ON THE LONGEST DISTANCE PROBLEM

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ABSTRACT. The shortest path between two points in the (flat) plane is a straight line. What about the longest? In this article we delve into this question and explore some of its subtleties.

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1. Introduction

Given two points A, B in the plane, there are infinitely many ways to join them. A classical, almost intuitive, theorem states that the path with minimal length is a straight line. While this may sound "obvious", a mathematical proof of this fact is, by no means, trivial; see [3] for a simpler approach of that problem.

The shortest distance principle is useful in many circumstances, but it might just happen that one is interested in paths with very long lengths. Think for instance about our digestive system, airport security lines, stock market fluctuations, etc. For those models, one wants to understand the longest path possible within a confined region. This is a relevant applied problem that require proper mathematical formulations. For related questions, we cite [1,4,5].

We start off by discussing a general mathematical model for the longest distance problem. We will be interested in graph paths of the form $(t, f(t))$, for some function $f\colon [0,1] \to \mathbb{R}$, joining two opposite corners of the unit square, $Q = [0,1]\times[0,1]$, namely the origin $A = (0,0)$ and the point $B = (1,1)$. Here comes the most important question we must address: what kind of functions should be allowed as "acceptable paths"? Well, the first natural requirement, as to avoid routes with jumps, would be continuity. Also, as to add some minimal structure, we assume continuous differentiability, up to finitely many points. Under such minimal assumptions, we can represent the arc-length of a given path by a simple mathematical formula:

$$\ell(f) := \int_0^1 \sqrt{1 + f'^2(t)}dt.$$

Let's give a name for the collection of all possible piecewise continuously differentiable paths $f\colon [0,1] \to \mathbb{R}$ joining $A = (0,0)$ and $B = (1,1)$:

$$\mathscr{P} := \left\{ \begin{array}{c} f\colon [0,1] \to \mathbb{R} \;\big|\; f \text{ is piecewise continuously differentiable in } (0,1), \\ \text{continuous at 0 and at 1, with } f(0) = 0 \text{ and } f(1) = 1. \end{array} \right\}$$

If no further assumption is required, one can easily cook up a function g belonging to the set $\mathscr{P}$ whose arc-length is infinite. To make the problem more interesting (and more realistic) we need to impose further constrains on the collection of admissible paths. In this article we will discuss the longest distance problem under monotonicity as well as under Lipschitz control on admissible paths.

2. Non-decreasing function

An effective way to assure arc-length finiteness is by preventing the function from zig-zagging. Mathematically this is equivalent to assuming the function $f(t)$ is non-decreasing. Since f is piecewise continuously differentiable, such a condition can be further represented as $f'(t) \geq 0$ for all $t \in (0,1)$ where f is differentiable. Let's give a name for the collection of such paths:

$$\mathscr{P}^+ := \left\{ f \in \mathscr{P} \;\middle|\; f'(t) \geq 0 \text{ for all } t \in (0,1) \text{ where } f \text{ is differentiable} \right\}.$$

At least now one can easily obtain an upper bound for the arc-length of paths in $\mathscr{P}^+$. We restore to a classical inequality, namely $\sqrt{1+a^2} \leq 1 + a$, for all $a \geq 0$. Thus, if $f \in \mathscr{P}^+$, one has:

$$\ell(f) = \int_0^1 \sqrt{1 + f'(t)^2} dt \leq \int_0^1 (1 + f'(t))\, dt = 1 + f(1) - f(0) = 2. \tag{1}$$

It turns out the longest arc-length problem over $\mathscr{P}^+$ is an interesting example of an optimization problem whose supremum is not attained, namely:

$$\sup \left\{ \ell(f) \;\middle|\; f \in \mathscr{P}^+ \right\} = 2 > \ell(f), \quad \forall f \in \mathscr{P}^+.$$

Indeed, since $\sqrt{1+a^2} < 1 + a$, provided $a > 0$ and $f'(t)$ cannot be identically zero, inequality (1) is actually strict; compare this conclusion with the results from [1,5]. That the supremum is in fact 2, follows by considering the sequence of functions $f_n(t) = t^n \in \mathscr{P}^+$ and verifying $\lim\limits_{n \to \infty} \ell(f_n) = 2$.

3. Non-decreasing functions with maximal slope

Let's now discuss the longest path problem for paths with maximum admissible angles. Mathematically this is equivalent to restricting the derivative of f,

$$\mathscr{P}_L^+ := \left\{ f \in \mathscr{P}^+ \;\middle|\; 0 \leq f'(t) \leq L \text{ for all } t \in (0,1) \text{ where } f \text{ is differentiable} \right\}.$$

Note that, while the classical Ascoli-Arzela Theorem assures $\mathscr{P}^+$ is sequentially locally compact in the uniform topology (see for instance [2]), the existence of a maximal length graph in $\mathscr{P}^+$ does not follow directly, as the functional ℓ is actually a function of f' (not of f). We need another, more constructive, approach to the problem and this is the contents of our next theorem.

Theorem 1. *Let $f \in \mathscr{P}_L^+$, then*

$$\ell(f) \leq 1 + \sqrt{1 + L^{-2}} - L^{-1}.$$

Furthermore, there are infinitely many maximizers, including

$$f_L(t) = \begin{cases} 0 & \textit{for} \quad 0 \leq t \leq 1 - L^{-1} \\ Lt - L & \textit{for} \quad 1 - L^{-1} \leq t \leq 1. \end{cases}$$

Proof. The idea is obtain a refined upper bound for the integrand $F(x) = \sqrt{1+x^2}$, that holds for all $0 \leq x \leq L$. More precisely, for a number $0 < a < 1$, depending on L we want to attain:

$$\sqrt{1+x^2} \leq 1 + a \frac{x}{\sqrt{1+x^2}} x, \tag{2}$$

for all $0 \leq x \leq L$. Note that if one takes $a = 1$, (2) holds for all $x \in (0, +\infty)$. This is because $F'(x) = \frac{x}{\sqrt{1+x^2}}$ is increasing in $(0, \infty)$ and thus, by the Mean Value Theorem, for some $0 < t_x < x$, we have:

$$F(x) - F(0) = F'(t_x)x \leq F'(x)x.$$

To find the smallest a possible such that (2) holds leads to the algebraic equation:

$$\sqrt{1+L^2} = 1 + a\frac{L}{\sqrt{1+L^2}}L, \tag{3}$$

which can be easily solved. Arguing as before, we estimate, for all $f \in \mathscr{P}_L$,

$$\ell(f) \leq \int_0^1 \left(1 + a\frac{L}{\sqrt{1+L^2}}f'(t)\right) dt = 1 + a\frac{L}{\sqrt{1+L^2}}. \tag{4}$$

Dividing by L the equality (3) we reach:

$$a\frac{L}{\sqrt{1+L^2}} = \frac{1}{L}\sqrt{1+L^2} - L^{-1} = \sqrt{1+L^{-2}} - L^{-1}.$$

Going back to (4) we reach:

$$\ell(f) \leq 1 + \sqrt{1+L^{-2}} - L^{-1} = \ell(f_L).$$

Since $f_L \in P_L$ we have proven f_L is a maximizer. To show there are infinitely many maximizers in $\mathscr{P}_L^+$, we opt for a visual proof, which can be easily translated into an analytical one:

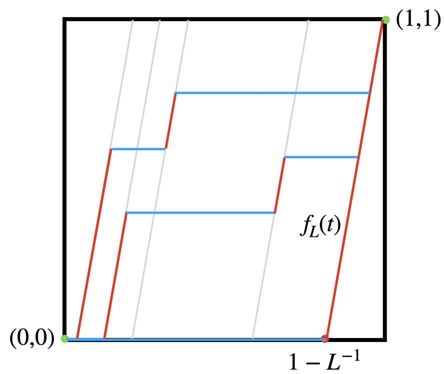

Figure 1. For any partition of the interval $[0, 1 - L^{-1}]$, one can construct a length extremal function in $\mathscr{P}_L^+$ by intercalating constant functions and affine functions with slope exactly L.

The proof of Theorem 1 is complete. □

We conclude this section by remarking that Theorem 1 can be easily generalized to non-decreasing functions defined on an arbitrary interval $[a, b]$ and taking values $f(a) = c$ and $f(b) = d$. For any $L > \frac{d-c}{b-a}$, the maximal length is attained by infinitely many maximizers, including:

$$g(t) = \begin{cases} c & \text{for} \quad a \leq t \leq b - L^{-1}(d-c) \\ L(t-b) - d & \text{for} \quad b - L^{-1}(d-c) \leq t \leq b. \end{cases}$$

When $L = \frac{d-c}{b-a}$, then it is easy to see that the corresponding set $\mathscr{P}_L^+$ is unitary and the only admissible function is $f(t) = \frac{d-c}{b-a}(t-a) + c$. These remarks will be useful in the next section.

4. The general problem with maximal slope

In this final section we tackle the longest path problem, allowing zig-zags but restricting the maximal slope of the graph. That is, we consider the set of admissible paths as:

$$\mathscr{P}_L := \left\{ f \in \mathscr{P} \;\middle|\; |f'(t)| \le L \text{ for all } t \in (0,1) \text{ where } f \text{ is differentiable} \right\}.$$

Given a function $f \in \mathscr{P}_L$, there exist points $0 = t_0 < t_1 < t_2 < \cdots < t_{k-1} < t_k = 1$ such that $f\big|_{[t_{j-1},t_j]}$ is monotone, say:

$$\begin{array}{c} f\big|_{[0,t_1]} \text{ is non-decreasing,} \\ f\big|_{[t_1,t_2]} \text{ is non-increasing,} \\ \vdots \\ f\big|_{[t_{k-1},t_k]} \text{ is non-increasing,} \\ f\big|_{[t_k,1]} \text{ is non-decreasing.} \end{array}$$

Let $f(t_i) = a_i$, $i = 1, 2, \cdots k$. Applying Theorem 1 to $f\big|_{[0,t_1]}$ (see remark after the Theorem) we conclude

$$\ell(f; 0, t_1) \le t_1 - \delta_1 + \sqrt{\delta_1^2 + a_1^2},$$

where $0 < \delta_1 \le t_1$ is such that:

$$\frac{a_1}{\delta_1} = L.$$

Hence we can further write:

$$\ell(f; 0, t_1) \le (t_1 - \delta_1) + \delta_1 \sqrt{1 + L^2}.$$

Similarly, for a number $0 < \delta_2 \le t_2 - t_1$, satisfying

$$\frac{|a_1 - a_2|}{\delta_2} = L.$$

we have

$$\begin{aligned} \ell(f; t_1, t_2) &\le (t_2 - \delta_2 - t_1) + \sqrt{\delta_2^2 + |a_1 - a_2|^2} \\ &= (t_2 - \delta_2 - t_1) + \delta_2 \sqrt{1 + L^2}. \end{aligned}$$

Continuing the analysis, we conclude:

$$\ell(f; t_{j-1}, t_j) \le (t_j - \delta_j - t_{j-1}) + \delta_j \sqrt{1 + L^2}.$$

Adding all such inequalities up, noting the telescopic sum involved, we reach:

$$\ell(f; 0, 1) \le 1 + \left(\sqrt{1 + L^2} - 1 \right) \sum_{j=1}^{k-1} \delta_j.$$

At this point it is probably a good idea to remind that for all such estimates above, we know the shape of the maximizers, Figure 1. Back to the analysis: one clearly has $\sum_{j=1}^{k-1} \delta_j \le 1$, so to optimize the RHS, we should make $\sum_{j=1}^{k-1} \delta_j = 1$, which is equivalent to "avoiding flat graphs" in each subinterval, i.e. $\delta_j = t_j - t_{j-1}$. In conclusion, the optimal configuration is a true zig-zagging function which always turns with maximal angle allowed. Let's state these results as a Theorem.

Theorem 2. *Let $L > 0$ be given and $f\colon [0,1] \to \mathbb{R}$ a piecewise continuously differentiable function satisfying*

$$f(0) = 0, \quad |f(1)| < L, \quad |f'(t)| \le L.$$

Then

$$\ell(f) \le \sqrt{1 + L^2}.$$

Furthermore, there are infinitely many extremal functions $f_\star$ verifying $\ell(f_\star) = \sqrt{1+L^2}$, constructed by turning with maximal allowed angle, i.e. $|f_\star'(t)| = L$ whenever $f_\star$ is differentiable.

Let's conclude with final remarks. It follows from Theorem 2 that the maximum length does not depend upon the number of turns the path takes, as long as it always turn with maximal allowed angle. This is, in principle, counter-intuitive. Also, it does not matter where you want to arrive at time $t = 1$, as long as $|f(t)| \le L$, the maximal length is always the same. Again this is a bit counter-intuitive. It is interesting to note that the maximum length, $\sqrt{1+L^2}$, equals the length of the straight line joining $(0,0)$ and $(1, L)$. This is easy to see geometrically: at each downward turning point, simply flip the graph upward, as to make the resulting function, $g(t)$ to verify $g'(t) = +L$ everywhere.

5. Declarations

Conflict of interest. The author declares that she has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability. Concerns regarding data availability does not apply to this manuscript.

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