REMARKS ON HADAMARD MATRICES AND A THEOREM OF MACPHAIL

KATIUSCIA B. TEIXEIRA

Abstract. In this note we discuss a unified approach to the unconditionally summable problem using combinatorial properties of Hadamard matrices. In particular we derive a constructive proof of the classical Macphail's Theorem on the existence of series in ℓ_1 that converge unconditionally but not absolutely.

1. Introduction

A square matrix $\mathbf{H} = [h_{ij}]_{n\times n}$ is called a Hadamard matrix if $h_{ij} = \pm 1$ and its columns (and thus its rows) vectors are mutually orthogonal. That is, if $u_1, \dots, u_n$ are the row vectors of a Hadamard matrix $\mathbf{H}$, then

$$\langle u_i, u_j \rangle = n\delta_{ij}$$

for all i, j. As usual δ_{ij} denotes the Kronecker delta. Equivalently, a square matrix, $\mathbf{H} = [h_{ij}]$ is a Hadamard matrix if $|h_{ij}| = 1$ and

$$\mathbf{H}\mathbf{H}^\top = n\mathbf{I}_n.$$

Hadamard matrices are important in several extrema problems for its combinatorial properties. In particular, Hadamard matrices have maximal determinant, $\pm n^{\frac{n}{2}}$, among all possible matrices whose entries are in $[-1, 1]$.

Given a Hadamard matrix $\mathbf{H}$ of order n, a well known construction due to Sylvester yields a Hadamard matrix of order $2n$, simply by partitioning $\mathbf{H}$ as:

$$\begin{bmatrix} \mathbf{H} & \mathbf{H} \\ \mathbf{H} & -\mathbf{H} \end{bmatrix}.$$

In particular, starting at the trivial 1×1 Hadamard matrix $\mathbf{H} = [1]$, following Sylvester's procedure, one can construct Hadamard matrices of order 2^n for all $n \geq 0$.

In this paper we are interested in combinatorial iterations of Hadamard matrices' entries, with connection to the classical problem of unconditionally convergent sequences in Banach spaces. A famous result originally due to Macphail [2] assures the existence of unconditionally convergent series in ℓ_1 that do not converge absolutely; for a general historical account of the unconditionally convergent problem, please see [1]. The problem has been revisited by many authors. Two constructive solutions have been recently obtained by Pellegrino and Silva in [3]. In that paper, the authors present two different proofs of Macphail's theorem, one using complex scalars *ala* Toeplitz [5] and another one using the so called Walsh system [6].

We shall present an alternative, simpler proof of Macphail's theorem using combinatorial properties of Hadamard matrices. Our solution encloses both approaches from [3]. Indeed, a closer inspection reveals that both constructions presented in [3] are, in a way, based on particular cases of Hadamard matrices. In turn, our proof unifies these seemingly disconnected constructions and sheds lights into the core connection Hadamard matrices have with the problem of unconditionally convergent sequences.

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Here is the main theorem proven in this note:

Theorem 1. *Let $k \geq 1$ be a natural number and $\mathbf{H}_k = \left[h_{rs}^{(k)}\right]$ be a Hadamard matrix of order $2^{k(k-1)}$. Let $p \in [1,2]$ and consider the sequence in ℓ_p defined by the following algorithm: for $j \in \left\{2^{k(k-1)}, 2^{k(k-1)}+1, \ldots, 2^{k(k-1)+1}-1\right\}$ set*

$$x^{(j)} := 2^{-k(k-1)\left(\frac{1}{2}+\frac{1}{p}+\frac{1}{k}\right)} \left(\sum_{s=1}^{2^{k(k-1)}} h_{rs}^{(k)} e_{2^{k(k-1)}+s-1}\right),$$

where e_j denotes the j-th canonical unit vectors in ℓ_p. If $j \notin \left\{2^{k(k-1)}, 2^{k(k-1)}+1, \ldots, 2^{k(k-1)+1}-1\right\}$, simply set $x^{(j)} = 0$. The resulting sequence, $\left(x^{(j)}\right)_{j=1}^{\infty}$ is unconditionally summable in ℓ_p; however

$$\sum_{j=1}^{\infty} \left\|x^{(j)}\right\|_p^{2-\varepsilon} = \infty,$$

for all $\varepsilon > 0$.

2. The proof

Following ideas from [4], we starting by noting that for all k and $m = 2^{k(k-1)}$, there holds

$$\sup\left\{\sum_{j=1}^{m}\left|\sum_{i=1}^{m} h_{ij}^{(k)} x_i\right| : \left\|(x_j)_{j=1}^{m}\right\|_{p^*} \leq 1\right\} \leq m^{\frac{1}{2}+\frac{1}{p}}. \tag{2.1}$$

Indeed, if u_i denotes the i-th row vector of $\mathbf{H}_k$, an application of the Cauchy-Schwarz inequality yields:

$$\begin{aligned}
\sum_{j=1}^{m}\left|\sum_{i=1}^{m} h_{ij}^{(k)} x_i\right| &\leq m^{1/2}\left(\sum_{j=1}^{m}\left|\sum_{i=1}^{m} h_{ij}^{(k)} x_i\right|^2\right)^{1/2} \\
&= m^{1/2}\left(\sum_{i,k=1}^{m} x_i \overline{x_k}\left\langle u_i, u_k\right\rangle\right)^{1/2} \\
&= m\left(\sum_{i=1}^{m} |x_i|^2\right)^{1/2} \\
&\leq m^{\frac{1}{2}+\frac{1}{p}}.
\end{aligned}$$

Next we shall confirm that the sequence defined in the statement of theorem 1 is indeed unconditionally convergent in ℓ_p, for all $p \in [1,2]$. For $n \in \mathbb{N}$, let k_n be the biggest integer such that $2^{k_n(k_n-1)} \leq n$. If $\varphi \in \ell_p^* = \ell_{p^*}$ is a continuous linear functional, we have:

$$\sum_{j=n}^{\infty}\left|\varphi\left(x^{(j)}\right)\right| \leq \sum_{j=2^{k_n(k_n-1)}}^{\infty}\left|\varphi\left(x^{(j)}\right)\right| = \sum_{k=k_n}^{\infty}\sum_{j=2^{k(k-1)}}^{2^{k(k-1)+1}-1}\left|\varphi\left(x^{(j)}\right)\right|.$$

Taking the supreme over all $\varphi \in \ell_{p^*}$ with norm less than or equal to 1, we reach:

$$\sup_{\|\varphi\|_{p^*}\leq 1}\sum_{j=n}^{\infty}\left|\varphi\left(x^{(j)}\right)\right| \leq \sum_{k=k_n}^{\infty}\left(\sup_{\|\varphi\|_{p^*}\leq 1}\sum_{j=2^{k(k-1)}}^{2^{k(k-1)+1}-1}\left|\varphi\left(x^{(j)}\right)\right|\right). \tag{2.2}$$

It now follows by the algorithm that defines $x^{(j)} := \left(x_m^{(j)}\right)_{m=1}^{\infty}$ that for $j = 2^{k(k-1)} + r - 1$, there holds

$$\sum_{j=2^{k(k-1)}}^{2^{k(k-1)+1}-1} \left|\varphi\left(x^{(j)}\right)\right| = \sum_{j=2^{k(k-1)}}^{2^{k(k-1)+1}-1} \left|\sum_{m=1}^{\infty} \varphi_m x_m^{(j)}\right|$$
$$= \sum_{r=1}^{2^{k(k-1)}} \left|\sum_{m=1}^{\infty} \varphi_m x_m^{\left(2^{k(k-1)}+r-1\right)}\right|$$
$$= 2^{-k(k-1)\left(\frac{1}{2}+\frac{1}{p}+\frac{1}{k}\right)} \sum_{r=1}^{2^{k(k-1)}} \left|\sum_{s=1}^{2^{k(k-1)}} h_{rs}^{(k)} \varphi_{2^{k(k-1)}+s-1}\right|. \tag{2.3}$$

Combining (2.3) with estimate (2.1), we conclude for any $\delta > 0$ given, there exists an $n_\delta \in \mathbb{N}$ such that

$$\sum_{k=k_n}^{\infty} \left(\sup_{\|\varphi\|_{p^*} \leq 1} \sum_{j=2^{k(k-1)}}^{2^{k(k-1)+1}-1} \left|\varphi\left(x^{(j)}\right)\right|\right)$$
$$= \sum_{k=k_n}^{\infty} \left(\sup_{\|\varphi\|_{p^*} \leq 1} 2^{-k(k-1)\left(\frac{1}{2}+\frac{1}{p}+\frac{1}{k}\right)} \sum_{r=1}^{2^{k(k-1)}} \left|\sum_{s=1}^{2^{k(k-1)}} h_{rs}^{(k)} \varphi_{2^{k(k-1)}+s-1}\right|\right)$$
$$= \sum_{k=k_n}^{\infty} \left(2^{-k(k-1)\left(\frac{1}{2}+\frac{1}{p}+\frac{1}{k}\right)} \sup_{\|\varphi\|_{p^*} \leq 1} \sum_{r=1}^{2^{k(k-1)}} \left|\sum_{s=1}^{2^{k(k-1)}} h_{rs}^{(k)} \varphi_{2^{k(k-1)}+s-1}\right|\right)$$
$$\leq \sum_{k=k_n}^{\infty} 2^{-k(k-1)\left(\frac{1}{2}+\frac{1}{p}+\frac{1}{k}\right)} \cdot 2^{k(k-1)\left(\frac{1}{2}+\frac{1}{p}\right)}$$
$$= \sum_{k=k_n}^{\infty} 2^{-(k-1)} < \delta, \tag{2.4}$$

whenever $n \geq n_\delta$. We have proven that

$$\sup_{\|\varphi\|_{p^*} \leq 1} \sum_{j=n}^{\infty} \left|\varphi\left(x^{(j)}\right)\right| < \delta,$$

whenever $n \geq n_\delta$, and hence $\left(x^{(j)}\right)_{j=1}^{\infty}$ is unconditionally summable.

Next we note that, for $j \in \left\{2^{k(k-1)}, \ldots, 2^{k(k-1)+1} - 1\right\}$,

$$\left\|x^{(j)}\right\|_p = \left\|\left(2^{k(k-1)}\right)^{-\left(\frac{1}{2}+\frac{1}{p}+\frac{1}{k}\right)} \left(\sum_{s=1}^{2^{k(k-1)}} h_{rs}^{(k)} e_{2^{k(k-1)}+s-1}\right)\right\|_p = 2^{-k(k-1)\left(\frac{1}{2}+\frac{1}{k}\right)}.$$

Clearly $\left\|x^{(j)}\right\|_p = 0$ if $j \notin \left\{2^{k(k-1)}, \ldots, 2^{k(k-1)+1} - 1\right\}$. For any $r > 0$, we compute

$$\sum_{j=1}^{\infty} \left\|x^{(j)}\right\|_p^r = \sum_{k=1}^{\infty} 2^{k(k-1)} \cdot 2^{-k(k-1)\left(\frac{r}{2}+\frac{r}{k}\right)} = \sum_{k=1}^{\infty} 2^{k(k-1)\left(1-\frac{r}{2}-\frac{r}{k}\right)}$$

and note that such a series diverges provided $r < 2$. The proof of Theorem 1 is complete. □.

References

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