On King-Serf Pair in Tournaments

Xiaoyun Lu

US Census Bureau

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A tournament $T = (V, E)$ of order $n$ is an orientation of complete graph $K_n$, where $V$ is the vertex set and $E$ is the arc set of $T$. 

A tournament of order 5
Theorem 1: (Rédei, 1934) Every tournament has a hamiltonian path.
Question 1: Given two subsets $X \subseteq V$, $Y \subseteq V$, is there a hamiltonian path starting from a vertex in $X$ and ending at a vertex in $Y$?
A **[X, Y]-path** is a path that starts at a vertex of X and ends at a vertex of Y. A **[X, Y]-graph** G is acyclic and every vertex z of G is in a [X, z]-path and also in a [z, Y]-path.

Note: A hamiltonian [X, Y]-path is an [X, Y]-graph.
Thomassen proved the following result in [3]:

**Theorem 2: (Thomassen, 1980)** Tournament $T$ has a hamiltonian $[x, y]$-path if and only if $T$ contains a spanning $[x, y]$-graph.
The following is an equivalent form:

**Theorem 3:** Tournament \( T \) has a hamiltonian \([X, Y]\)-path if and only if \( T \) contains a spanning \([X, Y]\)-graph.
Corollary: (Rédei, 1934) Every tournament has a hamiltonian path.

Proof. 
Let $X = Y = V$, then the spanning graph with empty arc set is an $[X, Y]$-graph.
A tournament $T$ is $X$-**hamiltonian crossing** if every hamiltonian path must have one end in $X$ and one end in $V - X$. 
If $|X| = |V - X| = 2$ and $T$ has an alternating cycle between $X$ and $V - X$, then $T$ is $X$-hamiltonian crossing. Let’s use $T_4$ to denote this family of tournaments. Every tournament in $T_4$ is a strong tournament.

Two tournaments in $T_4$
$X \Rightarrow Y$: every vertex of $X$ dominates every vertex of $Y$.

We call $V_1 \cup V_2 \cup \ldots \cup V_k = V$ a strong component decomposition of $T$ if $V_i \Rightarrow V_j$ for $i < j$ and the tournament induced by $V_i$ is strong for $1 \leq i \leq k$. 
Question 2: Which tournament is $X$ hamiltonian crossing?
Theorem 4: (Lu 2019) A tournament $T \notin T_4$ is $X$-hamiltonian crossing if and only if $T$ has a strong component decomposition $V = V_1 \cup \ldots \cup V_k$ such that $V_1 \subseteq X$ and $V_k \subseteq V - X$, or $V_1 \subseteq V - X$ and $V_k \subseteq X$, where $k > 1$. 
A **directed tree** is an orientation of a tree. An **out-branching** $R$ with root $x$ is a directed tree such that there is a unique directed path from $x$ to every $y \in V(R)$, and an **in-branching** $R$ with root $x$ is a directed tree such that there is a unique directed path from $y$ to $x$ for every $y \in V(R)$. The **depth** of $R$ is the length of a longest path in $R$. 
For a directed graph $G$, a vertex $x$ of $G$ is a **king** if every other vertex can be reached from $x$ by a directed path of length at most 2 and is a **serf** if $x$ can be reached from every other vertex by a directed path of length at most 2.

In other words, vertex $x$ is a king if there is a spanning out-branching of depth at most 2 with root $x$, and is a serf if there is a spanning in-branching of depth at most 2 with root $x$. We call $(x, y)$ a **king-serf pair** of $G$ if $x$ is a king and $y$ is a serf.
For tournament $T = (V, E)$ and $\{x, y\} \in V$, define:

$$[X, Y] = \{uv \in E(T) | u \in X, v \in Y\}$$

$$(X, Y) = \{uv | u \in X, v \in Y\}$$

$A = N^+(x) \cap N^-(y)$

$B = N^+(x) \cap N^+(y)$

$C = N^-(x) \cap N^-(y)$

$D = N^-(x) \cap N^+(y)$

We have:

$V(T) = \{x, y\} \cup A \cup B \cup C \cup D$. 
Recall that $X \Rightarrow Y$ means $X$ dominates $Y$. We partition set $D$ into disjoint subsets as follows:

- $D_0 = \{ z \in D | z \Rightarrow B, C \Rightarrow z \}$
- $D_1 = \{ z \in D | z \Rightarrow B, [z, C] \neq \emptyset \}$
- $D_2 = \{ z \in D | C \Rightarrow z, [B, z] \neq \emptyset \}$
- $D_3 = \{ z \in D | [B, z] \neq \emptyset, [z, C] \neq \emptyset \}$
X and Y are not present to make figure simpler.
We use $R(X_1, Y_1; X_2, Y_2)$ to denote a tournament $R$ with the following properties:
(1) $V(R) = X_1 \cup Y_1 \cup X_2 \cup Y_2$;
(2) $R$ is $X_1 \cup Y_1$ hamiltonian crossing; and
(3) $R$ has a strong component decomposition $V_1 \cup V_2 \cup \ldots \cup V_k$ such that, let $i$ be the largest index with $(X_1 \cup Y_1) \cap V_i \neq \emptyset$ and $j$ the smallest index with $(X_2 \cup Y_2) \cap V_j \neq \emptyset$, then $Y_1 \cap V_i = \emptyset$ and $Y_2 \cap V_j = \emptyset$. 
**Theorem 5: (Lu, 2019)** Let $T$ be a tournament and $(x, y)$ a king-serf pair in $T - xy$. Then there exists no hamiltonian $[x, y]$-path in $T$ if and only if $A = \{a_1, a_2\}$, $D_3 = \emptyset$, $|D_0| \geq 2$, $C \Rightarrow B$, $a_1 \Rightarrow C$, $C \Rightarrow a_2$, $B \Rightarrow a_1$, $a_2 \Rightarrow B$ and the subtournament $R$ induced by $D$ is a tournament $R(S_1, D_1; S_2, D_2)$, where $S_i = N^+(a_i) \cap D_0$ for $i \in \{1, 2\}$.

**Corollary:** If $(x, y)$ is a king-serf pair in $T - xy$, then $T$ has a hamiltonian $[x, y]$-path if $|A| \neq 2$. 

For $xy \in E(T)$ the exceptional classes are defined as:

1. $D = \{d\}$, $B = \emptyset$ and $A \cup C \Rightarrow d$; or
2. $D = \{d\}$, $C = \emptyset$ and $d \Rightarrow A \cup B$; or
3. $B = C = \emptyset$, $D = \{d_1, d_2\}$, $A = \{a\} \cup A_0$ with $d_1 \Rightarrow A_0$, $A_0 \Rightarrow d_2$ and arcs $\{ad_1, d_1d_2, d_2a\}$, where $|A_0| \leq 1$.

**Remark:** Class 3 contains only three tournaments. One is a regular tournament of order 5 and the other two are almost regular tournaments of order 6.
Theorem 6: [2] Let $T$ be a tournament of order $n \geq 3$. If $x$ is a vertex of maximum out-degree and $y$ a vertex of maximum in-degree then there exists a hamiltonian $[x, y]$-path if and only if $T$ is not in the exceptional classes.
References


Thank you very much!