Improper Integrals: an alternative criterion

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Introduction

The topic *Improper Integrals*, often introduced in the second course of Calculus, is an important, though difficult concept for students to grasp, viz. [**1**], [**2**], and [**3**].

In this article we discuss an alternative (geometric) criterion for an improper integral to diverge. While the criterion is indeed efficient and easy to apply, if one believes, like I do, that teaching Calculus is more than training students to manipulate formulas, then the opportunity to present and discuss the reasoning leading to such a result should be thought as more valuable than the criterion, per se.

Let us start off with a classical example of divergent integral:

$$\int_0^1 \frac{1}{x} dx = +\infty. \tag{1}$$

This can be obtained by — and it is often presented as — a direct application of the Fundamental Theorem of Calculus, i.e.:

$$\int_0^1 \frac{1}{x} dx := \lim_{t \to 0^+} \int_t^1 \frac{1}{x} dx = \lim_{t \to 0^+} [\ln 1 - \ln t] = +\infty.$$

While certainly powerful, such a solution hides a beautiful geometric interpretation rooted in the very motivation of integrals, namely the calculus of areas.

Indeed, if one looks that the area of the degenerating rectangles of base $[0, t]$ and height $\frac{1}{t}$, one immediately sees that they all have constant area 1, no matter how small $t > 0$ is. The fact that one can find rectangles with with arbitrarily small base and constant area explains geometrically why such an improper integral must diverge.

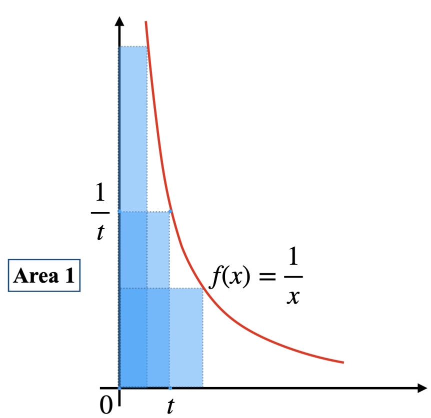

Figure 1. The geometric idea: since all such rectangles have area 1, the improper integral diverges.

The key point to note is that whether an improper integral converges or diverges depends upon the (sum of the) areas of those degenerating rectangles, near the vertical asymptote. If they all contributes with a constant amount, then the improper integral diverges.

Such a result can certainly be analyzed through Riemann sums, but we present here an alternative approach which may be easier to grasp:

$$\begin{aligned}\int_0^1 \frac{1}{x}dx &:= \lim_{t\to 0^+}\left(\int_t^{2t}\frac{1}{x}dx+\int_{2t}^1\frac{1}{x}dx\right)\\ &\geq \lim_{t\to 0^+}\left(\int_t^{2t}\frac{1}{2t}dx+\int_{2t}^1\frac{1}{x}dx\right)\\ &\geq \tfrac{1}{2}+\lim_{t\to 0^+}\left(\int_{2t}^1\frac{1}{x}dx\right)\\ &= \frac{1}{2}+\int_0^1\frac{1}{x}dx,\end{aligned}$$

where in the second line we have used that the function $\frac{1}{x}$ is decreasing and thus in $[t, 2t]$, one has $\frac{1}{x} \geq \frac{1}{2t}$. Clearly the inequality above implies $\displaystyle\int_0^1 \frac{1}{x}dx$ cannot be a real number, and hence such an improper integral diverges.

The reasoning above could possibly be better explained as a "contradiction argument", that is: assuming the improper integral $\displaystyle\int_0^1 \frac{1}{x}dx$ converges, say to a limit number L, one would end with the inequality: $L \geq \frac{1}{2} + L$, leading to a contradiction.

While rather simple, this argument yields an efficient, more direct and didactical criterion for divergent integrals, namely:

Theorem 1. *Let $f\colon (a, b] \to \mathbb{R}$ be a continuous function defined on a bounded interval. Assume for some $a < c < b$, the function $f(x)$ is non-increasing in (a, c). Then*

$$\lim_{x\to a^+}(x-a)f(x)=\mu>0 \qquad \Longrightarrow \qquad \int_a^b f(x)=+\infty.$$

Proof. The proof is just a mere generalization of the argument explained above. Indeed, one can write

$$\begin{aligned}\int_a^b f(x)dx &:= \lim_{t\to 0^+}\left(\int_{a+t}^{a+2t}f(x)dx+\int_{a+2t}^b f(x)dx\right)\\ &\geq \lim_{t\to 0^+}\left(\int_{a+t}^{a+2t}f(a+2t)dx+\int_{a+2t}^b f(x)dx\right)\\ &\geq \lim_{t\to 0^+} tf(a+2t)+\lim_{t\to 0}\left(\int_{a+2t}^b f(x)dx\right)\\ &= \lim_{x\to 0^+}\frac{(x-a)}{2}f(x)+\int_a^b f(x)dx\\ &= \frac{\mu}{2}+\int_a^b f(x)dx.\end{aligned}$$

In the second line we have used that for t small enough, $a + 2t < c$, along with the assumption that f is non-increasing in (a, c). In the third line, we computed $\lim_{x\to 0^+} \frac{(x-a)}{2} f(x)$ by the change of variables, $a + 2t = x$.

Arguing as before, since $\frac{\mu}{2}$ is positive, the above inequality implies that $\displaystyle\int_a^b f(x)dx$ cannot be a real number, and thus the improper integral must diverge. ■

Remark. It's worth noticing that one could use the "comparison theorem" to confirm the thesis of Theorem 1. Indeed, if one assumes $\lim_{x\to a^+} (x - a)f(x) = \mu > 0$, then, for some $\delta > 0$, we have $f(x) > \frac{\mu}{2} \cdot \frac{1}{x-a}$, for all $a < x < a + \delta$. Since $\int_a^{a+\delta} \frac{1}{x-a} dx = +\infty$, the conclusion follows by comparison. The argument presented above, though, is arguably more pedagogical and its visual representation more appealing.

Here are few examples elucidating the applicability of such a criterion.

Example 1. The improper integral $\int_0^1 \frac{1}{\sin x} dx$ diverges. Indeed, one simply calculates $\lim_{t\to 0^+} t \cdot \frac{1}{\sin t} = 1$.

Example 2. The improper integral $\int_0^1 \frac{1}{x^{1-x}} dx$ diverges. To verify that, one computes $\lim_{t\to 0^+} t \cdot \frac{1}{t^{1-t}} = \lim_{t\to 0^+} t^t = 1$.

Example 3. Easily one generalizes the previous Example to check that the improper integral $\int_0^1 \frac{1}{x^{1-\sqrt{x}}} dx$ diverges, and in fact $\int_0^1 \frac{1}{x^{1-x^\alpha}} dx$ diverges, for all $\alpha > 0$. That's because $\lim_{t\to 0^+} t \cdot \frac{1}{t^{1-t^\alpha}} = \lim_{t\to 0^+} t^{t^\alpha} = 1$.

The very same argument can be easily employed as a criterion for improper integrals of functions blowing-up at the right-end point of the interval. That is, if $f\colon (a, b) \to \mathbb{R}$ is a continuous function on a bounded interval with $\lim_{x\to b^-} f(x) = +\infty$. Then, if for some $a < c < b$, the function $f(x)$ is non-decreasing in (c, b), one has

$$\lim_{x\to b^-} (b - x)f(x) = \mu > 0 \quad \Longrightarrow \quad \int_a^b f(x) = +\infty.$$

Example 4. The improper integral $\int_0^{\pi/2} \tan x dx$ diverges. Indeed,

$$\lim_{t\to \frac{\pi}{2}^-} (\frac{\pi}{2} - t) \cdot \tan t = 1.$$

Similar analysis can also be done for nonnegative continuous functions $f\colon [a, \infty) \to \mathbb{R}$. If the degenerating rectangles of base $[a, x]$ and height $[0, f(x)]$ have uniform positive area as $x \to \infty$, then $\int_a^\infty f(x)dx$ must diverge. The argument is exactly the same. We invite the readers to state the result and modify the proof of Theorem 1 to verify it.

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