n}) \leq (n - 1)k + 2n + 2$</td>
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THM [Chen, Song, FFZ, 2019+]

For all $k \geq 1$ and $n \geq 3$,

$$GR_k(C_{2n}) = (n - 1)k + n + 1.$$
Gallai-Ramsey numbers of wheels

- $W_n$ denotes a wheel on $n + 1$ vertices.
- $W_3 = K_4$.

**THM [Song, Wei, FFZ, Zhao, 2019+]**

For all $k \geq 2$,

$$GR_k(W_4) = \begin{cases} 
14 \cdot 5^{\frac{k-2}{2}} + 1, & \text{if } k \text{ is even,} \\
28 \cdot 5^{\frac{k-3}{2}} + 1, & \text{if } k \text{ is odd.}
\end{cases}$$

**Conjecture (Song, 2019+)**

For all $n \geq 2$ and $k \geq 2$,

$$GR_k(W_{2n}) = \begin{cases} 
(R_2(W_{2n}) - 1) \cdot 5^{(k-2)/2} + 1 & \text{if } k \text{ is even} \\
2(R_2(W_{2n}) - 1) \cdot 5^{(k-3)/2} + 1 & \text{if } k \text{ is odd.}
\end{cases}$$
Let $G = (V, E)$ be a graph, and let $A, B \subseteq V(G)$ be disjoint.

- $(G, \tau)$ denotes a Gallai $k$-colored complete graph if $G$ is a complete graph and $\tau : E(G) \to [k]$ is a Gallai $k$-coloring.

- $A$ is **mc-complete** to $B$ under $(G, \tau)$ if all the edges between $A$ and $B$ in $G$ are colored the same color.

- $A$ is **red-complete** to $B$ if all the edges between $A$ and $B$ in $G$ are colored red under $(G, \tau)$.

![Diagram showing red-complete relationship between sets A and B]
For each \((G, \tau)\), we use \(G_i\) to denote the spanning subgraph of \(G\) with \(E(G_i) := \{e \in E(G) \mid \tau(e) = i\}\) for all \(i \in [k]\).

\(q_\tau(G, n)\) denotes the number of colors \(i \in [k]\) such that \(G_i\) has a component of order at least \(n\).

\[1 \leq q_\tau(G, n) \leq k.\]
For each \((G, \tau)\), we use \(G_i\) to denote the spanning subgraph of \(G\) with 
\[
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\[1 \leq q_\tau(G, n) \leq k.\]

**THM [Chen, Song, FFZ, 2019+]**

For every \((G, \tau)\) and every \(n\) with \(|G| \geq n \geq 3\), if 
\[|G| \geq (n - 1)q_\tau(G, n) + n + 1,\]
then \(G\) has a monochromatic copy of \(C_{2n}\).
For each \((G, \tau)\), we use \(G_i\) to denote the spanning subgraph of \(G\) with 
\[E(G_i) := \{e \in E(G) | \tau(e) = i\}\] for all \(i \in [k]\).

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**THM [Chen, Song, FFZ, 2019+]**

For every \((G, \tau)\) and every \(n\) with \(|G| \geq n \geq 3\), if
\[|G| \geq (n - 1)q_\tau(G, n) + n + 1,\] then \(G\) has a monochromatic copy of \(C_{2n}\).

**THM [Chen, Song, FFZ, 2019+]**

For all \(k \geq 1\) and \(n \geq 3\), \(GR_k(C_{2n}) = (n - 1)k + n + 1.\)
Proof sketch

THM [ Chen, Song, FFZ, 2018+]

For all \( k \geq 1 \) and \( n \geq 3 \), \( GR_k(C_{2n+1}) = n \cdot 2^k + 1 \).

Proof. Let \((G, \tau)\) be a Gallai \( k \)-colored \( K_{n \cdot 2^k + 1} \). Suppose \((G, \tau)\) contains no m.c. \( C_{2n+1} \).

- Choose \((G, \tau)\) with \( k \) minimum. Then \( k \geq 3 \).
Proof sketch

THM [Chen, Song, FFZ, 2018+]

For all $k \geq 1$ and $n \geq 3$, $GR_k(C_{2n+1}) = n \cdot 2^k + 1$.

Proof. Let $(G, \tau)$ be a Gallai $k$-colored $K_{n \cdot 2^k+1}$. Suppose $(G, \tau)$ contains no m.c. $C_{2n+1}$.

- Choose $(G, \tau)$ with $k$ minimum. Then $k \geq 3$.
- Let $X_1, \ldots, X_k$ be disjoint subsets of $V(G)$ such that for each $i \in [k]$, $X_i$ (possibly empty) is mc-complete in color $i$ to $V(G) \setminus \bigcup_{i=1}^k X_i$.
- Let $X := \bigcup_{i=1}^k X_i$. Choose $X$ such that $|X| \leq (k+1)n$ is as large as possible.

- $|X_i| \leq n - 1$ and $X_i = \emptyset$ for some $i \in [k]$, so $|X| \leq (k - 1)(n - 1)$. 
Proof sketch

Consider the Gallai Partition of the graph $G\setminus X$ with parts $V_1, \ldots, V_P$, where $p \geq 2$ and $|V_1| \leq \cdots \leq |V_P|$.
Proof sketch

Proof sketch

\[ X \leq n - 1 \] (the most difficult case). This means \[ q \tau (G - X, n) \leq 2 \]. Assume \[ |B| \leq |R| \]. By the stronger result for even cycles, there exists either a blue \( C_2 \) or red \( C_2 \) in \( G - X \). Then we will have \[ |X \cup V_p| \leq n + (k - 2)(n - 1) \].

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Gallai-Ramsey numbers of cycles
Proof sketch

- $|V_p| \leq n - 1$ (the most difficulty case). This means $q_\tau(G - X, n) \leq 2$. Assume $|B| \leq |R|$. By **stronger result of even cycles**, there exists either a blue $C_{2n}$ or red $C_{2n}$ in $G - X$. Then we will have $|X \cup V_p| \leq n + (k - 2)(n - 1)$. 

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Gallai-Ramsey numbers of cycles
Proof sketch

$|V_p| \leq n - 1$ (the most difficulty case). This means $q_\tau(G - X, n) \leq 2$. Assume $|B| \leq |R|$.

From $|X \cup V_p| \leq n + (k - 2)(n - 1)$, $|R| \geq \frac{|G| - |X \cup V_p|}{2} > 3n - 1$, by stronger result of even cycles again, there exists a blue $C_{2n} \subseteq R$. We will extent this blue $C_{2n}$ to blue $C_{2n+1}$, if not we will have a red $C_{2n+1}$.
THANK YOU!