IMPROVED BOUNDS FOR THE COMPLEX POLYNOMIAL BOHNENBLUST–HILLE INEQUALITY

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Abstract. In this note, we make an observation that yields improved estimates in the Bohnenblust–Hille inequality for complex homogeneous polynomials.

1. Introduction

The Bohnenblust–Hille inequalities (BH for short) for multilinear forms and homogeneous polynomials have been extensively investigated in the last decades. The understanding of the sharp constants in such inequalities is an important open question in the field, with applications ranging from Quantum Information Theory and Theoretical Computer Science (see [1, 6]) to Complex Analysis ([4]).

In the classical paper [3], Bohnenblust and Hille showed that for each m there exists a constant $D_m \geq 1$ such that for every m-homogeneous polynomial $\sum_{|\alpha|=m} a_\alpha z^\alpha$ defined over the complex field $\mathbb{C}$, there holds

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \leq D_m \sup_{z\in\mathbb{D}^n} \left|\sum_{|\alpha|=m} a_\alpha z^\alpha\right|;$$

the exponent $2m/(m+1)$ cannot be improved.

The best known information on the lower bounds for D_m can be found in [7] and the best upper bounds were obtained in [2, Theorem 5.2]:

$$D_m \leq \min_{1\leq k\leq m-1} \left\{ \left(1+\frac{1}{k}\right)^{\frac{m-k}{2}} \times \frac{m^m}{(m-k)^{m-k}} \times \sqrt{\frac{(m-k)!}{m!}} \times \prod_{j=2}^{k} \Gamma\left(2-\frac{1}{j}\right)^{\frac{j}{2-2j}} \right\} \tag{1.1}$$

and, as a consequence, the authors conclude that the constants D_m have a subexponential growth.

In this note we improve (1.1) by introducing the contractive factor $\frac{(k+1)^{\frac{k+1}{2}}}{2^k k^{\frac{k}{2}}}$ in the estimate, namely we show that

$$D_m \leq \min_{1\leq k\leq m-1} \left\{ \left(1+\frac{1}{k}\right)^{\frac{m-k}{2}} \times \frac{m^m}{(m-k)^{m-k}} \times \sqrt{\frac{(m-k)!}{m!}} \times \frac{(k+1)^{\frac{k+1}{2}}}{2^k k^{\frac{k}{2}}} \times \prod_{j=2}^{k} \Gamma\left(2-\frac{1}{j}\right)^{\frac{j}{2-2j}} \right\}. \tag{1.2}$$

The following table compares both estimates:

m	D_m in (1.1)	D_m in (1.2)
15	1032.4	1020.1
50	6.8×10^6	2.7×10^6
100	2.7×10^{10}	6.5×10^9
200	5.7×10^{15}	6.9×10^{14}
500	8.4×10^{26}	1.9×10^{25}
1000	1.0×10^{40}	3.8×10^{37}

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2. The proof

The proof follows closely the reasoning employed in [2, Theorem 5.2]. The key observation is to incorporate the results of [5] in the analysis. Here are the details:

Let $L \in \mathcal{L}\left({}^m\ell_n^\infty;\mathbb{C}\right)$ denote the space of m-linear forms defined over ℓ_n^∞ with values in $\mathbb{C}$. It follows from [5, Theorem 1] that if $L \in \mathcal{L}\left({}^m\ell_n^\infty;\mathbb{C}\right)$ then

$$L\left(x_1^{n_1},...,x_k^{n_k}\right) \leq \frac{n_1!...n_k!m^m}{n_1^{n_1}...n_k^{n_k}m!}\left\|\widehat{L}\right\|, \tag{2.1}$$

for all unit vectors $x_1,\ldots,x_k$. As usual ℓ_n^∞ denotes $\mathbb{C}^n$ with the supremum norm, x^n denotes $\left(x,\overset{n\text{ times}}{\ldots},x\right)$ and $\widehat{L}$ is the homogeneous polynomial associated to L, i.e.,

$$\widehat{L}(z) = L\left(z,...,z\right).$$

Now, from [5, Corollary 5], if $L \in \mathcal{L}\left({}^m l_n^\infty;\mathbb{C}\right)$, then

$$\|L\| \leq \frac{m^{\frac{m}{2}}\left(m+1\right)^{\frac{m+1}{2}}}{2^m m!}\left\|\widehat{L}\right\|. \tag{2.2}$$

Following the notation of [2], let $P\colon \ell_n^\infty \to \mathbb{C}$ be an m-homogeneous polynomial and let L denotes its associated m-linear form. For a fixed $z \in \mathbb{D}_1$, define the k-linear operator

$$L_z\left(x_1,\ldots,x_k\right) := L\left(x_1,\ldots,x_k,z,\ldots,z\right).$$

Clearly one has

$$\widehat{L_z}(w) = L\left(w,\ldots,w,z,\ldots,z\right),$$

and thus from (2.2) one can write

$$\begin{aligned}
\|L_z\| &\leq \frac{k^{\frac{k}{2}}\left(k+1\right)^{\frac{k+1}{2}}}{2^k k!}\left\|\widehat{L_z}\right\| \\
&= \frac{k^{\frac{k}{2}}\left(k+1\right)^{\frac{k+1}{2}}}{2^k k!}\sup_{\|w\|=1}\left|L\left(w,....,w,z,...,z\right)\right| \\
&\overset{(2.1)}{\leq} \frac{k^{\frac{k}{2}}\left(k+1\right)^{\frac{k+1}{2}}}{2^k k!}\left(\frac{k!\left(m-k\right)!}{k^k\left(m-k\right)^{m-k}}\frac{m^m}{m!}\|P\|\right) \\
&= \frac{\left(k+1\right)^{\frac{k+1}{2}}}{2^k}\frac{\left(m-k\right)!}{k^{\frac{k}{2}}\left(m-k\right)^{m-k}}\frac{m^m}{m!}\|P\|.
\end{aligned}$$

Hence, one can estimate:

$$\sup_{\|w_1\|=1,\ldots,\|w_k\|=1}\left|L\left(w_1,\ldots,w_k,z,\ldots,z\right)\right| \leq \frac{\left(k+1\right)^{\frac{k+1}{2}}}{2^k}\frac{\left(m-k\right)!}{k^{\frac{k}{2}}\left(m-k\right)^{m-k}}\frac{m^m}{m!}\|P\|. \tag{2.3}$$

Follow the lines of the proof of [2, Theorem 5.2], just replacing (2.1) by the new estimate (2.3), we readily obtain (1.2), and the proof is complete.

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