Evolution of the Scattering Coefficients of the Camassa–Holm Equation, for General Initial Data

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We consider the Camassa–Holm equation for general initial data, particularly when the potential in the scattering problem of the Lax pair, \( m + \kappa \), becomes negative over a finite region. We show that the direct scattering problem of the eigenvalue problem of the Lax pair for this equation may be solved by dividing the spatial infinite interval into a union of separate intervals. Inside each of these intervals, the initial potential is uniformly either positive or negative. Due to this, one can define Jost functions inside each interval, each of which will have a uniform asymptotic form. We then demonstrate that one can obtain the \( t \)-evolution of the scattering coefficients of the scattering matrix of each interval. In the process, we also demonstrate that the evolution of the zeros of \( m + \kappa \) can be given entirely in terms of limits of the scattering coefficients at singular points.

1. Introduction

The Camassa–Holm (CH) equation [1]

\[
\begin{align*}
  m_t + um_x + 2u_x(m + \kappa) &= 0, & t > 0, x \in \mathbb{R}, \\
  m &= u - u_{xx},
\end{align*}
\]

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describes unidirectional propagation of shallow water waves on a flat bottom where \(u(x, t)\) is the scaled height of the free surface, \(\kappa(>0)\) is the scaled critical shallow water speed, and \(m(x, t)\) is the conjugate momentum of \(u\). This equation is integrable, and has the Lax pair [1]

\[
v_{xx} - \frac{1}{4} v + \frac{m + \kappa}{2\lambda} v = 0, \tag{2a}
\]

\[
v_t + uv_x + \lambda v_x + \left(\alpha - \frac{1}{2} u_x\right) v = 0, \tag{2b}
\]

where \(\lambda\) is the spectral parameter and \(\alpha\) is an arbitrary constant (used to adjust the oscillation frequency of the phase of the Jost functions). The integrability condition for (2) is (1), with \(\lambda\) independent of time.

Among the interesting features of this and related equations is the existence of peaked soliton solutions [1–3] which, for \(\kappa = 0\), although the solution is continuous, nevertheless has a discontinuity in the slope at the peak [1]. Due to the interesting nature of these solutions, there have been many studies of the \(\kappa = 0\) case [4–8], as well as others, which include applications to geometry [9] and studies of the periodic case [1, 4, 6, 10]. Multisoliton solutions have been given a parametric representation by Matsuno [11]. Interestingly, this equation has also served as a counter-example to the Painlevé test for integrability [12].

More recently, Boyd [13] has studied the transition from a smooth traveling wave to a peaked wave \((\kappa \to 0)\), and A. Parker [14] has shown how to develop a direct method of solution. Water waves are not the only application of this equation, since it has also been found to occur in elasticity [15] and in non-Newtonian fluids [16].

As one can see from (2a), the \(x\)-component of this Lax pair has a form where the inverse of the spectral parameter multiplies the potential term, \(m + \kappa\), whereas in the Schrödinger equation, the spectral parameter and the potential are additive. This gives rise to a different set of analytical and asymptotic properties for the eigenfunctions of this spectral problem [17]. In particular, whenever the potential has a change in sign, the analytical and asymptotic properties are not uniform in \(x\). The direct and inverse scattering problem for (2a), when the initial data for \(m + \kappa\) changes sign, has not been addressed. However, in the case when the initial data for \(m + \kappa\) never changes sign, both the direct and inverse scattering problem, on the infinite interval, had been earlier addressed [2], and further described in a series of recent papers [18–20]. In this case, eigenfunctions can always be chosen so that the analytical and asymptotic properties will be uniform in \(x\).

An important feature of the CH equation is that one can transform from an Eulerian coordinate to a Lagrangian coordinate, upon which (2a) transforms into the Schrödinger equation [19]. Then providing \(m + \kappa\) remains of one sign, one can proceed as in the solution of the KdV equation [21]. However,
whenever $m + \kappa$ has simple zeros, the direct and inverse scattering problems of (2a) become significantly more complex. In addition to singularities appearing, there is, effectively, a rotation of $\pi/2$ in the analytical and asymptotic properties of the eigenfunction of (2a), with these properties not being uniform in $x$.

Relevant to the above, it has been pointed out by McKean [22, 23] that whenever the initial data of $m + \kappa$ has any positive region to the left of any negative region, then the solution of (1) will break in a finite time. Thus we can anticipate that when the initial data do contain a negative region, then the solution of the more general initial value problem would contain some new and interesting features. This same class of spectral problems (where the potential multiplies the spectral parameter) has been around for some time, since it also appears in the Lax pair of the Harry–Dym equation [24] and other related equations [2, 25], the oscillating two-stream instability (OTSI) problem [26, 27], as well as in the related degenerate two-photon propagation (DTPP) system [28–30]. The latter two are important physical integrable systems, which with this result, can now be re-attacked.

In this paper, we define a set of scattering coefficients, and also present the time evolution of those scattering coefficients, for the spectral problem (2a), when the potential of that spectral problem changes sign. This therefore covers the general class of initial value problems for the CH equation, as well as the same for the OTSI and the DTPP systems. The general formulation of an Inverse Scattering Transform (IST) for the CH equation, valid for general initial data on the infinite interval, provided only that the initial data for $m$ vanish sufficiently rapid at infinity (the usual Faddeev conditions), will not be addressed in this publication. Rather, here we shall only address the direct scattering problem and the evolution of the scattering coefficients.

Our approach will be as follows. Given the initial data for $m$, we break the infinite interval into intervals of positive and negative values of the initial data for $m + \kappa$. This is necessary if one is to have uniform analytical and asymptotical properties for the Jost functions, inside any interval. In the absence of such uniform analytical properties, the usual method of solution for the inverse scattering problem appears intractable. Inside each of these intervals, we can solve the direct and inverse scattering problems, independent of the other intervals, and define appropriate scattering data for the initial data inside that interval [29]. The real problem now becomes how to evolve these independent sets of scattering data in time. This problem can indeed be solved. By the use of the second component of the Lax pair, (2b), one can determine the evolution in time of the scattering coefficients of each interval. (The scattering coefficients are the coefficients, $a, \bar{a}, b, \text{and } \bar{b}$ to be introduced below. These coefficients are not the “scattering data.” But the scattering data are a reduced set which can be obtained from the scattering coefficients.) How to correctly evolve and couple the $t$-evolutions of the various scattering coefficients will be shown below. Following upon this procedure, one could
then formulate and solve the inverse scattering problem inside each interval at any later time, and each could be solved independent of the other intervals, with an independent reconstruction of \( m \) inside each interval. The union of the solutions in each interval, over all intervals, would give the total solution for \( m \) on the infinite interval.

Of course, given this result, one could construct the complete scattering matrix for the entire infinite interval, as a product of the individual scattering matrices from each interval. However how useful this could be in constructing an inverse scattering solution for the entire interval remains to be seen. Our approach is to use known techniques to solve the full problem. Then in hindsight, perhaps a general solution for this inverse scattering problem on the full infinite interval could be formulated.

In Section 2, we shall describe a key important feature, which can be gleaned directly from the CH equation. This is the fact that the zeros of \( m + \kappa \) will move with the flow. In Section 3, we will show that one can obtain the evolution of the flow, \( u \), at a zero of \( m + \kappa \), in terms of two functions of \( t \). This will be important for determining the time evolution of the individual scattering coefficients. In Section 4, we will describe the Jost functions and the solution of the direct scattering problem for all intervals. In Section 5, we will obtain the evolution of the scattering coefficients in each of these intervals. In Section 6, we will obtain the evolution of the zeros of \( m + \kappa \), from simply a knowledge of the scattering coefficients, and will also related this evolution directly to \( m \) itself. In Section 7, we make some concluding remarks and outline future work.

2. Background

Let us next expand on some key points essential to the development of a general IST for the CH equation. First, for the CH equation, zeros of \( m + \kappa \) will move with the flow [19]. This follows from (1a), which can be rewritten as

\[
(\partial_t + u \partial_x)(m + \kappa) + 2(m + \kappa)\partial_x u = 0. \tag{3}
\]

Note that whenever \( m + \kappa = 0 \), that point moves with the flow. As a consequence of this, Lagrangian coordinates, \((\chi, \tau)\), can be defined [19] where,

\[
\partial_x \chi = \sqrt{|m + \kappa|}, \quad \tau = t. \tag{4}
\]

These coordinates have the property that the \( \chi \) value of a zero of \( m + \kappa \) remains invariant in \( \tau \). Thus in the Lagrangian frame, assuming only simple zeros of \( m + \kappa \), one may label the zeros of the initial data of \( m + \kappa \) with \( \chi_a < \chi_b < \cdots < \chi_z \) providing that all zeros are simple (i.e., \( \partial_x m|_{\chi_j} \neq 0 \)). (Note that since we take \( m \) to vanish at \( x = \pm \infty \), then we must have an even number of simple zeros.) Furthermore, the \( \chi \)-separation between these zeros is maintained, since \( \int_{x_a}^{x_b} \sqrt{|m + \kappa|} \, dx \) is a constant of the motion, where \( x_a \) and \( x_b \)
are the Eulerian values of any two successive zeros. Given the Lagrangian values of the zeros of the initial data for \( m + \kappa \), it follows that one may transform the location of these zeros back to the Eulerian frame [19]. Thus the infinite interval in either frame can be broken up into a set of successive intervals, each of a fixed \( \chi \)-length, and consisting of alternating intervals of positive and negative \( m + \kappa \). Inside each of these intervals, we can solve the direct scattering problem for the scattering coefficients for that interval.

We note that since the Lagrangian variable, \( \chi \), is simply a coordinate transformation on \( x \), one can actually treat this problem with either coordinate, depending on one’s preference. If one uses Lagrangian coordinate, \( \chi \), then the zeros will be stationary. However, the signs of the coefficients in the transformed Lax pair, (2), will vary from interval to interval, due to the magnitudes in (4). If one uses the Eulerian coordinate, \( x \), then the zeros will move, but they will never cross. However, the structure of, and the signs in, the Lax pair, (2), will be uniform throughout all intervals. We shall choose to use the Eulerian form. In this case, the zeros of \( m + \kappa \) will be designated by \( x_a(t) < x_b(t) < \cdots < x_z(t) \). Their values at \( t = 0 \) can be obtained from zeros of the initial data of \( m + \kappa \).

To illustrate the above, consider a simple nontrivial example where \( m + \kappa \) has only two simple zeros, such as shown in Figure 1, which is a plot of \( m(x, 0) \) versus \( x \). Note that we shall always consider the physical case where, as \( x \to \pm \infty \), \( m(x, t) \) vanishes. Here we see that we have three intervals; \( -\infty < x \leq x_a \) where \( m + \kappa \geq 0 \), \( x_a \leq x \leq x_b \) where \( m + \kappa \leq 0 \), and \( x_b \leq x < \infty \) where again \( m + \kappa \geq 0 \).

3. Evolution of the flow at a zero of \( m + \kappa \)

The evolution of \( u \) at a zero of \( m + \kappa \) follows from the CH equation. We assume that \( u(x, t) \) to be regular at a zero of \( m + \kappa \), and therefore expand \( u \) in a Taylor series as

\[
u(x, t) = \eta_0 + (x - x_0)\eta_1 + \frac{1}{2!}(x - x_0)^2\eta_2 + \cdots \]  

(5)
where \( x_0(t) \) is the location of the zero concerned, as a function of \( t \), and the \( \eta \)'s are functions of \( t \) also. Taking this, and then by (1b), we have

\[
m(x, t) = \eta_{0,2} + \eta_{1,3}(x - x_0) + \frac{1}{2!}\eta_{2,4}(x - x_0)^2 + \cdots \tag{6}
\]

where we have defined

\[
\eta_{\ell, \ell+2} = \eta_\ell - \eta_{\ell+2}. \tag{7}
\]

Now we determine the evolution of these coefficients. First, since \( m(x_0(t), t) + \kappa = 0 \) in order to be a zero, it follows from (6) that \( \eta_{0,2} + \kappa = 0 \), or

\[
\eta_2 = \eta_0 + \kappa. \tag{8}
\]

The time evolution of the remaining coefficients follow from (1b). Inserting (5) and (6) into (1b) then gives

\[
\partial_t x_0 = \eta_0 \tag{9}
\]

\[
\partial_t \eta_{1,3} + 3\eta_1 \eta_{1,3} = 0 \tag{10a}
\]

\[
\partial_t \eta_{2,4} + 4\eta_1 \eta_{2,4} + 5\eta_2 \eta_{1,3} = 0 \tag{10b}
\]

etc. Clearly, all the coefficients in (5) can be determined in terms of two functions, \( \eta_0 \) and \( \eta_1 \), and the initial data for \( u \).

4. Direct scattering problem

In this section, we shall set up the direct scattering problem and define the scattering coefficients. There are several manners in which this could be done. On the finite intervals, one could set up the Jost functions to be defined as in a periodic problem. Or one could set them up with independent boundary conditions like (1, 0) and (0, 1). However, what we have found to be best is to set them up exactly as plane waves, like one would do for the infinite interval. To do this, we take the boundary conditions on each Jost function to be exactly those of a plane wave with some wave vector \( k \), when \( m = 0 \). We will need to relate this wave-vector to the spectra parameter \( \lambda \), and to do this, we simply consider the limit of \( x \to \pm \infty \) where \( m \) vanishes. Then inserting such a plane wave into (2a), upon setting \( m = 0 \), we have that \( k \) and \( \lambda \) will be related by

\[
\lambda = \frac{2\kappa}{4k^2 + 1}. \tag{11}
\]

Here, we shall treat the example case, such as shown in Figure 1, where \( m + \kappa \) has only two zeros at \( x = x_b \) and \( x = x_c \). Extension of the method to more general cases is obvious. Let us first consider the left interval where \( x < x_b \). At the left end, we define the Jost functions \( \phi_1 \) and \( \hat{\phi}_1 \) by
where the subscripts on the Jost functions, and on the scattering coefficients, will refer to those quantities in this interval \((-\infty < x \leq x_b)\), which we shall refer to as “Interval #1”.

At the right boundary of this interval, at \(x = x_b\), we define two other Jost functions, \(\psi_1\) and \(\bar{\psi}_1\), such that they appear, at the boundary, to be plane waves. Thus we take

\[
\psi_1(x_b) = e^{ikx_b}, \quad \partial_x \psi_1(x_b) = ike^{ikx_b},
\]

\[\bar{\psi}_1(x_b) = e^{-ikx_b}, \quad \partial_x \bar{\psi}_1(x_b) = -ike^{-ikx_b}.\]

With these definitions, using the Green’s function for (2a), standard techniques [18–20, 31] and the Faddeev conditions, one can show that inside Interval #1, \((-\infty < x < x_b)\)

1. \(\phi_1e^{ikx}\) and \(\psi_1e^{-ikx}\) are analytic in the upper half complex \(k\)-plane,
2. \(\bar{\phi}_1e^{-ikx}\) and \(\bar{\psi}_1e^{ikx}\) are analytic in the lower half complex \(k\)-plane.

Furthermore, each set of Jost functions is a linearly independent set on the real \(k\) axis (except for the usual singularity at \(k = 0\)). Thus we may relate the two sets as follows, which defines the “scattering coefficients” on this interval.

\[
\phi_1 = a_1(k, t)\bar{\psi}_1 + b_1(k, t)\psi_1,
\]

\[
\bar{\phi}_1 = \bar{a}_1(k, t)\psi_1 + \bar{b}_1(k, t)\bar{\psi}_1.
\]

The inverse of this is

\[
\psi_1 = a_1(k, t)\bar{\phi}_1 - \bar{b}_1(k, t)\phi_1,
\]

\[
\bar{\psi}_1 = \bar{a}_1(k, t)\phi_1 - b_1(k, t)\bar{\phi}_1.
\]

From the Wronskian of (2a), we have that, for real \(k\),

\[
a_1\bar{a}_1 - b_1\bar{b}_1 = 1.
\]

From the above relations, and the fact that the interval is a left semi-infinite interval, it follows that [31]

1. Both \(a_1\) and \(b_1\) are analytic in the upper-half complex \(k\)-plane,
2. Both \(\bar{a}_1\) and \(\bar{b}_1\) are analytic in the lower-half complex \(k\)-plane.

One should note that on the infinite interval, \(b_1\) and \(\bar{b}_1\) are not analytic and usually only exist on the real \(k\) axis (and with the usual \(1/k\) singularity at \(k = 0\)). However since we have only a semi-infinite interval where \(a_1\) and \(\bar{b}_1\) can be defined at a finite value of \(x\), and since \(\phi_1e^{ikx}\) and \((\partial_x \phi_1)e^{ikx}\) are both analytic
in the upper-half complex $k$-plane, then so are these scattering coefficients. Similarly for $\tilde{a}_1$ and $\tilde{b}_1$ in the lower half plane.

In a similar manner, we can define the scattering coefficients for the other intervals. First, we define our intervals.

\begin{align}
\text{Interval } #1: & \quad -\infty < x \leq x_b(t) \\
\text{Interval } #2: & \quad x_b(t) \leq x \leq x_c(t) \\
\text{Interval } #3: & \quad x_c(t) \leq x < \infty. 
\end{align}

Let us now define the remainder of our Jost functions. In general, we will take $\phi(\psi)$ to be the Jost function with boundary conditions on the left (right). We shall continue to use subscripts as needed to label the Jost functions and scattering coefficients according to the above intervals. Two of these intervals are semi-infinite. We have already defined the Jost functions for Interval #1. For Interval #3, the $\psi$ Jost function will be defined by

\begin{align}
\psi_3(x \to +\infty) & \to e^{ikx}, \\
\tilde{\psi}_3(x \to +\infty) & \to e^{-ikx}.
\end{align}

All other boundary conditions will be at a zero of $m + \kappa$. At these boundaries, we take

\begin{align}
\phi_2(x_b) & = e^{-ikx_b}, \quad \partial_x \phi_2(x_b) = -ike^{-ikx_b}, \\
\bar{\phi}_2(x_b) & = e^{ikx_b}, \quad \partial_x \bar{\phi}_2(x_b) = ike^{ikx_b}, \\
\psi_2(x_c) & = e^{ikx_c}, \quad \partial_x \psi_2(x_c) = ike^{ikx_c}, \\
\bar{\psi}_2(x_c) & = e^{-ikx_c}, \quad \partial_x \bar{\psi}_2(x_c) = -ike^{-ikx_c}, \\
\phi_3(x_c) & = e^{-ikx_c}, \quad \partial_x \phi_3(x_c) = -ike^{-ikx_c}, \\
\bar{\phi}_3(x_c) & = e^{ikx_c}, \quad \partial_x \bar{\phi}_3(x_c) = ike^{ikx_c}.
\end{align}

Similar to the above statements for Interval #1, we have the following properties for the Jost functions inside Interval #2, $(x_b < x < x_c)$

1. $\phi_2 e^{ikx}$, $\bar{\phi}_2 e^{-ikx}$, $\psi_2 e^{-ikx}$, $\bar{\psi}_2 e^{ikx}$ are all analytic in the entire complex $k$-plane,

For the Jost functions inside Interval #3, $(x_c < x < +\infty)$, we have

1. $\phi_3 e^{ikx}$ and $\psi_3 e^{-ikx}$ are analytic in the upper half complex $k$-plane,
2. $\bar{\phi}_3 e^{-ikx}$ and $\bar{\psi}_3 e^{ikx}$ are analytic in the lower half complex $k$-plane.

The scattering coefficients are again defined as in Interval #1. For the middle interval, Interval #2, we take

\begin{align}
\phi_2 & = a_2 \bar{\psi}_2 + b_2 \psi_2, \\
\bar{\phi}_2 & = \bar{a}_2 \psi_2 + \bar{b}_2 \bar{\psi}_2,
\end{align}
where

- \( a_2, \bar{a}_2, b_2, \bar{b}_2 \) are entire functions of \( k \).

From the Wronskian,

\[
\bar{a}_2 a_2 - \bar{b}_2 b_2 = 1,
\]

we obtain the inverse relations

\[
\psi_2 = a_2 \phi_2 - \bar{b}_2 \phi_2,
\]
\[
\bar{\psi}_2 = \bar{a}_2 \phi_2 - b_2 \phi_2.
\]

On Interval #3, we take

\[
\psi_3 = a_3 \phi_3 - \bar{b}_3 \phi_3,
\]
\[
\bar{\psi}_3 = \bar{a}_3 \phi_3 - b_3 \phi_3.
\]

where

1. \( a_3 \) and \( \bar{b}_3 \) are analytic in the upper-half complex \( k \)-plane,
2. \( \bar{a}_3 \) and \( b_3 \) are analytic in the lower-half complex \( k \)-plane.

From the Wronskian,

\[
\bar{a}_3 a_3 - \bar{b}_3 b_3 = 1,
\]

which gives the inverse of (29) to be

\[
\phi_3 = a_3 \bar{\psi}_3 + b_3 \psi_3,
\]
\[
\bar{\phi}_3 = \bar{a}_3 \psi_3 + \bar{b}_3 \overline{\psi}_3.
\]

The above have defined the scattering coefficients inside each interval. From these scattering coefficients, one could define the scattering data [31], and then inside any interval, proceed to solve the appropriate Marchenko equations for either the semi-infinite interval or the finite interval [29]. Thus the potentials could be reconstructed from only the scattering data in that interval, and this reconstruction could be done at any instant of time. However, it is also certainly clear that these scattering coefficients must be coupled in some way, at least in their time evolution. Otherwise the separate intervals would evolve independent of each other, in violation of the known nature of characteristics of PDEs. Thus it is essential to address the time evolution of these scattering coefficients, which we do in the next section.

### 5. The time evolution of the scattering coefficients

By the use of (2b), one can obtain the time evolution of the scattering coefficients. Starting with Interval #1, as \( x \to -\infty \), in order to satisfy (12) and (13), we must take
\[ \alpha \phi = i \lambda \kappa, \quad \alpha \bar{\phi} = -i \lambda \kappa \] 

where \( \alpha \phi \) is the value of \( \alpha \) for \( \phi_1 \) and \( \alpha \bar{\phi} \) is the same for \( \bar{\phi}_1 \).

At the right end, we evaluate (2b) at \( x = x_b(t) \), which is where \( m + \kappa = 0 \), by means of the expansion (5), the values in (14) and (15), and the definitions of the scattering coefficients in (15), to express \( \phi \) in terms of \( \psi, \bar{\psi} \) and the scattering coefficients. One obtains the following evolutions for the scattering coefficients \( a_1 \) and \( b_1 \).

\[
\frac{\partial}{\partial t} a_1 = i \eta_{0b} \frac{\kappa}{2k\lambda} a_1 + \frac{1}{2} \left( \eta_{1b} + i \eta_{0b} \right) b_1 e^{2ikx_b}, 
\]

\[
\frac{\partial}{\partial t} b_1 = \frac{1}{2} \left( \eta_{1b} - i \frac{\kappa}{2k} \eta_{0b} \right) a_1 e^{-2ikx_b} + i \frac{\kappa}{2k} \left[ \lambda - 2\kappa - \frac{\eta_{0b}}{\kappa} \right] b_1,
\]

where \( \eta_{0b} \) and \( \eta_{1b} \) are the values of \( \eta_0 \) and \( \eta_1 \) at the left zero, \( x_b \).

The evolutions for \( \bar{a}_1 \) and \( \bar{b}_1 \) are exactly the same as for (33) and (34), except that one replace \( a_1 \) and \( b_1 \) by \( \bar{a}_1 \) and \( \bar{b}_1 \), and \( i \) by \( -i \). Thus

\[
\frac{\partial}{\partial t} \bar{a}_1 = -i \eta_{0b} \frac{\kappa}{2k\lambda} \bar{a}_1 + \frac{1}{2} \left( \eta_{1b} - i \frac{\kappa}{2k} \eta_{0b} \right) \bar{b}_1 e^{-2ikx_b},
\]

\[
\frac{\partial}{\partial t} \bar{b}_1 = \frac{1}{2} \left( \eta_{1b} + i \frac{\kappa}{2k} \eta_{0b} \right) \bar{a}_1 e^{2ikx_b} - \frac{1}{2} \left( \eta_{1b} - i \frac{\kappa}{2k} \eta_{0b} \right) \bar{b}_1.
\]

In exactly the same manner, one can find the evolutions of the other scattering coefficients. Letting \( \eta_{0c} \) and \( \eta_{1c} \) be the values of \( \eta_0 \) and \( \eta_1 \) at the right zero, \( x_c \), we then find that

\[
\frac{\partial}{\partial t} a_2 = i(\eta_{0c} - \eta_{0b}) \frac{\kappa}{2k\lambda} a_2 + \frac{2\eta_{1c} k + i \eta_{0c}}{4k} e^{2ikx_c} b_2 - \frac{2\eta_{1b} k - i \eta_{0b}}{4k} e^{-2ikx_b} \bar{b}_2,
\]

\[
\frac{\partial}{\partial t} b_2 = \frac{1}{2} \left( \eta_{1c} + i \frac{\kappa}{2k} \eta_{0c} \right) a_2 e^{-2ikx_c} - \frac{1}{2} \left( \eta_{1b} - i \frac{\kappa}{2k} \eta_{0b} \right) \bar{a}_2 e^{2ikx_b} + \frac{i}{2k} \left[ \lambda - 2\kappa - \frac{\eta_{0b}}{\kappa} \right] b_2,
\]

\[
\frac{\partial}{\partial t} \bar{a}_2 = -i(\eta_{0c} - \eta_{0b}) \frac{\kappa}{2k\lambda} \bar{a}_2 - \frac{2\eta_{1b} k + i \eta_{0b}}{4k} \bar{a}_2 e^{2ikx_b} + \frac{2\eta_{1c} k - i \eta_{0c}}{4k} e^{-2ikx_c} \bar{b}_2,
\]

\[
\frac{\partial}{\partial t} \bar{b}_2 = -\frac{1}{2} \left( \eta_{1b} + i \frac{\kappa}{2k} \eta_{0b} \right) a_2 e^{2ikx_b} + \frac{1}{2} \left( \eta_{1c} + i \frac{\kappa}{2k} \eta_{0c} \right) \bar{a}_2 e^{2ikx_c} - \frac{i}{2k} \left[ \lambda - 2\kappa - \frac{\eta_{0b}}{\kappa} \right] \bar{b}_2,
\]
and

\[ \partial_t a_3 = -i \eta_{0c} \frac{\kappa}{2k \lambda} a_3 - \frac{1}{2} \left( \eta_{1c} - i \frac{\eta_{0c}}{2k} \right) b_3 e^{-2ikx}, \quad (41) \]

\[ \partial_t b_3 = -\frac{1}{2} \left( \eta_{1c} - i \frac{\eta_{0c}}{2k} \right) \bar{a}_3 e^{-2ikx} + \frac{i}{2k} \left( \lambda - 2\kappa - \frac{\eta_{0c} \kappa}{\lambda} \right) b_3, \quad (42) \]

\[ \partial_t \bar{a}_3 = i \eta_{0c} \frac{\kappa}{2k \lambda} \bar{a}_3 - \frac{1}{2} \left( \eta_{1c} + i \frac{\eta_{0c}}{2k} \right) b_3 e^{2ikx}, \quad (43) \]

\[ \partial_t \bar{b}_3 = -\frac{1}{2} \left( \eta_{1c} + i \frac{\eta_{0c}}{2k} \right) a_3 e^{2ikx} - \frac{i}{2k} \left[ \lambda - 2\kappa - \frac{\eta_{0c} \kappa}{\lambda} \right] \bar{b}_3. \quad (44) \]

In the above, the evolution of the four \( \eta \)'s are still unstated, and how this needs to be stated will be described in the next section.

6. The time evolutions of the \( \eta \)'s

At this point, the only thing remaining before one can determine the evolution of the scattering coefficients are the four functions \( \eta_{0b}, \eta_{1b}, \eta_{0c}, \) and \( \eta_{1c} \). These quantities are nothing more than the values of \( u \) and \( u_x \) at the zeros of \( m + \kappa \). Normally, one could not determine these until after the solution for \( u(x, t) \) itself was determined. However, we shall now find that these functions can also be determined by the evolutions of the scattering coefficients as given by (33)–(44), combined with the analytical properties of the scattering coefficients, as detailed earlier. To see this, let us first obtain the structure of the scattering coefficients about the singular points.

Consider (11) when \( k = \pm i/2 \). At these points in the complex plane, \( 1/\lambda = 0 \). Then according to (2a), the potential term vanishes, and the solution for all Jost functions becomes trivial. One then can ascertain that in the upper-half complex \( k \)-plane, at \( k = \pm i/2 \), we have \( a = 1 + \mathcal{O}(\lambda^{-1}) \) and \( b = \mathcal{O}(\lambda^{-1}) \) in all three intervals, provided that the appropriate \( a \) and \( b \) are known to exist in that half-plane. In particular, let us designate these limits as follows. First for the limit of \( k \to +i/2 \), let us take

\[ a_1 \to 1 - \frac{1}{2\lambda} I_{ab}^{(0)} + \mathcal{O}(\lambda^{-2}), \quad b_1 \to \frac{1}{2\lambda} I_{ab}^{(+1)} + \mathcal{O}(\lambda^{-2}), \quad (45) \]

\[ a_2 \to 1 - \frac{1}{2\lambda} I_{bc}^{(0)} + \mathcal{O}(\lambda^{-2}), \quad b_2 \to \frac{1}{2\lambda} I_{bc}^{(+1)} + \mathcal{O}(\lambda^{-2}), \quad (46) \]

\[ \bar{a}_2 \to 1 + \frac{1}{2\lambda} I_{bc}^{(0)} + \mathcal{O}(\lambda^{-2}), \quad \bar{b}_2 \to -\frac{1}{2\lambda} I_{bc}^{(-1)} + \mathcal{O}(\lambda^{-2}), \quad (47) \]

\[ a_3 \to 1 - \frac{1}{2\lambda} I_{cd}^{(0)} + \mathcal{O}(\lambda^{-2}), \quad \bar{b}_3 \to -\frac{1}{2\lambda} I_{cd}^{(-1)} + \mathcal{O}(\lambda^{-2}). \quad (48) \]
while in the lower half-plane, in the limit of $k \to -i/2$, we will take
\[ \bar{a}_1 \to 1 - \frac{1}{2\lambda} I_{ab}^{(0)} + \mathcal{O}(\lambda^{-2}), \quad \bar{b}_1 \to \frac{1}{2\lambda} I_{ab}^{(+1)} + \mathcal{O}(\lambda^{-2}), \]  
(49)\]
\[ a_2 \to 1 + \frac{1}{2\lambda} I_{bc}^{(0)} + \mathcal{O}(\lambda^{-2}), \quad b_2 \to -\frac{1}{2\lambda} I_{bc}^{(-1)} + \mathcal{O}(\lambda^{-2}), \]  
(50)\]
\[ \bar{a}_2 \to 1 - \frac{1}{2\lambda} I_{bc}^{(0)} + \mathcal{O}(\lambda^{-2}), \quad \bar{b}_2 \to \frac{1}{2\lambda} I_{bc}^{(+1)} + \mathcal{O}(\lambda^{-2}), \]  
(51)\]
\[ \bar{a}_3 \to 1 - \frac{1}{2\lambda} I_{cd}^{(0)} + \mathcal{O}(\lambda^{-2}), \quad b_3 \to -\frac{1}{2\lambda} I_{cd}^{(-1)} + \mathcal{O}(\lambda^{-2}), \]  
(52)\]
where all the $I$’s are defined by these limits. Thus these $I$’s are seen to be obtainable from the scattering data.

On the other hand, we may return to (2a), construct the appropriate Green’s function, and obtain a solution for the Jost functions, and thereby the scattering coefficients, about the singular points $k = \pm i/2$, in an asymptotic series of $1/\lambda$. Doing so, one finds that these $I$’s can also be related to the dynamical variable $m$ by
\[ I_{ij}^{(\gamma)} = \int_{x_i}^{x_j} m(x, t) e^{\gamma x} \, dx, \]  
(53)\]
where $\gamma = \pm 1$ or $0$, $x_a = -\infty$, $x_d = +\infty$, and as before, $x_b$ and $x_c$ are the two zeros of $m + \kappa$. So, the $I$’s may be interpreted either as scattering data, or integrals of $m$, as in (53).

Now, let us put our attention on Interval #1. There we have that both $a_1$ and $b_1$ are analytic in the upper-half complex $k$-plane. Therefore they must possess a Taylor series expansion about any point in this half plane. (Note that we are assuming that the Faddeev conditions will continue to be satisfied, at least for some finite time during the evolution, and that the solution will not break at $t = 0$.) Inserting such an expansion into (33) and (34), one finds that the evolution of $b_1$ will in general develop a nonzero value at the singular point $k = i/2$. This is in violation of the above expansion for $b_1$ and will immediately lead to the instantaneous creation of an essential singularity at the singular point, in violation of the stated analytical properties of $b_1$. This can be avoided only if, in the limit of $1/\lambda \to 0$, the right-hand side of (34) vanishes. Using the above expansions, one can show that this coefficient must be
\[ e^{x_b}(\eta_{1b} - \eta_{0b}) + I_{ab}^{(+1)} = 0. \]  
(54)\]
If one would carry out the same for the singularity in the lower-half plane, one would find that the exact same condition also arises.

The same can also be done in the other intervals, noticing that in the second interval, all coefficients must be entire, and thus $b_2$ must not be allowed to
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Develop essential singularities are either of these points, \( k = i/2 \) and \( k = -i/2 \). This then gives the conditions

\[
(\eta_{1c} - \eta_{0c})e^{x_c} - (\eta_{1b} - \eta_{0b})e^{x_b} + I_{bc}^{(+1)} = 0,
\]

while in the third interval, we find

\[
e^{x_c}(\eta_{1c} + \eta_{0c}) - I_{cd}^{(-1)} = 0.
\]

Solving these four conditions for the \( \eta \)'s, we obtain

\[
\eta_{0b} = \frac{1}{2} e^{-x_b} I_{ab}^{(+1)} + \frac{1}{2} e^{x_b} I_{bd}^{(-1)},
\]

\[
\eta_{1b} = -\frac{1}{2} e^{-x_b} I_{ab}^{(+1)} + \frac{1}{2} e^{x_b} I_{bd}^{(-1)},
\]

\[
\eta_{0c} = \frac{1}{2} e^{-x_c} I_{ac}^{(+1)} + \frac{1}{2} e^{x_c} I_{cd}^{(-1)},
\]

\[
\eta_{1c} = -\frac{1}{2} e^{-x_c} I_{ac}^{(+1)} + \frac{1}{2} e^{x_c} I_{cd}^{(-1)}.
\]

Thus by obtaining the \( I \)'s from the limits of the scattering coefficients about the singular points, we have the \( \eta \)'s. Once we have the \( \eta \)'s, we then have the evolution of the scattering coefficients, entirely in terms of the same scattering coefficients and their initial values. Note that by (9), the evolution of the \( x_j \)'s are given by (58) and (60), which therefore can also be obtained from the scattering coefficients.

To make this point further, consider solving an arbitrary initial value problem by means of this composite IST. Taking the initial data, we solve (2a) for the initial scattering coefficients. Then we integrate the scattering coefficients forward in time, using (33)–(44), taking care that the \( \eta \)'s in the coefficients of these equations always satisfy (58)–(61), where the \( I \)'s are obtained from the limits in (45)–(48) or (49)–(52). Then clearly, one can obtain the scattering coefficients at any later time, unless the solution for \( u \) happens to break. We then obtain the evolution of the scattering coefficients.

We note in passing that due to (53), which is obtained from (2a), the \( I \)'s, for \( \gamma = \pm 1 \), are exactly integrable, because \( me^{\pm x} = \partial_x [e^{\pm x} (\pm u - u_x)] \), due to (1b). Using this relation and (5), one may readily verify that (54)–(57) are indeed correct, which were obtained instead by requiring the scattering coefficients to retain their appropriate analytical properties as they evolved. This is not surprising, since by requiring the scattering coefficients to retain their analytical properties during their evolution, we are also requiring the potential in (1b) to continue to remain as it should be.
7. Conclusions

Let us now review and summarize the situation. As demonstrated above, inside each interval, one can obtain a set of first-order differential equations for the evolution of the scattering coefficients, as in (33) and (34). These equations will be dependent on the position of the zeros, the \( x_n \)'s, and the first two \( \eta \)'s at each zero. One then finds that on each semi-infinite interval, there will be exactly one relation, as in (54) and (57). On every internal finite interval, one will obtain two other such independent relations, as in (33) and (34). A simple counting then shows that one has sufficient data to determine all functions, as well as the evolution of the scattering coefficients.

Another feature of these evolution equations for the scattering coefficients that should be noted is that they are linear in the scattering coefficients, provided one is not in a neighborhood of either of the singular points \( k = \pm i/2 \). However in a neighborhood of these singular points, the equations actually become nonlinear, since the values of the \( \eta \)'s in the coefficients of these equations are to be given in terms of the limits of the \( b \)'s, as given in (45)–(48) or (49)–(52).

Although we do now have the equations for these evolutions, how practical they will be as a means for solving the time evolution of the scattering data, as well the same of the CH equation, remains to be seen. However, the above does demonstrate that in principle, it is workable. The least that one can say, is that when this method of solution is finally and fully detailed, we will obtain a new understanding of the CH equation, as well as a new and deeper understanding of the method of the IST. Perhaps we will also be able to better understand why the CH equation satisfies only the weak Painlevé property [12].

In concluding this section, we should make some comments on some possible complications that will appear in deriving the inverse scattering equations for this general initial value problem of the CH equation. For example, in any neighborhood of the zeros of \( m + \kappa \), the asymptotics of the Jost functions are not as clean and simple as for the regular Schrödinger equation, where one has the simple result that \( \phi e^{ikx} \to 1 \) uniformly in \( x \) as \( |k| \to \infty \) in the upper half \( k \)-plane. Rather, considering (2a) when one is in a neighborhood of a zero of \( m + \kappa \), for \( |k| \) large, one sees that any Jost function will have a boundary layer of the structure of an Airy function. At any \( x \) away from such a zero, one does have \( \phi e^{ikx} \) approaching a constant limit [17] as \( |k| \to \infty \) for fixed \( x \). However, if one is exactly at such a zero, then one is always inside the boundary layer. Thus the limit is not uniform in \( x \) for the closed interval. We note that this may be an important consideration in deriving the Marchenko equations for this system, as well as what has been called the “linear dispersion relations” [31], which is the same as the solution of the associated Riemann–Hilbert problem. The key point here is that the scattering coefficients are defined by the values of the Jost functions exactly at these zeros, which is always inside these boundary layers.
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References


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