VECTOR-VALUED KAHANE–SALEM–ZYGMUND INEQUALITIES WITH ASYMPTOTICALLY BOUNDED CONSTANTS

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ABSTRACT. We establish new vector-valued Kahane–Salem–Zygmund inequalities, with asymptotically bounded constants, on spaces with unconditional Schauder basis.

1. INTRODUCTION

Multilinear forms or polynomials with ± 1 coefficients and small norms play key roles in several fields of mathematics and its applications, viz. [3, 4, 8, 9, 11]. To some extend, they should be thought as extrema of some sort of optimization problems. For instance, the classical Littlewood's $4/3$ inequality yields

$$\left(\sum_{i,j=1}^{\infty}\left|T\left(e_i,e_j\right)\right|^{4/3}\right)^{3/4}\leq\sqrt{2}\sup\left\{\left|T\left(x,y\right)\right|:\|x\|\leq 1\text{ and }\|y\|\leq 1\right\},$$

for all bilinear forms $T\colon c_0\times c_0\longrightarrow\mathbb{R}$. The exponent $4/3$ and the constant $\sqrt{2}$ are optimal and the optimality of such parameters are attained by suitable bilinear forms with coefficients ± 1.

Similarly, Bohnenblust and Hille [8] constructed an m-linear form $A_n\colon\ell_\infty^n\times\cdots\times\ell_\infty^n\longrightarrow\mathbb{C}$ with complex coefficients with modulus 1 satisfying

$$\|A_n\|\leq n^{(m+1)/2}$$

and they showed that the exponent $(m+1)/2$ is optimal, i.e., it cannot be replaced by a smaller one. This result plays a fundamental role in the investigation of the famous Bohr radius problem. In the 1970's and 1980's, similar inequalities were investigated in [4, 6, 7, 10, 14] using probabilistic techniques and their statements can be summarized as follows. If we denote $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ and represent $\mathbb{K}^n$ endowed with the ℓ_p-norm by ℓ_p^n, for all positive integers m,n and $p_1,\ldots,p_m\in[1,\infty]$, there exists an m-linear form $A_n\colon\ell_{p_1}^n\times\cdots\times\ell_{p_m}^n\longrightarrow\mathbb{K}$ of the type

$$A_n(z^{(1)},\ldots,z^{(m)})=\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n}\pm z_{i_1}^{(1)}\cdots z_{i_m}^{(m)},$$

such that

$$\|A_n\|\leq C_m n^{\frac{1}{\min\{\max\{2,p_1^*\},\ldots,\max\{2,p_m^*\}\}}+\sum_{k=1}^{m}\max\left\{\frac{1}{2}-\frac{1}{p_k},0\right\}}$$

for a certain constant C_m. Above and henceforth, as usual, we consider $1/\infty=0$, the conjugate of p is denoted by p^*, i.e., $p^*=p/(p-1)$.

Inequalities of that type are called nowadays called Kahane–Salem–Zygmund inequalities (KSZ for short). In the case of bilinear forms, Bennett's approach [5, Proposition 3.2] is more general, allowing different dimensions at the domain of the bilinear forms. More precisely, Bennett's inequality

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claims that, for all $p_1, p_2 \in [1, \infty]$ and all positive integers n_1, n_2, there exists a bilinear form $A_{n_1,n_2} \colon \ell_{p_1}^{n_1} \times \ell_{p_2}^{n_2} \longrightarrow \mathbb{R}$ with coefficients ± 1 satisfying

$$\|A_{n_1,n_2}\| \leq C \max \left\{ n_2^{1/p_2^*} n_1^{\max\left\{\frac{1}{2} - \frac{1}{p_1}, 0\right\}}, n_1^{1/p_1^*} n_2^{\max\left\{\frac{1}{2} - \frac{1}{p_2}, 0\right\}} \right\}, \tag{1.1}$$

where C is a constant depending only on p_1 and p_2. The KSZ inequality (following Bennett's style) was recently extended to m-linear forms ([2, 13]) as follows: let $m, n_1, \ldots, n_m$ be positive integers and $p_1, \ldots, p_m \in [1, \infty]$. There exist a constant C_m and an m-linear form $A_{n_1,\ldots,n_m} \colon \ell_{p_1}^{n_1} \times \cdots \times \ell_{p_m}^{n_m} \longrightarrow \mathbb{K}$ of the type

$$A_{n_1,\ldots,n_m}\left(z^{(1)}, \ldots, z^{(m)}\right) = \sum_{i_1=1}^{n_1} \cdots \sum_{i_m=1}^{n_m} \pm z_{i_1}^{(1)} \cdots z_{i_m}^{(m)},$$

such that

$$\|A_{n_1,\ldots,n_m}\| \leq C_m \max_{k=1,\ldots,m} \left\{ n_k^{\frac{1}{\min\{\max\{2, p_1^*\}, \ldots, \max\{2, p_m^*\}\}}} \right\} \prod_{k=1}^{m} n_k^{\max\left\{\frac{1}{2} - \frac{1}{p_k}, 0\right\}}. \tag{1.2}$$

A simple calculation shows that when $m = 2$ and $p_1, p_2 \in [2, \infty]$, the inequality (1.2) recovers (1.1). It turns out that, while powerful, the probabilistic approach generates very large bounding constants, viz. $C_m > \sqrt{m!}$.

The program to obtain improved (smaller) constants for these inequalities has become an important tread of research. For instance, in [12], the original KSZ inequality was investigated by means of deterministic methods that allowed to show that in some cases the constants are asymptotically dominated by 1. Here is the precise statement:

Theorem 1.1. *([12, Corollary 1.2]) Let a positive integer m and $\varepsilon > 0$ be given. There exists a positive integer N such that, for all $n \geq N$, there exists an m-linear form $A_n \colon \ell_\infty^n \times \cdots \times \ell_\infty^n \longrightarrow \mathbb{K}$ of the type*

$$A_n(z^{(1)}, \ldots, z^{(m)}) = \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} \pm z_{i_1}^{(1)} \cdots z_{i_m}^{(m)},$$

such that

$$\|A_n\| \leq (1 + \varepsilon)\, n^{\frac{m+1}{2}}.$$

The main goal of this current paper is to generalize Theorem 1.1 in several ways. Initially we are interested in inequalities over more general Banach spaces. This will be attained by replacing the ℓ_∞^n space by n-dimensional subspaces of a Banach space with unconditional Schauder basis. Next we extend the role of ℓ_s^n in the original inequality to allow n-dimensional subspaces of a Banach space of cotype s, with unconditional Schauder basis.

Before we can state our main result, let's denote the infimum of the cotypes assumed by a Banach space E by $\cot(E)$. The cotype q constant of E will be represented hereafter by $C_q(E)$. Here is the main theorem of this paper:

Theorem 1.2. *Let a positive integer m and $\varepsilon > 0$ be given and $E^{(1)}, \ldots, E^{(m)}$ be infinite-dimensional Banach spaces with normalized unconditional Schauder basis $\left(z_j^{(1)}\right)_{j=1}^{\infty}, \ldots, \left(z_j^{(m)}\right)_{j=1}^{\infty}$ and constants $K_1, \ldots, K_m$, respectively. Let also F be Banach space with cotype $q = \cot(F)$ and normalized unconditional Schauder basis $\left(z_j^{(m+1)}\right)_{j=1}^{\infty}$ with constant K_{m+1}. Let us denote*

$$E_k^{(i)} := \operatorname{span}\{z_1^{(i)}, \ldots, z_k^{(i)}\}$$

for all $i = 1, \ldots, m$. *Then, there exists a positive integer* N *such that, for all* $n \geq N$, *there exists an* m*-linear form* $A_n \colon E_n^{(1)} \times \cdots \times E_n^{(m)} \longrightarrow F^*$ *with*

$$A_n(z_{j_1}^{(1)}, \ldots, z_{j_m}^{(m)}) \left(z_{j_{m+1}}^{(m+1)} \right) = \pm 1,$$

for all $j_1, \ldots, j_m, j_{m+1} \in \{1, \ldots, n\}$, *satisfying*

$$\|A_n\| \leq (K_1 \cdots K_{m+1} C_q(F) + \varepsilon)\, n^{\frac{m+2}{2} - \frac{1}{q}}.$$

Moreover, the exponent $\frac{m+2}{2} - \frac{1}{q}$ *is optimal, in the sense that it cannot be improved by a smaller one keeping the generality of the statement.*

When $F^* = \ell_s$ and $E^{(1)} = \cdots = E^{(m)} = c_0$, we recover [1, Lemma 6.2]; however with a much more precise constant as the original estimate in [1] yields $C_m > \sqrt{m!}$.

The second main result we will proof in this article generalizes Bennett's inequality (1.1) for $p_1, p_2 \in [2, \infty]$ and $n_1 = n_2$ as follows:

Theorem 1.3. *Let* $\varepsilon > 0$ *and* $q_1, q_2 \in [2, \infty]$. *For* $k = 1, 2$, *let* $E^{(k)}$ *be a Banach space of cotype* $q_k = \cot\left(E^{(k)}\right)$ *and normalized unconditional Schauder basis* $\left(z_j^{(k)}\right)_{j=1}^{\infty}$ *with constant* K_k. *There exists a positive integer* N *such that, whenever* $n > N$, *there is a bilinear form* $A \colon E_n^{(1)} \times E_n^{(2)} \longrightarrow \mathbb{K}$ *of the type*

$$A(z^{(1)}, z^{(2)}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \pm\, z_i^{(1)} z_j^{(2)},$$

such that

$$\|A\| \leq \left(K_1 K_2 C_{q_1}(E^{(1)}) C_{q_2}(E^{(2)}) + \varepsilon \right) n^{\frac{3}{2} - \frac{1}{q_1} - \frac{1}{q_2}}$$

The rest of the paper is organized as follows: in Section 2 we establish the proof of Theorem 1.2. In fact we shall establish a slightly more general result. In section 3, we will discuss the proof of Theorem 1.3.

2. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. However, for the sake of completeness, we will actually establish a more general result, of whom Theorem 1.2 is a direct consequence. The precise statement of the result to be proven in this section is as follows:

Theorem 2.1. *Let a positive integer* m *and* $\varepsilon > 0$ *be given and* $E^{(1)}, \ldots, E^{(m)}$ *be Banach spaces with normalized unconditional Schauder basis* $\left(z_j^{(1)}\right)_{j=1}^{\infty}, \ldots, \left(z_j^{(m)}\right)_{j=1}^{\infty}$ *and constants* $K_1, \ldots, K_m$, *respectively. Let also* F *be an infinite-dimensional Banach space of cotype* q *and unconditional Schauder basis* $\left(z_j^{(m+1)}\right)_{j=1}^{\infty}$ *with constant* K_{m+1}. *Then, there exists a positive integer* N *such that, for all* $n_1, \ldots, n_m \geq N$, *there exists an* m*-linear operator* $A_{n_1,\ldots,n_m} \colon E_{n_1}^{(1)} \times \cdots \times E_{n_m}^{(m)} \longrightarrow F^*$ *with*

$$A_{n_1,\ldots,n_m}(z_{j_1}^{(1)}, \ldots, z_{j_m}^{(m)}) \left(z_{j_{m+1}}^{(m+1)} \right) = \pm 1,$$

for all $j_k \in \{1, \ldots, n_k\}$ *and* $j \in \{1, \ldots, \min\{n_1, \ldots, n_m\}\}$, *satisfying*

$$\|A_{n_1,\ldots,n_m}\| \leq (K_1 \cdots K_{m+1} C_q(F) + \varepsilon) \min\{n_1, \ldots, n_m\}^{\frac{1}{2} - \frac{1}{q}} \max\{n_1, \ldots, n_m\}^{1/2} \prod_{j=1}^{m} n_j^{1/2}.$$

Let $X_1, \ldots, X_m, Y$ be Banach spaces. We recall the following isometric isomorphism between the space $\mathcal{L}(X_1, \ldots, X_m; Y^*)$ of all continuous m-linear operators from $X_1 \times \cdots \times X_m$ to Y^* and the space $\mathcal{L}(Y, X_1, \ldots, X_m; \mathbb{K})$ of all $(m+1)$-linear forms from $Y \times X_1 \times \cdots \times X_m$ to $\mathbb{K}$.

$$\Psi \colon \mathcal{L}(Y, X_1, \ldots, X_m; \mathbb{K}) \longrightarrow \mathcal{L}(X_1, \ldots, X_m; Y^*)$$
$$\Psi(A)(x_1, \ldots, x_m)(y) = A(y, x_1, \ldots, x_m).$$

In view of such an isometric isomorphism, it suffices to prove the following:

If $E^{(1)}$ is an infinite-dimensional Banach space of cotype q and unconditional Schauder basis $\left(z_j^{(1)}\right)_{j=1}^{\infty}$ with constant K_1 and $E^{(2)}, \ldots, E^{(m)}$ are infinite-dimensional Banach spaces with unconditional Schauder basis $\left(z_j^{(2)}\right)_{j=1}^{\infty}, \ldots, \left(z_j^{(m)}\right)_{j=1}^{\infty}$ and constants $K_2, \ldots, K_m$, respectively, then, there exists a positive integer N with the following property:

- For all $n_2, \ldots, n_m \geq N$ and $n_1 = \min\{n_2, \ldots, n_m\}$, there exists an m-linear form $A_{n_1,\ldots,n_m} \colon E_{n_1}^{(1)} \times \cdots \times E_{n_m}^{(m)} \longrightarrow \mathbb{K}$ of the type
$$A_{n_1,\ldots,n_m}(z^{(1)}, \ldots, z^{(m)}) = \sum_{i_1=1}^{n_1} \cdots \sum_{i_m=1}^{n_m} \pm z_{i_1}^{(1)} \cdots z_{i_m}^{(m)},$$
satisfying

$$\|A_{n_1,\ldots,n_m}\| \leq \left(K_1 \cdots K_m C_q(E^{(1)}) + \varepsilon\right) n_1^{\frac{1}{2}-\frac{1}{q}} \max\{n_2, \ldots, n_m\}^{1/2} \prod_{j=2}^{m} n_j^{1/2}. \tag{2.1}$$

We then proceed to prove the existence of a constant $A_{n_1,\ldots,n_m}$ satisfying (2.1). Since $E^{(1)}$ has cotype q, the identity $\mathrm{id} \colon E^{(1)} \longrightarrow E^{(1)}$ is absolutely $(q;1)$-summing with constant $C_q(E^{(1)})$:

$$\left(\sum_{j=1}^{n} |a_j|^q\right)^{1/q} = \left(\sum_{j=1}^{n} \left\| id\left(a_j z_j^{(1)}\right)\right\|^q\right)^{1/q} \leq C_q(E^{(1)}) \sup_{\varepsilon_j = \pm 1} \left\|\sum_{j=1}^{n} \varepsilon_j a_j z_j^{(1)}\right\|.$$

Since K_1 is the constant of the unconditional basis $\left(z_j^{(1)}\right)_{j=1}^{\infty}$, we have

$$\sup_{\varepsilon_j = \pm 1} \left\|\sum_{j=1}^{n} \varepsilon_j a_j z_j^{(1)}\right\| \leq K_1 \left\|\sum_{j=1}^{n} a_j z_j^{(1)}\right\|.$$

Therefore,

$$\left\|\sum_{j=1}^{n} a_j z_j^{(1)}\right\| \leq 1 \Rightarrow \left(\sum_{j=1}^{n} |a_j|^q\right)^{1/q} \leq K_1 C_q(E^{(1)}) \Rightarrow \left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2} \leq K_1 C_q(E^{(1)}) n^{\frac{1}{2}-\frac{1}{q}}.$$

We also know that

$$\left\|\sum_{j=1}^{n} a_j z_j^{(k)}\right\| \leq 1 \Rightarrow |a_j| \leq K_k$$

for all $k = 2, \ldots, m$.

With no loss of generality, we can assume

$$n_2 \leq \cdots \leq n_m.$$

Let us first suppose that for each $k = 2, \ldots, m$, there is a Hadamard matrix $\mathbf{H}_{n_k}$ with order n_k.

For $k = 2, \ldots, m$, let $u_i^{(k)}$, $i = 1, \ldots, n_k$, be the rows of $\mathbf{H}_{n_k}$; hence

$$\left\langle u_i^{(k)}, u_j^{(k)}\right\rangle = n_k \delta_{ij}. \tag{2.2}$$

Let us consider the square matrices of order n_m defined by

$$\left[h_{ij}^{(k)}\right]_{n_m\times n_m} := \begin{bmatrix} \mathbf{H}_{n_k} & \mathbf{0}_{n_k\times(n_m-n_k)} \\ \mathbf{0}_{(n_m-n_k)\times n_k} & \mathbf{0}_{(n_m-n_k)\times(n_m-n_k)} \end{bmatrix}$$

for each $k=2,\ldots,m$. Consider the m-linear form $A\colon E_{n_1}^{(1)}\times\cdots\times E_{n_m}^{(m)}\longrightarrow\mathbb{K}$ defined by

$$A\left(x^{(1)},\ldots,x^{(m)}\right)=\sum_{i_1=1}^{n_1}\cdots\sum_{i_m=1}^{n_m} h_{i_1i_2}^{(2)}\cdots h_{i_{m-1}i_m}^{(m)} x_{i_1}^{(1)}\cdots x_{i_m}^{(m)},$$

where

$$x^{(k)}=\sum_{j=1}^{n_k} x_j^{(k)} z_j^{(k)}.$$

For all $k=1,\ldots,m$, given $x^{(k)}=\sum_{j=1}^{n_k} x_j^{(k)} z_j^{(k)}$ in the unit ball of $E^{(k)}$, let us denote

$$y^{(k)}=\sum_{j=1}^{n_m} y_j^{(k)} z_j^{(k)},$$

with $y_j^{(k)}=0$ for all $j=n_k+1,\ldots,n_m$ and $y_j^{(k)}=x_j^{(k)}$ for all $j=1,\ldots,n_k$. Then, by the Hölder inequality,

$$\begin{aligned}
\left|A\left(x^{(1)},\ldots,x^{(m)}\right)\right| &= \left|\sum_{i_1,\ldots,i_m=1}^{n_m}\left(\prod_{r=2}^{m} h_{i_{r-1}i_r}^{(r)}\right)\left(\prod_{s=1}^{m} y_{i_s}^{(s)}\right)\right| \\
&\le \sum_{i_m=1}^{n_m}\left|\sum_{i_1,\ldots,i_{m-1}=1}^{n_m}\left(\prod_{r=2}^{m} h_{i_{r-1}i_r}^{(r)}\right)\left(\prod_{s=1}^{m-1} y_{i_s}^{(s)}\right)\right|\left|y_{i_m}^{(m)}\right| \\
&\le \left(\sum_{i_m=1}^{n_m}|y_{i_m}^{(m)}|^2\right)^{1/2}\cdot\left(\sum_{i_m=1}^{n_m}\left|\sum_{i_1,\ldots,i_{m-1}=1}^{n_m}\left(\prod_{r=2}^{m} h_{i_{r-1}i_r}^{(r)}\right)\left(\prod_{s=1}^{m-1} y_{i_s}^{(s)}\right)\right|^2\right)^{1/2} \\
&\le K_m n_m^{1/2}\left(\sum_{i_m=1}^{n_m}\sum_{\substack{i_1,\ldots,i_{m-1}=1\\ j_1,\ldots,j_{m-1}=1}}^{n_m}\left(\prod_{r=2}^{m} h_{i_{r-1}i_r}^{(r)} h_{j_{r-1}j_r}^{(r)}\right)\left(\prod_{s=1}^{m-1} y_{i_s}^{(s)}\overline{y_{j_s}^{(s)}}\right)\right)^{1/2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left|A\left(x^{(1)},\ldots,x^{(m)}\right)\right| &\le K_m n_m^{1/2}\left(\sum_{\substack{i_1,\ldots,i_{m-1}=1\\ j_1,\ldots,j_{m-1}=1}}^{n_m}\left(\prod_{r=2}^{m-1} h_{i_{r-1}i_r}^{(r)} h_{j_{r-1}j_r}^{(r)}\right)\left(\prod_{s=1}^{m-1} y_{i_s}^{(s)}\overline{y_{j_s}^{(s)}}\right)\sum_{i_m=1}^{n_m} h_{i_{m-1}i_m}^{(m)} h_{j_{m-1}i_m}^{(m)}\right)^{1/2} \\
&= K_m n_m^{1/2}\left(\sum_{\substack{i_1,\ldots,i_{m-1}=1\\ j_1,\ldots,j_{m-1}=1}}^{n_m}\left(\prod_{r=2}^{m-1} h_{i_{r-1}i_r}^{(r)} h_{j_{r-1}j_r}^{(r)}\right)\left(\prod_{s=1}^{m-1} y_{i_s}^{(s)}\overline{y_{j_s}^{(s)}}\right)\left\langle u_{i_{m-1}}^{(m)},u_{j_{m-1}}^{(m)}\right\rangle\right)^{1/2}
\end{aligned}$$

and, by (2.2), we have

$$\left|A\left(x^{(1)},\ldots,x^{(m)}\right)\right| \le K_m n_m^{1/2}\left(\sum_{i_{m-1}=1}^{n_m}\sum_{\substack{i_1,\ldots,i_{m-2}=1\\ j_1,\ldots,j_{m-2}=1}}^{n_m}\left(\prod_{r=2}^{m-1} h_{i_{r-1}i_r}^{(r)} h_{j_{r-1}j_r}^{(r)}\right)\left(\prod_{s=1}^{m-2} y_{i_s}^{(s)}\overline{y_{j_s}^{(s)}}\right)\left|y_{i_{m-1}}^{(m-1)}\right|^2 n_m\right)^{1/2}$$

$$\le K_{m-1}K_m n_m \left(\sum_{\substack{i_1,\ldots,i_{m-2}=1 \\ j_1,\ldots,j_{m-2}=1}}^{n_m} \left(\prod_{r=2}^{m-2} h^{(r)}_{i_{r-1}i_r} h^{(r)}_{j_{r-1}j_r} \right) \left(\prod_{s=1}^{m-2} y^{(s)}_{i_s} \overline{y^{(s)}_{j_s}} \right) \sum_{i_{m-1}=1}^{n_m} h^{(m-1)}_{i_{m-2}i_{m-1}} h^{(m-1)}_{j_{m-2}i_{m-1}} \right)^{1/2}$$

$$= K_{m-1}K_m n_m \left(\sum_{\substack{i_1,\ldots,i_{m-2}=1 \\ j_1,\ldots,j_{m-2}=1}}^{n_m} \left(\prod_{r=2}^{m-2} h^{(r)}_{i_{r-1}i_r} h^{(r)}_{j_{r-1}j_r} \right) \left(\prod_{s=1}^{m-2} y^{(s)}_{i_s} \overline{y^{(s)}_{j_s}} \right) \left\langle u^{(m-1)}_{i_{m-2}}, u^{(m-1)}_{j_{m-2}} \right\rangle \right)^{1/2} .$$

Since

$$\left(\sum_{\substack{i_1,\ldots,i_{m-2}=1 \\ j_1,\ldots,j_{m-2}=1}}^{n_m} \left(\prod_{r=2}^{m-2} h^{(r)}_{i_{r-1}i_r} h^{(r)}_{j_{r-1}j_r} \right) \left(\prod_{s=1}^{m-2} y^{(s)}_{i_s} \overline{y^{(s)}_{j_s}} \right) \left\langle u^{(m-1)}_{i_{m-2}}, u^{(m-1)}_{j_{m-2}} \right\rangle \right)^{1/2}$$

$$= n_{m-1}^{1/2} \left(\sum_{i_{m-2}=1}^{n_m} \sum_{\substack{i_1,\ldots,i_{m-3}=1 \\ j_1,\ldots,j_{m-3}=1}}^{n_m} \left(\prod_{r=2}^{m-2} h^{(r)}_{i_{r-1}i_r} h^{(r)}_{j_{r-1}j_r} \right) \left(\prod_{s=1}^{m-2} y^{(s)}_{i_s} \overline{y^{(s)}_{j_s}} \right) \right)^{1/2}$$

$$= n_{m-1}^{1/2} \left(\sum_{i_{m-2}=1}^{n_m} |y^{(m-2)}_{i_{m-2}}|^2 \sum_{\substack{i_1,\ldots,i_{m-3}=1 \\ j_1,\ldots,j_{m-3}=1}}^{n_m} \left(\prod_{r=2}^{m-2} h^{(r)}_{i_{r-1}i_r} h^{(r)}_{j_{r-1}j_r} \right) \left(\prod_{s=1}^{m-3} y^{(s)}_{i_s} \overline{y^{(s)}_{j_s}} \right) \right)^{1/2} ,$$

we have

$$\left| A\left(x^{(1)}, \ldots, x^{(m)} \right) \right| \le \left(\prod_{j=m-2}^{m} K_j \right) n_m n_{m-1}^{1/2} \left(\sum_{i_{m-2}=1}^{n_m} \sum_{\substack{i_1,\ldots,i_{m-3}=1 \\ j_1,\ldots,j_{m-3}=1}}^{n_m} \left(\prod_{r=2}^{m-2} h^{(r)}_{i_{r-1}i_r} h^{(r)}_{j_{r-1}j_r} \right) \left(\prod_{s=1}^{m-3} y^{(s)}_{i_s} \overline{y^{(s)}_{j_s}} \right) \right)^{1/2}$$

and, repeating this procedure, we finally obtain

$$\begin{aligned} \left| A\left(x^{(1)}, \ldots, x^{(m)} \right) \right| &\le \left(\prod_{j=2}^{m} K_j \right) n_2^{1/2} \cdots n_{m-1}^{1/2} n_m \left(\sum_{i_1=1}^{n_m} \left| y^{(1)}_{i_1} \right|^2 \right)^{1/2} \\ &\le \left(\prod_{j=1}^{m} K_j \right) C_q(E^{(1)}) n_1^{\frac{1}{2}-\frac{1}{q}} n_m^{1/2} \prod_{j=2}^{m} n_j^{1/2}. \end{aligned}$$

Let us deal with the general case.

Recall that a set of positive integers A is said to be asymptotically dense in $\mathbb{N}$ if for all $\varepsilon > 0$, there exists a positive integer n_ε such that for all $m \ge n_\varepsilon$ there is $n \in A$ satisfying

$$m \le n \le m\left(1+\varepsilon\right).$$

The next result is folklore, but we present a proof for the sake of completeness.

Lemma 2.2. *The set of orders of Hadamard matrices is asymptotically dense in* $\mathbb{N}$.

Proof. Since for all i, j there are Hadamard matrices of order $4^i 12^j$, it suffices to show that $A := \left\{4^i 12^j : i, j \in \{0, 1, 2, 3, \ldots\}\right\}$ is asymptotically dense in $\mathbb{N}$. An immediate consequence of the classical Dirichlet's approximation theorem on Diophantine approximation says that, fixed an irrational number $\alpha > 0$ and given an arbitrary $\varepsilon > 0$, there exists a positive integer $n_\varepsilon \in \mathbb{N}$ such that, for all $x \geq n_\varepsilon$, we can find $a, b \in \{0, 1, 2, 3, \ldots\}$ satisfying

$$0 < a + b\alpha - x < \varepsilon.$$

It is easy to check that $\log_4 3$ is irrational, for all k. So, geting $\alpha = 1 + \log_4 3$, given $0 < \varepsilon < 1$, there is a sufficiently large positive integer n_ε such that, if $N_\varepsilon \geq 4^{n_\varepsilon}$, then, whenever $m \geq N_\varepsilon$, there are $i, j \in \{0, 1, 2, 3, \ldots\}$ such that

$$0 \leq i + j\left(1 + \log_4 3\right) - \log_4 m \leq \log_4\left(1 + \varepsilon\right)$$

or, equivalently,

$$\log_4 m \leq i + j\left(1 + \log_4 3\right) \leq \log_4\left(\left(1 + \varepsilon\right) m\right).$$

Since

$$i + j\left(1 + \log_4 3\right) = \log_4\left(4\right)^i \left(12\right)^j,$$

there are $i, j \in \{0, 1, 2, 3, \ldots\}$ such that

$$\log_4 m \leq \log_4\left(4\right)^i \left(12\right)^j \leq \log_4\left(\left(1 + \varepsilon\right) m\right),$$

whenever $m \geq N_\varepsilon$, which yields the thesis of the Lemma. $\square$

By the previous lemma, for each

$$\delta = \left(1 + \varepsilon\left(\left(\prod_{j=1}^{m} K_j\right) C_q(E^{(1)})\right)^{-1}\right)^{\frac{1}{\frac{m+1}{2} - \frac{1}{q}}} - 1 > 0,$$

there is a positive integer N such that, for each $k = 1, \ldots, m$, whenever $n_k > N$ is an integer, there exists a Hadamard matrix of order t_k satisfying

$$n_k \leq t_k < \left(1 + \delta\right) n_k.$$

Notice that, without loss of generality, we can assume $t_1 \leq \cdots \leq t_m$. Thus, there is an m-linear form $A_0 \colon E^{(t_1)} \times \cdots \times E^{(t_m)} \longrightarrow \mathbb{K}$ with coefficients ± 1 such that

$$\|A_0\| \leq \left(\prod_{j=1}^{m} K_j\right) C_q(E^{(1)}) t_1^{\frac{1}{2} - \frac{1}{q}} t_m^{1/2} \prod_{k=2}^{m} t_k^{1/2}.$$

If $n_k = t_k$ for each k, it is sufficient to make $A_{n_1, \ldots, n_m} = A_0$. Otherwise, let us consider the m-linear form

$$A_{n_1, \ldots, n_m} \colon E_{n_1}^{(1)} \times \cdots \times E_{n_m}^{(m)} \longrightarrow \mathbb{K}$$

defined by

$$A_{n_1, \ldots, n_m}\left(\sum_{j=1}^{n_1} a_j^{(1)} z_j^{(1)}, \ldots, \sum_{j=1}^{n_m} a_j^{(m)} z_j^{(m)}\right) = A_0\left(\sum_{j=1}^{t_1} a_j^{(1)} z_j^{(1)}, \ldots, \sum_{j=1}^{t_m} a_j^{(m)} z_j^{(m)}\right),$$

where $a_j^{(k)} = 0$ for all $k = n_k + 1, \ldots, t_k$. Then, given

$$\sum_{j=1}^{n_k} a_j^{(k)} z_j^{(k)} \in B_{E_{n_k}^{(k)}}, k = 1, \ldots, m,$$

we have

$$\left|A_{n_1, \ldots, n_m}\left(\sum_{j=1}^{n_1} a_j^{(1)} z_j^{(1)}, \ldots, \sum_{j=1}^{n_m} a_j^{(m)} z_j^{(m)}\right)\right| \leq \|A_0\| \leq \left(\prod_{j=1}^{m} K_j\right) C_q(E^{(1)}) t_1^{\frac{1}{2} - \frac{1}{q}} t_m^{1/2} \prod_{j=2}^{m} t_j^{1/2}$$

$$\le \left(\prod_{j=1}^{m} K_j\right) C_q(E^{(1)})\,(1+\delta)^{\frac{1}{2}-\frac{1}{q}}\, n_1^{\frac{1}{2}-\frac{1}{q}}\,(1+\delta)^{1/2}\, n_m^{1/2} \prod_{j=2}^{m} (1+\delta)^{1/2}\, n_j^{1/2}$$

$$\le \left(\prod_{j=1}^{m} K_j\right) C_q(E^{(1)})\,(1+\delta)^{\frac{m+1}{2}-\frac{1}{q}}\, n_1^{\frac{1}{2}-\frac{1}{q}} n_m^{1/2} \prod_{j=2}^{m} n_j^{1/2}$$

$$= \left(\left(\prod_{j=1}^{m} K_j\right) C_q(E^{(1)}) + \varepsilon\right) n_1^{\frac{1}{2}-\frac{1}{q}} n_m^{1/2} \prod_{j=2}^{m} n_j^{1/2}$$

and this completes the proof. □

3. Proof of Theorem 1.3

As in the proof of the previous result, for $k = 1, 2$, we have

$$\left\|\sum_{j=1}^{n} a_j z_j^{(k)}\right\| \le 1 \Rightarrow \left(\sum_{j=1}^{n} |a_j|^{q_k}\right)^{1/q_k} \le K_k C_{q_k}(E^{(k)}) \Rightarrow \left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2} \le K_k C_{q_k}(E^{(k)}) n^{\frac{1}{2}-\frac{1}{q_k}}.$$

Let $[h_{ij}]_{n\times n}$ be a Hadamard matrix of order n. It is easy to see that the bilinear form $A_0\colon E_n^{(1)} \times E_n^{(2)} \longrightarrow \mathbb{K}$ given by

$$A_0(x^{(1)}, x^{(2)}) = \sum_{i=1}^{n}\sum_{j=1}^{n} h_{ij} a_i^{(1)} a_j^{(2)}, \tag{3.1}$$

where

$$x^{(k)} = \sum_{j=1}^{n} a_j^{(k)} z_j^{(k)}.$$

has norm

$$\|A_0\| \le K_1 K_2 C_{q_1}(E^{(1)}) C_{q_2}(E^{(2)}) n^{\frac{3}{2}-\frac{1}{q_1}-\frac{1}{q_2}}.$$

In fact, if $x^{(k)} = \sum_{j=1}^{n} a_j^{(k)} z_j^{(k)} \in B_{E_{n_k}^{(k)}}$, $k = 1, 2$, from the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\left|A_0\left(x^{(1)}, x^{(2)}\right)\right| &\le \sum_{j=1}^{n}\left|\sum_{i=1}^{n} h_{ij} a_i^{(1)}\right| \left|a_j^{(2)}\right| \\
&\le \left(\sum_{j=1}^{n}\left|a_j^{(2)}\right|^2\right)^{1/2} \cdot \left(\sum_{j=1}^{n}\left|\sum_{i=1}^{n} h_{ij} a_i^{(1)}\right|^2\right)^{1/2} \\
&\le K_2 C_{q_2}(E^{(2)}) n^{\frac{1}{2}-\frac{1}{q_2}} \left(\sum_{j=1}^{n}\sum_{i,k=1}^{n} h_{ij} h_{kj} a_i^{(1)} \overline{a_k^{(1)}}\right)^{1/2} \\
&= K_2 C_{q_2}(E^{(2)}) n^{\frac{1}{2}-\frac{1}{q_2}} \left(\sum_{i,k=1}^{n} a_i^{(1)} \overline{a_k^{(1)}} n \delta_{ik}\right)^{1/2} \\
&\le K_1 K_2 C_{q_1}(E^{(1)}) C_{q_2}(E^{(2)}) n^{\frac{3}{2}-\frac{1}{q_1}-\frac{1}{q_2}}.
\end{aligned}$$

We shall show that, for other values of n, we have the same inequality, with the addition of the "asymptotic factor" $(1+\varepsilon)$.

Given $\delta > 0$, there is a positive integer N such that, whenever $n > N$, there exists a Hadamard order t that complies

$$n \le t \le n\,(1+\delta)\,. \tag{3.2}$$

Let

$$\delta = \left(1 + \varepsilon (K_1 K_2 C_{q_1}(E^{(1)}) C_{q_2}(E^{(2)})^{-1}\right)^{\frac{1}{\frac{3}{2} - \frac{1}{p_1} - \frac{1}{p_2}}} - 1 > 0.$$

Now, let us consider some Hadamard matrix $[h_{ij}]_{t\times t}$ of order t. Let $A_0 \colon E_t^{(1)} \times E_t^{(2)} \longrightarrow \mathbb{K}$ be as in (3.1). Hence,

$$\|A_0\| \leq K_1 K_2 C_{q_1}(E^{(1)}) C_{q_2}(E^{(2)}) t^{\frac{3}{2} - \frac{1}{p_1} - \frac{1}{p_2}}.$$

If $n = t$, we make $A = A_0$. If $n < t$, we define

$$A \colon E_n^{(1)} \times E_n^{(2)} \longrightarrow \mathbb{K},$$

taking

$$A\left(\sum_{j=1}^{n} a_j^{(1)} z_j^{(1)}, \sum_{j=1}^{n} a_j^{(2)} z_j^{(2)}\right) = A_0\left(\sum_{j=1}^{t} a_j^{(1)} z_j^{(1)}, \sum_{j=1}^{t} a_j^{(2)} z_j^{(2)}\right),$$

where $a_j^{(k)} = 0$ for all $k = n+1, \ldots, t$. Therefore, given $x^{(k)} = \sum_{j=1}^{n} a_j^{(1)} z_j^{(1)} \in B_{E_n^{(k)}}$, by (3.2), we have

$$\begin{aligned}
\left|A\left(x^{(1)}, x^{(2)}\right)\right| &\leq \|A_0\| \\
&\leq K_1 K_2 C_{q_1}(E^{(1)}) C_{q_2}(E^{(2)}) t^{\frac{3}{2} - \frac{1}{p_1} - \frac{1}{p_2}} \\
&\leq K_1 K_2 C_{q_1}(E^{(1)}) C_{q_2}(E^{(2)}) (1+\delta)^{\frac{3}{2} - \frac{1}{p_1} - \frac{1}{p_2}} n^{\frac{3}{2} - \frac{1}{p_1} - \frac{1}{p_2}} \\
&= K_1 K_2 C_{q_1}(E^{(1)}) C_{q_2}(E^{(2)}) (1+\varepsilon) n^{\frac{3}{2} - \frac{1}{p_1} - \frac{1}{p_2}},
\end{aligned}$$

as desired. The proof of Theorem 1.3 is complete. ☐

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