The typical structure of Gallai colorings and their extremal graphs

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2. Gallai colorings of complete graphs

3. Erdős-Rothchild-type problems

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Introduction
An edge coloring of a graph $G$ is a Gallai coloring if it contains no rainbow triangle, that is, no triangle is colored with three distinct colors. [Gyárfás, Simonyi]

Gallai colorings occur in relation of deep structural properties of fundamental objects.

- The theory of partially ordered sets [Gallai]
- Some applications in Information theory [Körner, Simonyi]
- Generalizations of the perfect graph theorem [Cameron, Edmonds, Lovász]
The study of Gallai colorings has a rich history.

- Structural results
- Ramsey-type results

**Extremal perspective:**

♣ How many Gallai $r$-colorings are there?

♣ Can we describe the typical structure of Gallai $r$-colorings?
Gallai colorings of complete graphs
Given $r$ colors, the number of 2-colorings of $K_n$ is exactly

$$\binom{r}{2} \left( 2^{\binom{n}{2}} - 2 \right) + r = \binom{r}{2} 2^{\binom{n}{2}} - r(r - 2).$$
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If we further consider all Gallai colorings of $K_n$ using exactly 3 colors, in which one of the colors $i$ is used only once, the number of them is exactly

$$\binom{n}{2} \left( 2^{\binom{n}{2}} - (n - 1) - 2 \right).$$
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Therefore, the trivial lower bound is

$$\left( \binom{r}{2} + 2^{-n} \right) 2^{\binom{n}{2}}.$$
### Upper bounds

**Theorem (Falgas-Ravry, O’Connell and Uzzell, 2018)**

The number of Gallai 3-colorings of $K_n$ is $2^{(1+o(1)) \binom{n}{2}}$.

**Multicolor container method!**

**Theorem (Benevides, Hoppen and Sampaio, 2017)**

For all $n \geq 2$, the number of Gallai 3-colorings of $K_n$ is at most

$$\frac{3}{2} (n - 1)! \cdot 2^{\binom{n}{2}}.$$

**Theorem (Bastos, Benevides, Mota and Sau, 2019+)**

For all $n \geq 2$, the number of Gallai 3-colorings of $K_n$ is at most

$$7(n + 1) \cdot 2^{\binom{n}{2}}.$$
**Theorem (Balogh and Li, 2019+)**

For every integer $r \geq 3$, there exists $n_0$ such that for all $n > n_0$, the number of Gallai $r$-colorings of $K_n$ is at most

$$\left( \binom{r}{2} + 2^{-\frac{n}{4 \log^2 n}} \right) 2^{\binom{n}{2}}.$$
Our results

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Recall that the number of Gallai $r$-colorings using at most 2 colors are around $\binom{r}{2} 2^{\binom{n}{2}}$.

Theorem (Balogh and Li, 2018+)
For every integer $r \geq 3$, almost all Gallai $r$-colorings of the complete graph are 2-colorings.
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Bastos, Benevides, and Han also proved the above result for $r \leq 2^{n/4300}$. 
Erdős-Rothchild-type problems
Erdős-Rothchild problems

Erdős-Rothchild problem, 1974

Which \( n \)-vertex graph has the maximum number of two-edge-colorings without monochromatic triangles?

Erdős and Rothchild believed that \( K_{\lfloor n/2\rfloor,\lceil n/2\rceil} \) is the only extremal graph. In 1996, Yuster confirmed that Erdős and Rothchild’s conjecture for sufficiently large \( n \).
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Let an \((r, F)\)-coloring of a graph \( G \) be an \( r \)-coloring of its edges without any monochromatic copies of \( F \).

Theorem (Alon, Balogh, Keevash and Sudakov, 2004)

Turán graph \( T_k(n) \) admits the largest number of \((r, K_{k+1})\)-coloring for \( k \geq 3 \) and \( r = \{2, 3\} \).

Technique: Szemerédi’s regularity lemma and the stability method.
More variants

More variants:

- \( k \)-uniform hypergraphs;
- Boolean lattice (Sperner’s Theorem);
- Sum-free sets.

Another natural generalization of Erdős-Rothchild problem is to consider other color patterns.

For a \( r \)-colored graph \( \hat{F} \), a graph \( G \) on \( n \) vertices is called \((r, \hat{F})\)-extremal if it admits the largest number of \( r \)-colorings which contain no subgraph whose color pattern is isomorphic to \( \hat{F} \).

- \( T_{k-1}(n) \) is \((2, \hat{F})\)-extremal, where \( \hat{F} \) is a 2-coloring of a clique \( K_k \) that uses both colors [Balogh, 2006];
- \( r \geq 3, \hat{F} \) be an \( r \)-coloring of \( K_k \) which is not monochromatic [Benevides, Hoppen, Sampaio, Lefmann, Odermann and etc].
A graph $G$ on $n$ vertices is called Gallai $r$-extremal if the number of Gallai $r$-colorings of $G$ is the maximum over all graphs on $n$ vertices.

**Theorem (Hoppen, Lefmann, and Odermann, 2017)**

For all $r \geq 10$ and $n \geq 5$, the only Gallai $r$-extremal graph of order $n$ is the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

**Theorem (Hoppen, Lefmann, and Odermann, 2017)**

For all $r \geq 5$, there exists $n_0$ such that for all $n > n_0$, the only Gallai $r$-extremal graph of order $n$ is the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.
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What happens for $r \in \{3, 4\}$?
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What happens for $r \in \{3, 4\}$?
Theorem (Benevides, Hoppen and Sampaio, 2017)

There exists $n_0$ such that the following hold for all $n > n_0$.

- For all $\delta > 0$, if $G$ is a graph of order $n$, then the number of Gallai 3-colorings of $G$ is at most $2^{(1+\delta)n^2/2}$.

- For all $\xi > 0$, if $G$ is a graph of order $n$, and $e(G) \leq (1 - \xi)\binom{n}{2}$, then the number of Gallai 3-colorings of $G$ is at most $2^{\binom{n}{2}}$.

Theorem (Hoppen, Lefmann, and Odermann, 2017)

There exists $n_0$ such that the following hold for all $n > n_0$. For all $\delta > 0$, if $G$ is a graph of order $n$, then the number of Gallai 4-colorings of $G$ is at most $4^{(1+\delta)n^2/4}$.
Our results

**Conjecture** (Benevides, Hoppen and Sampaio, 2017): The only Gallai 3-extremal graph is the complete graph $K_n$.

**Conjecture** (Hoppen, Lefmann, and Odermann, 2017): The only Gallai 4-extremal graph is the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.
Conjecture (Benevides, Hoppen and Sampaio, 2017): The only Gallai 3-extremal graph is the complete graph $K_n$.

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**Theorem (Balogh and Li, 2019+)**

There exists $n_0$ such that for all $n > n_0$, among all graphs of order $n$, the complete graph $K_n$ is the unique Gallai 3-extremal graph.

**Theorem (Balogh and Li, 2019+)**

There exists $n_0$ such that for all $n > n_0$, among all graphs of order $n$, the complete bipartite graph $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ is the unique Gallai 4-extremal graph.
Proof idea for dense graphs
An *r-template* of order $n$ is a function $P : E(K_n) \rightarrow \{0, 1\}^r$, associating to each edge $e$ of $K_n$ a list of colors $P(e) \subseteq [r]$; we refer to this set $P(e)$ as the *palette* available at $e$.

Let $P_1, P_2$ be two $r$-templates of order $n$. $P_1$ is a *subtemplate* of $P_2$ (or $P_1$ is contained in $P_2$) if $P_1(e) \subseteq P_2(e)$ for every edge $e \in E(K_n)$.

For an $r$-template $P$, denoted by $RT(P)$ the number of subtemplates of $P$ that are rainbow triangles.
Using the hypergraph container theorem, we obtain the following.

**Theorem**

For every \( r \geq 3 \), there exists a constant \( c = c(r) \) and a collection \( C \) of \( r \)-templates of order \( n \) such that

- every rainbow triangle-free \( r \)-template of order \( n \) is a subtemplate of some \( P \in C \);
- for every \( P \in C \), \( RT(P) \leq n^{-1/3} \binom{n}{3} \);
- \( |C| \leq 2^{cn^{-1/3} \log^2 n} \binom{n}{2} \).
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For every $r \geq 3$, there exists a constant $c = c(r)$ and a collection $C$ of $r$-templates of order $n$ such that

- every rainbow triangle-free $r$-template of order $n$ is a subtemplate of some $P \in C$;
- for every $P \in C$, $RT(P) \leq n^{-1/3} \binom{n}{3}$;
- $|C| \leq 2^{c n^{-1/3} \log^2 n \binom{n}{2}}$.

Every Gallai $r$-coloring of a $n$-vertex graph $G$ is a subtemplate of some $P \in C$. 
Lemma 1

For $n^{-1/6} \ll \delta \ll 1$, let $P$ be a $r$-template in $C$ with more than $2^{(1-\delta)(\binom{n}{2})}$ Gallai $r$-colorings. Then the number of triangles $T$ of $K_n$ with $\sum_{e \in T} |P(e)| = 6$ and $P(e) = P(e')$ for every $e, e' \in T$ is at least $(1 - 6\delta)\binom{n}{3}$.
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**Lemma 2**

For $n^{-1/6} \ll \delta \ll 1$, let $P$ be a $r$-template in $C$ with more than $2^{(1-\delta)\binom{n}{2}}$ Gallai $r$-colorings. Then there exist two colors $i, j \in [r]$ such that the number of edges of $K_n$ with palette $\{i, j\}$ is at least $(1 - 6r^4\delta)\binom{n}{2}$. 
Properties of containers

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**Lemma 2**

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- Partition all the containers into $\binom{r}{2}$ classes such that for the containers in the class $C_{i,j}$, almost all edges use the palette $\{i, j\}$;
- From Lemma 2, we can easily conclude that the number of colorings contained in some $P \in C_{i,j}$ is at most $2^{(1+o(1))\binom{n}{2}}$;
- The total number of colorings contained in $C$ is at most $\binom{r}{2}2^{(1+o(1))\binom{n}{2}}$. 
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- Partition all the containers into $\binom{r}{2}$ classes such that for the containers in the class $C_{i,j}$, almost all edges use the palette $\{i, j\}$;
- Based on Lemma 2, further analysis shows that the number of Gallai colorings contained in some $P \in C_{i,j}$ is at most $(1 + o(1))2\binom{n}{2}$;
- The total number of Gallai colorings contained in $C$ is at most $(1 + o(1))\binom{r}{2}2\binom{n}{2}$. 
Thanks!