On the topology of punctured spaces

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Abstract. We give a short, elementary proof of the fact that singletons are not topologically negligible for a relevant family of metric spaces. The idea is geometric and, in particular, it offers alternative solutions to some classical theorems of that nature.

Given a topological space (X, τ), an important question regarding its intrinsic topology is whether its punctured version, i.e. $X \setminus \{p\}$, is homeomorphic (or not) to X. Such an investigation is connected, for instance, with fixed point properties of subsets of X and it has attracted warm attention for decades.

While somewhat intuitive, problems of that nature are in general subtle and their proofs often involve rather heavy machinery. For instance, that the punctured Euclidean space, $\mathbb{R}^n \setminus \{0\}$, is not homeomorphic to $\mathbb{R}^n$ requires tools from algebraic topology, see for instance [**2**]. On the other hand, a deep theorem proven by Klee in the 1950's assures that an infinite dimensional normed vector space is always homeomophic to its punctured version, see [**3, 4**]. In fact one can delete entire compact subsets and even more general subregions of an infinite dimensional vector space without changing its topology, see for instance [**5, 1**].

By the classical Riesz's theorem, local compactness, i.e. the property that bounded, closed sets are compact, characterizes finite dimensionality of a normed vector space. Thus, *in a somewhat baffling way*, local compactness turns out to be the ultimate topological obstruction preventing a vector space from being homeomorphic to its punctured version.

The discussion above motivates the main result of this note. Namely we are interested in a criteria that prevents a metric space from being homeomorphic to its punctured version. When projected onto the Euclidean space, our result offers an elementary proof of the fact that $\mathbb{R}^n \setminus \{0\}$ is not homeomorphic to $\mathbb{R}^n$, based on local compactness of the space.

In this paper we use the standard notation for open balls in a metric space (X, d), i.e. $B_R(q) := \left\{ v \in X \;\middle|\; d(v, q) < R \right\}$.

Theorem 1. *Let (X, d) be a complete, locally compact metric space and assume $X \setminus B_R(q)$ is path connected, for some $q \in X$ and all $R > 0$ large enough. Then there is no homeomorphism between X and its punctured version, $X \setminus \{p\}$.*

Proof. We argue by contradiction, that is, we suppose it is possible to find a homeomorphism $\varphi \colon X \setminus \{p\} \to X$. Initially we note that if $(p_n)_{n \in \mathbb{N}}$ is a sequence in $X \setminus \{p\}$ converging to p, then its image is unbounded, i.e.

$$\lim_{n \to \infty} d(p_n, p) = 0 \quad \implies \quad \lim_{n \to \infty} d(\varphi(p_n), p) = +\infty, \tag{1}$$

as $n \to \infty$. Indeed, let us assume, seeking a contradiction, that for a subsequence $(p_{n_j})_j$ there holds

$$d(\varphi(p_{n_j}), p) \le C_0 < +\infty.$$

Hence, by local compactness, passing to a further subsequence if necessary, one would have

$$\lim_{j\to\infty} \varphi(p_{n_j}) = q \in X. \tag{2}$$

Applying the continuous map φ^{-1} to (2) one would reach:

$$p = \lim_{j\to\infty} p_{n_j} = \lim_{j\to\infty} \varphi^{-1} \circ \varphi(p_{n_j}) = \varphi^{-1}(q),$$

but this is a contradiction, since $\varphi\colon X \setminus \{p\} \to X$ is a bijection, and thus $\varphi^{-1}(q)$ is a point in $X \setminus \{p\}$, and thus in particular it cannot be p.

A similar argument by contradiction yields the following conclusion:

$$\lim_{n\to\infty} d(p_n, p) = +\infty \quad \Longrightarrow \quad \lim_{n\to\infty} d(\varphi(p_n), p) = +\infty. \tag{3}$$

Indeed, if for some subsequence $d(\varphi(p_{n_j}), p)$ remained bounded, then, passing to another subsequence, if necessary,

$$\lim_{j\to\infty} \varphi(p_{n_j}) = s \in X. \tag{4}$$

Let $z := \varphi^{-1}(q) \in X \setminus \{p\}$. We estimate:

$$d(p_{n_j}, p) = d(\varphi^{-1}(\varphi(p_{n_j}), p) = d(z, p) + \mathrm{o}(1) < \infty,$$

which is a contradiction to the hypothesis made on (3). Next, let $\lambda > 0$ be a positive number and consider the set:

$$\mathbb{S}^X_\lambda := \left\{ v \in X \setminus \{p\} \;\middle|\; d(v, p) = \lambda \right\}.$$

This is clearly a bounded, closed subset of $X \setminus \{p\}$. By the local compactness hypothesis on (X, d), $\mathbb{S}^X_\lambda$ is then a compact set. Let $K := \varphi(\mathbb{S}^X_\lambda)$; a compact (and thus bounded) subset of X. For $R > 0$ large enough, we have $K \subset B_R(q)$. From (1) there exists a point $z_1 \in X \setminus \{p\}$ verifying:

$$0 < d(z_1, p) < \frac{\lambda}{4} \quad \text{and} \quad \varphi(z_1) \notin B_R(q). \tag{5}$$

Similarly, it follows from (3) that there exists a point $z_2 \in X \setminus \{p\}$ satisfying:

$$d(z_2, p) > 4\lambda \quad \text{and} \quad \varphi(z_2) \notin B_R(q). \tag{6}$$

By assumption, $X \setminus B_R(q)$ is path connected, hence there exists a continuous path, say $\varrho\colon [0,1] \to X \setminus B_R(q)$ satisfying:

$$\varrho(0) = \varphi(z_1) \quad \text{and} \quad \varrho(1) = \varphi(z_2).$$

The path $\mu := \varphi^{-1} \circ \varrho\colon [0,1] \to X \setminus \{p\}$ is continuous. Readily one checks that $d(\mu(0), p) < \frac{\lambda}{4}$ and $d(\mu(1), p) > 4\lambda$.

It now follows by the Intermediate Value Theorem for continuous functions that there exists a time $t_0 \in (0,1)$ such that

$$d(\mu(t_0), p) = \lambda.$$

But then $\varrho(t_0) = \varphi(\mu(t_0)) \in K \subset B_R(q)$ which finally leads to a contradiction. ■

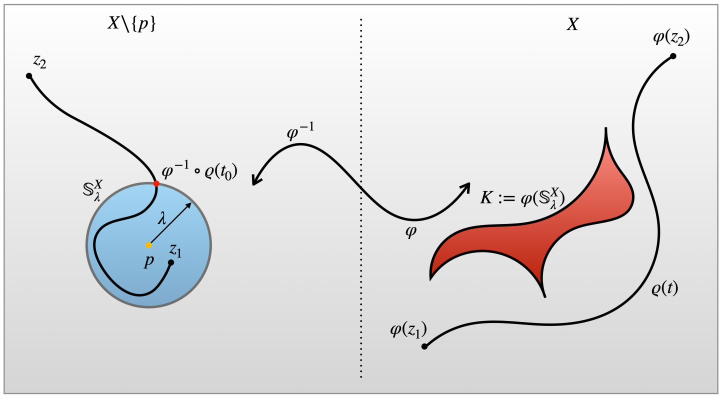

Figure 1. Geometric idea of the proof.

We conclude with two comments. First, let us explain the geometric insight behind the proof, described in Figure 1. The idea relies on the possibility of finding a point z_1 close to p and a point z_2 far from p whose images, $\varphi(z_1)$ and $\varphi(z_2)$, can be joined by a path ϱ which avoids all together the image of the compact set $\mathbb{S}^X_\lambda$. However, this is a contradiction as the pulled-back path, $\varphi^{-1}\circ\varrho$, must cross $\mathbb{S}^X_\lambda$.

The prototypical metric space satisfying the conditions of Theorem 1 is the Euclidean space $\mathbb{R}^n$, $n>1$, endowed with a norm. Indeed, it is simple to check that when $n>1$, the set $\mathbb{R}^n\setminus B_R(q)$ is path-connected for any $q\in\mathbb{R}^n$ and $R>0$. One way to see this is by using the fact that the sphere $\mathbb{S}^{n-1}_R(q) := \{y\in\mathbb{R}^n \mid \|y-q\|=R\}$ is path-connected. This is a very classical result, see for instance [**6**], but here is easy way to check this. With no loss of generality, take $q=0$. If $v,w\in\mathbb{S}^{n-1}_R$ are linearly independent vectors, the path $\gamma(t)=R\frac{(1-t)v+tw}{\|(1-t)v+tw\|}\in\mathbb{S}^{n-1}_R$ and joins v and w. If v and w are linearly dependent, then choose a third vector $\nu\in\mathbb{S}^{n-1}_R$ such that both $\{v,\nu\}$ and $\{\nu,w\}$ are linearly independent sets; we can do this since $n>1$. The concatenation of the paths joining v and ν and ν and w yields a continuous path that joins v and w.

Now, similarly, if $x,y\notin B_R(q)$, consider the straight line, ℓ, joining x and y, $\ell(t)=x+t(y-x)$, $0\le t\le 1$. If ℓ does not pass through $B_R(q)$, we take it as the path joining x and y. If ℓ does pass through $B_R(q)$, then there exist two points v_x and v_y in $\mathbb{S}^{n-1}_R(q)$, such that $\ell(t_x)=v_x$ and $\ell(t_y)=v_y$, $0<t_x<t_y$. Let γ be a path $\mathbb{S}^{n-1}_R(q)$ joining v_x and v_y. The concatenation of the paths $\ell\big|_{[0,t_x]}$, γ, and $\ell\big|_{[t_y,1]}$ is a path joining x and y that does not intersect with $B_R(q)$, see Figure 2.

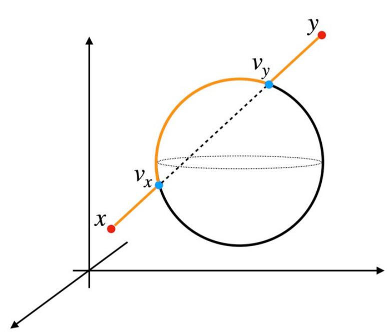

Figure 2. $\mathbb{R}^n\setminus B_R(q)$ is path connected, $n>1$.

For the sake of completeness, we comment that the real line $\mathbb{R}$ does not fall into the hypotheses of Theorem 1; however that $\mathbb{R}$ is not homeomorphic to $\mathbb{R} \setminus \{p\}$ follows directly by connectedness considerations.

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