THE KHINCHIN INEQUALITY FOR MULTIPLE SUMS REVISITED

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Abstract. We present a self-contained proof of the Khinchin inequality for multiple sums, which avoids advanced results from Probability theory. Not only our new proof is more accessible, but also it sheds lights on some properties of the inequality which may yield further generalizations.

1. Introduction

Khinchin's inequality is an important analytical tool with a charming probabilistic flavor. Established in the 1920's, [12], Khinchin's inequality was originally motivated by the investigation of the rate of convergence in E. Borel's strong law of large numbers.

Through the years, Khinchin's inequality has found a plethora of striking applications in a number of fields including: harmonic and functional analysis, viz. [1, 2, 3], partial differential equations, viz. [6, 18, 19], stochastic processes, viz. [7, 14], computer sciences, viz. [9, 8], and number theory, viz. [4, 16], to cite a few. It has several extensions and generalizations and we shall focus on its statement for multiple sums, referred as the multiple Khinchin inequality, which reads as follows:

Theorem 1.1 (Multiple Khinchin inequality)**.** *Let $(a_{i_1\ldots i_m})_{i_1,\ldots,i_m=1}^{n}$ be real or complex scalars and $0<p<\infty$. Then there are constants A_p and B_p such that*

$$\begin{aligned} A_p^m\left(\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n}\left|a_{i_1\ldots i_m}\right|^2\right)^{1/2} &\leq \left\|\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n} r_{i_1}\left(t_1\right)\cdots r_{i_m}\left(t_m\right)a_{i_1\ldots i_m}\right\|_{L_p([0,1]^m)} \\ &\leq B_p^m\left(\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n}\left|a_{i_1\ldots i_m}\right|^2\right)^{1/2}, \end{aligned}$$

where $r_n(t)$ denotes the Rademacher functions, $r_n(t)=\operatorname{sgn}\left(\sin 2^{n+1}\pi t\right)$.

The probabilistic flavor of Khinchin's inequality is certainly appealing, but also intricate. Heuristically, the inequality informs about the behavior of a sort "random walks", as follows. Suppose we are given n real (or complex) numbers $a_1,\ldots,a_n$ and a fair coin. We shall produce a new sequence of numbers, α_j, through the following rules. Flip a coin. If it comes up heads, set $\alpha_1=a_1$; if it comes up tails, set $\alpha_1=-a_1$. Repeating the process, after having flipped the coin k times, set

$$\alpha_{k+1}=\alpha_k+a_{k+1},$$

if the $(k+1)$ flip comes up heads; otherwise set

$$\alpha_{k+1}=\alpha_k-a_{k+1}.$$

The question one is interested in is how does the absolute value of the generated sequence α_j grows. That is, after n steps, what should be the expected value of $|\alpha_n|$?

Khinchin's inequality shows, in a very precise sense, that this quantity is close to the ℓ_2-norm of the original sequence, (a_n). It is a remarkable result.

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To better appreciate this interpretation, one should look at the structure of the Rademacher functions, $r_n\colon [0,1] \to [0,1]$, defined analytically as

$$r_n(t) = \operatorname{sgn}\left(\sin 2^{n+1}\pi t\right),$$

for all positive integers n. They are in the core of Khinchin's inequality and our first main goal is to discuss some of their key properties, which are sometimes not formally proven in the literature. This will allow us to establish a self-contained proof of Khinchin's inequality, in its multiple (extended) version; see for instance [15] and the references therein.

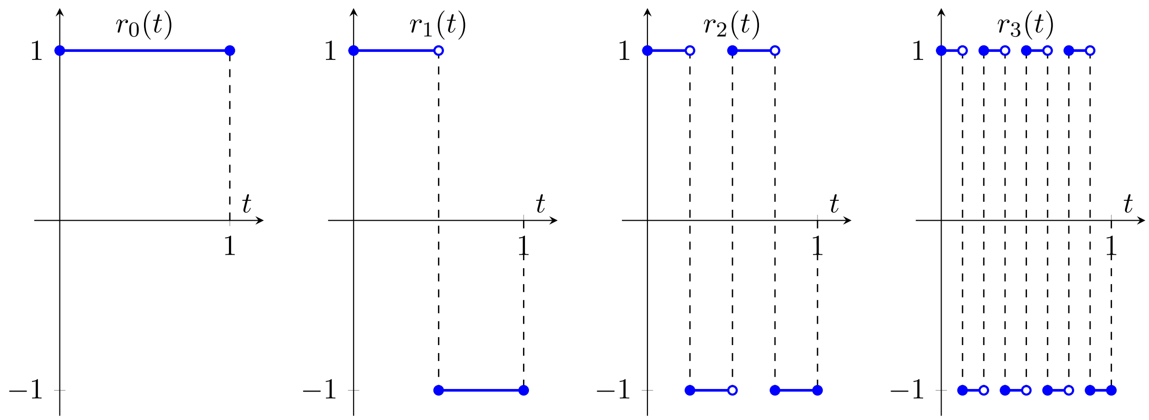

FIGURE 1. Rademacher functions.

The (natural) question regarding optimal constants for Khinchin's inequality, $p > 2$, has been tackled by Szarek in [17] and Haagerup in [11]. The problem boils down to solving the optimization problems

$$\min \int_0^1 \left| \sum_{k=1}^{n} a_k r_k\,(t) \right|^p dt$$

and

$$\max \int_0^1 \left| \sum_{k=1}^{n} a_k r_k\,(t) \right|^p dt$$

where the minimum and the maximum are taken over the unit sphere of ℓ_2, i.e. the sequences satisfying $\sum\limits_{k=1}^{n} |a_k|^2 = 1$. Through a delicate analysis, Haagerup ultimately managed to prove that:

$$A_p = \begin{cases} 2^{\frac{p-2}{2}}, & \text{if } 0 < p \le p_0, \\ 2^{\frac{p-2}{2}} \dfrac{\Gamma\left(\dfrac{p+1}{2}\right)}{\Gamma\left(\dfrac{3}{2}\right)}, & \text{if } p_0 \le p \le 2, \end{cases}$$

where $p_0 \in (1,2)$ is determined by

$$\Gamma\left(\frac{p_0+1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

The optimal constants of the multiple Khinchin inequality are A_p^m and B_p^m (see [13]).

Our proof of Khinchin's inequality relies on the multi-orthogonality of the Rademacher functions; a result that is well understood by the experts, but its proof is often omitted in the literature, see for instance [5, page 10]. We offer an easy proof of this property, which is interesting by its own. This is the content of Section 2. In Section 3 we prove the multiple version of Khinchin's inequality by means of an elementary tools. Not only our new proof is more accessible to (say) graduate students, but also it sheds lights on properties of the inequality which might yield further generalizations.

2. Multi-orthogonality of the Rademacher functions

We begin by proving a folkloric property of the Rademacher functions which is usually not proven in the literature (see [5, page 10]).

Lemma 2.1. *Let $(n_j)_{j=1}^{\infty}$ be an increasing sequence of natural numbers. If $0 < n_1 < n_2 < \cdots < n_k$ and $k, p_1, p_2, \ldots, p_k$ are positive integers, then*

$$\int_0^1 r_{n_1}^{p_1}(t) \cdot r_{n_2}^{p_2}(t) \cdot \cdots \cdot r_{n_k}^{p_k}(t)\, dt = \begin{cases} 1, & \text{if each } p_j \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let

$$g(t) = \prod_{i=1}^{k} r_{n_i}^{p_i}(t) = r_{n_\ell}^{p_\ell}(t)\, f(t),$$

where ℓ is the largest integer in the interval $[1,k]$ where p_ℓ is odd, and where

$$f(t) = \prod_{\substack{1 \le i \le k \\ i \ne \ell}} r_{n_i}^{p_i}(t) = \prod_{1 \le i \le \ell - 1} r_{n_i}^{p_i}(t).$$

We note that $f(t)$ is constant along intervals of the form

$$I_j := \left[\frac{j-1}{2^{n_\ell - 1}}, \frac{j}{2^{n_\ell - 1}}\right];$$

yet,

$$\int_{I_j} r_{n_\ell}^{p_\ell}(t)\, dt = 0;$$

thus, $\int_{I_j} g(t)\, dt = 0$, and so $\int_0^1 g(t)\, dt = 0$. □

3. The proof of Khinchin's inequality for multiple sums

The classical proof of Khinchin's inequality relies on the fact that the Rademacher functions are independent random variables, and thus

$$\int_0^1 \prod_{n=1}^{m} \exp\left(a_n r_n(t)\right) dt = \prod_{n=1}^{m} \int_0^1 \exp\left(a_n r_n(t)\right) dt, \tag{3.1}$$

for any real scalars $a_1, \ldots, a_m$.

Our first goal is to show (3.1) by means of self-contained, elementary arguments, and thus avoiding the usage of probabilistic tools. This is the contents of the upcoming Lemma 3.1.

We start with few elementary considerations. Let $(a_n)_{n=1}^{\infty} \in \ell_2$ be an arbitrary sequence. Define $S_0 = 0$ and $S_n = \sum_{k=1}^{n} a_k r_k(t)$ for $n \ge 1$. We readily obtain

$$\|S_m - S_n\|_{L_2[0,1]}^2 = \int_0^1 \left(\sum_{i=n+1}^{m} a_i r_i(t)\right)\left(\sum_{j=n+1}^{m} \overline{a_j} r_j(t)\right) dt = \sum_{k=n+1}^{m} |a_k|^2. \tag{3.2}$$

In particular, $(S_n)_{n=1}^{\infty}$ is a Cauchy sequence and thus it converges strongly in $L_2[0,1]$. Setting $n=0$ and letting $m\to\infty$ in (3.2) yields

$$\int_0^1 \left|\sum_{n=1}^{\infty} a_n r_n(t)\right|^2 dt = \sum_{n=1}^{\infty} |a_n|^2 .$$

Next, for all positive integers m and each $j=1,\dots,2^m$, set

$$c_j^{(m)} := \sum_{n=1}^{m} a_n r_n\left(t_j^{(m)}\right),$$

where

$$\frac{j-1}{2^m} < t_j^{(m)} < \frac{j}{2^m}.$$

Note that the definition of $c_j^{(m)}$ does not depend on the choice of $t_j^{(m)}$. We claim that

$$c_k^{(m+1)} = \begin{cases} c_j^{(m)} + a_{m+1}, \text{ if } k = 2j-1, \\ c_j^{(m)} - a_{m+1}, \text{ if } k = 2j. \end{cases}$$

In fact, if $k=2j$, there holds

$$c_k^{(m+1)} = a_{m+1} r_{m+1}\left(t_k^{(m+1)}\right) + \sum_{n=1}^{m} a_n r_n\left(t_k^{(m+1)}\right)$$

and, in this case,

$$\frac{j-1}{2^m} < \frac{k-1}{2^{m+1}} < t_k^{(m+1)} < \frac{k}{2^{m+1}} = \frac{j}{2^m}.$$

Hence

$$r_n\left(t_k^{(m+1)}\right) = r_n\left(t_j^{(m)}\right),$$

for all $n=1,\dots,m$ and, since $k\geq 2$ is even, we also have

$$r_{m+1}\left(t_k^{(m+1)}\right) = -1.$$

Therefore,

$$c_k^{(m+1)} = -a_{m+1} + \sum_{n=1}^{m} a_n r_n\left(t_j^{(m)}\right) = c_j^{(m)} - a_{m+1}.$$

The case $k=2j-1$ is analogous.

Lemma 3.1. *For all $a_1,\dots,a_m\in\mathbb{R}$, we have*

$$\int_0^1 \prod_{n=1}^{m} \exp\left(a_n r_n(t)\right) dt = \prod_{n=1}^{m} \int_0^1 \exp\left(a_n r_n(t)\right) dt.$$

Proof. Notice that

$$\prod_{n=1}^{m} \exp\left(a_n r_n(t)\right) = \prod_{n=1}^{m}\sum_{i=0}^{\infty} \frac{\left(a_n r_n(t)\right)^i}{i!} = \sum_{i_1,\dots,i_m\geq 0} \frac{a_1^{i_1}\cdots a_m^{i_m} r_1^{i_1}(t)\cdots r_m^{i_m}(t)}{i_1!\cdots i_m!}. \tag{3.3}$$

Integrating both sides of (3.3) from 0 to 1, and using Lemma 2.1, we deduce

$$\begin{aligned}\int_0^1 \prod_{n=1}^m \exp\left(a_n r_n(t)\right) dt &= \sum_{i_1,\dots,i_m\geq 0} \frac{a_1^{i_1}\cdots a_m^{i_m}}{i_1!\cdots i_m!}\int_0^1 r_1^{i_1}(t)\cdots r_m^{i_m}(t)\, dt \\ &= \sum_{i_1,\dots,i_m\geq 0} \frac{a_1^{2i_1}\cdots a_m^{2i_m}}{2i_1!\cdots 2i_m!} \\ &= \prod_{n=1}^m \cosh(a_n) \\ &= \prod_{n=1}^m \int_0^1 \exp\left(a_n r_n(t)\right) dt\end{aligned}$$

and the proof is done. □

3.1. **The first step of the proof.** The proof of Theorem 1.1 is done by induction on m. The first step of the proof (the case $m=1$) is the classical Khinchin inequality; its proof is standard and can be found in many books.

- First case: $2 \leq p < \infty$ and real scalars.
  Let $a_1, \dots, a_m$ be non-simultaneously null real numbers. Define

$$f(t) = \sum_{n=1}^m a_n r_n(t),$$

and set

$$g(t) = \frac{f(t)}{\|f\|_{L_2[0,1]}} = \sum_{n=1}^m b_n r_n(t),$$

where $b_n = a_n / \|f\|_{L_2[0,1]}$. Clearly

$$\sum_{n=1}^m b_n^2 = \|g\|_{L_2[0,1]}^2 = 1. \tag{3.4}$$

From Lemma 3.1, we have

$$\int_0^1 \exp(g(t))\, dt = \int_0^1 \prod_{n=1}^m \exp(b_n r_n(t))\, dt = \prod_{n=1}^m \int_0^1 \exp(b_n r_n(t))\, dt.$$

Hence, since each $r_n(t)$ equals 1 and -1 in sets of measure $1/2$, we have

$$\int_0^1 \exp(g(t))\, dt = \prod_{n=1}^m \frac{1}{2}\left(\exp(b_n) + \exp(-b_n)\right) = \prod_{n=1}^m \cosh(b_n). \tag{3.5}$$

Using the expansions of $\cosh(x)$ and $\exp\left(x^2/2\right)$ in power series, we reach

$$\cosh(x) = \sum_{n=0}^\infty \frac{x^{2n}}{(2n)!} \leq \sum_{n=0}^\infty \frac{x^{2n}}{n!2^n} = \exp\left(\frac{x^2}{2}\right). \tag{3.6}$$

By (3.4), (3.5), and (3.6), we have

$$\int_0^1 \exp(g(t))\, dt \leq \prod_{n=1}^m \exp\left(\frac{b_n^2}{2}\right) = \exp\left(\frac{1}{2}\sum_{n=1}^m b_n^2\right) = e^{1/2}. \tag{3.7}$$

Similarly, we conclude that

$$\int_0^1 \exp(-g(t))\, dt \leq e^{1/2}. \tag{3.8}$$

If k is a positive integer, there holds

$$|y|^k \leq k! \left(1 + \frac{|y|^k}{k!}\right) \leq k! e^{|y|}, \tag{3.9}$$

for all $y \in \mathbb{R}$. Thus, by (3.7), (3.8) and (3.9), we can estimate

$$\int_0^1 |g(t)|^k \, dt \leq k! \int_0^1 \exp(|g(t)|) \, dt \leq k! \int_0^1 (\exp(g(t)) + \exp(-g(t))) \, dt \leq 2k! e^{1/2}.$$

Since we are considering the case $2 \leq p < \infty$, by the monotonicity of the L_p norms, we can further estimate

$$\|g\|_{L_p[0,1]} \leq \|g\|_{L_{\lceil p \rceil}[0,1]} = \left(\int_0^1 |g(t)|^{\lceil p \rceil} \, dt\right)^{1/\lceil p \rceil} \leq \left(2 \lceil p \rceil! e^{1/2}\right)^{1/\lceil p \rceil}, \tag{3.10}$$

where $\lceil p \rceil$ is the smallest integer bigger than or equal to p. Recall that $a_n = b_n \|f\|_{L_2[0,1]}$, so from (3.10) we have

$$\begin{aligned} \|f\|_{L_2[0,1]} \leq \|f\|_{L_p[0,1]} &= \left\| \sum_{n=1}^m b_n \|f\|_{L_2[0,1]} r_n \right\|_{L_p[0,1]} \\ &= \|f\|_{L_2[0,1]} \|g\|_{L_p[0,1]} \leq \left(2 \lceil p \rceil! e^{1/2}\right)^{1/\lceil p \rceil} \|f\|_{L_2[0,1]}, \end{aligned}$$

i.e.,

$$\left(\sum_{n=1}^m |a_n|^2\right)^{1/2} \leq \left(\int_0^1 \left|\sum_{n=1}^m a_n r_n(t)\right|^p dt\right)^{1/p} \leq \left(2 \lceil p \rceil! e^{1/2}\right)^{1/\lceil p \rceil} \left(\sum_{n=1}^m |a_n|^2\right)^{1/2}. \tag{3.11}$$

Now, let $(a_n)_{n=1}^\infty \in \ell_2$ be non-null sequence. By (3.11), taking $S_n = \sum_{k=1}^n a_k r_k$, we conclude $(S_n)_{n=1}^\infty$ is a convergent sequence. Letting $m \to \infty$ in (3.11) yields

$$\left(\sum_{n=1}^\infty |a_n|^2\right)^{1/2} \leq \left(\int_0^1 \left|\sum_{n=1}^\infty a_n r_n(t)\right|^p dt\right)^{1/p} \leq \left(2 \lceil p \rceil! e^{1/2}\right)^{1/\lceil p \rceil} \left(\sum_{n=1}^\infty |a_n|^2\right)^{1/2}.$$

• Second case: $2 \leq p < \infty$ and complex scalars. It suffices to consider $(a_n)_{n=1}^\infty = (b_n)_{n=1}^\infty + i(c_n)_{n=1}^\infty \in \ell_2$, with $(b_n)_{n=1}^\infty$ and $(c_n)_{n=1}^\infty$ sequences of real scalars and use the previous case.

• Third case: $0 < p < 2$. Let $f = \sum_{n=1}^m a_n r_n$ and define

$$\theta := \left(2 - \frac{p}{2}\right)^{-1},$$

so that

$$p\theta + 4(1 - \theta) = 2. \tag{3.12}$$

Also, as $0<p<2$, we have $1<1/\theta<2$. By the Hölder inequality, we estimate

$$\begin{aligned}\int_0^1 |f(t)|^2\,dt &= \int_0^1 |f(t)|^{p\theta}\,|f(t)|^{4(1-\theta)}\,dt\\ &\le \left(\int_0^1 \left[|f(t)|^{p\theta}\right]^{1/\theta} dt\right)^{\theta}\left(\int_0^1 \left[|f(t)|^{4(1-\theta)}\right]^{1/(1-\theta)} dt\right)^{1-\theta}\\ &= \left(\int_0^1 |f(t)|^{p}\,dt\right)^{\theta}\left(\int_0^1 |f(t)|^{4}\,dt\right)^{1-\theta}\\ &= \left[\left(\int_0^1 |f(t)|^{p}\,dt\right)^{1/p}\right]^{p\theta}\left[\left(\int_0^1 |f(t)|^{4}\,dt\right)^{1/4}\right]^{4(1-\theta)}.\end{aligned}$$

Applying thecase $p=4$ (already established), we have $\|f\|_{L_4[0,1]} \le B_4\,\|f\|_{L_2[0,1]}$ and

$$\|f\|_{L_2[0,1]}^2 \le \|f\|_{L_p[0,1]}^{p\theta}\left(B_4\,\|f\|_{L_2[0,1]}\right)^{4(1-\theta)}.$$

Hence,

$$B_4^{-4(1-\theta)}\,\|f\|_{L_2[0,1]}^{2-4(1-\theta)} \le \|f\|_{L_p[0,1]}^{p\theta}$$

and, in view of (3.12), we obtain

$$B_4^{4(\theta-1)/p\theta}\,\|f\|_{L_2[0,1]} \le \|f\|_{L_p[0,1]}.$$

Since

$$\frac{4(\theta-1)}{p\theta} = 2-\frac{4}{p},$$

we have

$$B_4^{2-\frac{4}{p}}\,\|f\|_{L_2[0,1]} \le \|f\|_{L_p[0,1]}.$$

Applying the monotonicity of the L_p norms once again, we get

$$B_4^{2-\frac{4}{p}}\left(\sum_{n=1}^m |a_n|^2\right)^{1/2} \le \left(\int_0^1\left|\sum_{n=1}^m a_n r_n(t)\right|^p dt\right)^{1/p} \le \left(\sum_{n=1}^m |a_n|^2\right)^{1/2}.$$

3.2. **The induction step.** By Lemma 2.1 we have

$$\left(\sum_{i_1=1}^n \cdots \sum_{i_m=1}^n |a_{i_1\ldots i_m}|^2\right)^{1/2} = \left\|\sum_{i_1=1}^n \cdots \sum_{i_m=1}^n r_{i_1}(t_1)\cdots r_{i_m}(t_m)\,a_{i_1\ldots i_m}\right\|_{L_2([0,1]^m)}. \tag{3.13}$$

Note that

$$\begin{aligned}&\int_{[0,1]^m}\left|\sum_{i_1=1}^n \cdots \sum_{i_m=1}^n r_{i_1}(t_1)\cdots r_{i_m}(t_m)\,a_{i_1\ldots i_m}\right|^p dt_1\ldots dt_m\\ &= \frac{1}{2^{nm}}\sum_{k_1=1}^{2^n}\cdots\sum_{k_m=1}^{2^n}\left|\sum_{i_1=1}^n \cdots \sum_{i_m=1}^n r_{i_1}\left(t_1^{(k_1)}\right)\cdots r_{i_m}\left(t_m^{(k_m)}\right) a_{i_1\ldots i_m}\right|^p,\end{aligned} \tag{3.14}$$

for all

$$\frac{k_s-1}{2^n} \le t_s^{(k_s)} < \frac{k_s}{2^n}, \qquad k_s = 1,2,3,\ldots,2^n \text{ and } s=1,\ldots,m.$$

Suppose Theorem 1.1 has been verifies for a certain $m\ge 1$. Let's prove the induction step, $m+1$.

- First case: $0 < p \le 2$. Let $(j_1, \ldots, j_m) = (i_2, \ldots, i_{m+1})$ and, for each $i_1 \in \{1, \ldots, n\}$, let
$$c^{(i_1)}_{j_1 \ldots j_m} = a_{i_1 \ldots i_{m+1}}.$$

Thus

$$\left(\sum_{i_1=1}^{n} \cdots \sum_{i_{m+1}=1}^{n} \left| a_{i_1 \ldots i_{m+1}} \right|^2 \right)^{1/2} = \left(\sum_{i_1=1}^{n} \left[\left(\sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} \left| c^{(i_1)}_{j_1 \ldots j_m} \right|^2 \right)^{1/2} \right]^2 \right)^{1/2}. \tag{3.15}$$

By the induction hypothesis, for each $i_1 \in \{1, \ldots, n\}$, we have

$$\begin{aligned} &\left(\sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} \left| c^{(i_1)}_{j_1 \ldots j_m} \right|^2 \right)^{1/2} \\ &\le A_p^{-m} \left(\int_{[0,1]^m} \left| \sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} r_{j_1}(x_1) \cdots r_{j_m}(x_m)\, c^{(i_1)}_{j_1 \ldots j_m} \right|^p dx_1 \ldots dx_m \right)^{1/p}. \end{aligned} \tag{3.16}$$

Notice that

$$\begin{aligned} &\int_{[0,1]^m} \left| \sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} r_{j_1}(x_1) \cdots r_{j_m}(x_{m-1})\, c^{(i_1)}_{j_1 \ldots j_m} \right|^p dx_1 \ldots dx_m \\ &= 2^{-nm} \sum_{k_m=1}^{2^n} \cdots \sum_{k_1=1}^{2^n} \left| \sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} r_{j_1}\left(x_1^{(k_1)}\right) \cdots r_{j_{m-1}}\left(x_m^{(k_m)}\right) c^{(i_1)}_{j_1 \ldots j_m} \right|^p, \end{aligned} \tag{3.17}$$

where

$$\frac{k_s - 1}{2^n} \le x_s^{(k_s)} < \frac{k_s}{2^n}, \qquad k_s = 1, 2, 3, \ldots, 2^n \text{ and } s = 1, \ldots, m.$$

Let

$$b_{i_1 k_m \ldots k_1} = \sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} r_{j_1}\left(x_1^{(k_1)}\right) \cdots r_{j_m}\left(x_m^{(k_m)}\right) c^{(i_1)}_{j_1 \ldots j_m}.$$

By (3.15), (3.16) and (3.17), we have

$$\begin{aligned} &A_p^m 2^{\frac{nm}{p}} \left(\sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} \left| a_{i_1 \ldots i_{m+1}} \right|^2 \right)^{1/2} \\ &\le \left(\sum_{i_1=1}^{n} \left(\sum_{k_m=1}^{2^n} \cdots \sum_{k_1=1}^{2^n} \left| \sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} r_{j_1}\left(x_1^{(k_1)}\right) \cdots r_{j_m}\left(x_m^{(k_m)}\right) c^{(i_1)}_{j_1 \ldots j_m} \right|^p \right)^{2/p} \right)^{1/2} \end{aligned} \tag{3.18}$$

and, since $p \le 2$, by the Minkowski inequality (see [10, Corollary 5.4.2]) we have

$$\begin{aligned} &\left(\sum_{i_1=1}^{n} \left(\sum_{k_m=1}^{2^n} \cdots \sum_{k_1=1}^{2^n} \left| \sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} r_{j_1}\left(x_1^{(k_1)}\right) \cdots r_{j_m}\left(x_m^{(k_m)}\right) c^{(i_1)}_{j_1 \ldots j_m} \right|^p \right)^{2/p} \right)^{1/2} \\ &= \left(\sum_{i_1=1}^{n} \left(\sum_{k_m=1}^{2^n} \cdots \sum_{k_1=1}^{2^n} \left| b_{i_1 k_m \ldots k_1} \right|^p \right)^{2/p} \right)^{1/2} \\ &\le \left(\sum_{k_m=1}^{2^n} \cdots \sum_{k_1=1}^{2^n} \left[\left(\sum_{i_1=1}^{n} \left| b_{i_1 k_m \ldots k_1} \right|^2 \right)^{1/2} \right]^p \right)^{1/p} \end{aligned} \tag{3.19}$$

By the induction hypothesis, i.e. Khinchin's inequality for each $k_1, \ldots, k_m \in \{1, 2, 3, \ldots, 2^n\}$, we reach

$$\begin{aligned}
&\left(\sum_{i_1=1}^{n} \left|b_{i_1 k_m \ldots k_1}\right|^2\right)^{1/2} \\
&\leq A_p^{-1} \left(\int_0^1 \left|\sum_{i_1=1}^{n} r_{i_1}(t_1)\, b_{i_1 k_m \ldots k_1}\right|^p dt_1\right)^{1/p} \\
&= A_p^{-1} \left(\int_0^1 \left|\sum_{i_1=1}^{n}\sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} r_{i_1}(t_1)\, r_{j_1}\left(x_1^{(k_1)}\right) \cdots r_{j_m}\left(x_m^{(k_m)}\right) c_{j_1 \ldots j_m}^{(i_1)}\right|^p dt_1\right)^{1/p} \\
&= A_p^{-1} \left(\int_0^1 \left|\sum_{i_1=1}^{n}\sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} r_{i_1}(t_1)\, r_{i_2}\left(x_1^{(k_1)}\right) \cdots r_{i_{m+1}}\left(x_m^{(k_m)}\right) a_{i_1 \ldots i_{m+1}}\right|^p dt_1\right)^{1/p}. \qquad (3.20)
\end{aligned}$$

Therefore, by (3.18), (3.19) and (3.20) we have

$$\begin{aligned}
&A_p^{m+1} 2^{\frac{nm}{p}} \left(\sum_{i_1=1}^{n} \cdots \sum_{i_{m+1}=1}^{n} \left|a_{i_1 \ldots i_{m+1}}\right|^2\right)^{1/2} \\
&\leq \left(\sum_{k_m=1}^{2^n} \cdots \sum_{k_1=1}^{2^n} \int_0^1 \left|\sum_{i_1=1}^{n}\sum_{i_2=1}^{n} \cdots \sum_{i_{m+1}=1}^{n} r_{i_1}(t_1)\, r_{i_2}\left(x_1^{(k_1)}\right) \cdots r_{i_{m+1}}\left(x_m^{(k_m)}\right) a_{i_1 \ldots i_{m+1}}\right|^p dt_1\right)^{1/p} \\
&\overset{(3.14)}{=} 2^{\frac{nm}{p}} \left(\int_{[0,1]^{m+1}} \left|\sum_{i_1=1}^{n}\sum_{i_2=1}^{n} \cdots \sum_{i_{m+1}=1}^{n} r_{i_1}(t_1) \cdots r_{i_{m+1}}(t_{m+1})\, a_{i_1 \ldots i_{m+1}}\right|^p dt_1 \ldots dt_{m+1}\right)^{1/p}.
\end{aligned}$$

On the other hand, since $\|\cdot\|_{L_p\left([0,1]^{m+1}\right)} \leq \|\cdot\|_{L_2\left([0,1]^{m+1}\right)}$ the other side of the inequality follows immediately from (3.13).

• Second case: $2 < p < \infty$. Since $\|\cdot\|_{L_2\left([0,1]^{m+1}\right)} \leq \|\cdot\|_{L_p\left([0,1]^{m+1}\right)}$, the right hand side of the inequality follows from (3.13) . Now, for each $k = 1, 2, 3, \ldots, 2^n$, let

$$\frac{k-1}{2^n} \leq t_{m+1}^{(k)} < \frac{k}{2^n} \qquad \text{e} \qquad b_{i_1 \ldots i_m}^{(k)} = \sum_{i_{m+1}=1}^{n} r_{i_{m+1}}\left(t_{m+1}^{(k)}\right) a_{i_1 \ldots i_{m+1}}.$$

Hence, for all $t_1, \ldots, t_m$, we have

$$\begin{aligned}
&\int_0^1 \left|\sum_{i_1=1}^{n} \cdots \sum_{i_{m+1}=1}^{n} r_{i_1}(t_1) \cdots r_{i_{m+1}}(t_{m+1})\, a_{i_1 \ldots i_{m+1}}\right|^p dt_{m+1} \\
&= \int_0^1 \left|\sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} r_{i_1}(t_1) \cdots r_{i_m}(t_m) \sum_{i_{m+1}=1}^{n} r_{i_{m+1}}(t_{m+1})\, a_{i_1 \ldots i_{m+1}}\right|^p dt_{m+1} \\
&= \frac{1}{2^n} \sum_{k=1}^{2^n} \left|\sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} r_{i_1}(t_1) \cdots r_{i_m}(t_m) \sum_{i_{m+1}=1}^{n} r_{i_{m+1}}\left(t_{m+1}^{(k)}\right) a_{i_1 \ldots i_{m+1}}\right|^p \\
&= \frac{1}{2^n} \sum_{k=1}^{2^n} \left|\sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} r_{i_1}(t_1) \cdots r_{i_m}(t_m)\, b_{i_1 \ldots i_m}^{(k)}\right|^p
\end{aligned}$$

and, consequently,

$$\begin{aligned}
&\left(\int_{[0,1]^{m+1}}\left|\sum_{i_1=1}^{n}\cdots\sum_{i_{m+1}=1}^{n} r_{i_1}(t_1)\cdots r_{i_{m+1}}(t_{m+1})\,a_{i_1\ldots i_{m+1}}\right|^p dt_1\ldots dt_{m+1}\right)^{1/p}\\
&=\left(\int_{[0,1]^m}\left(\int_0^1\left|\sum_{i_1=1}^{n}\cdots\sum_{i_{m+1}=1}^{n} r_{i_1}(t_1)\cdots r_{i_{m+1}}(t_{m+1})\,a_{i_1\ldots i_{m+1}}\right|^p dt_{m+1}\right) dt_1\ldots dt_m\right)^{1/p}\\
&=\left(\frac{1}{2^n}\sum_{k=1}^{2^n}\int_{[0,1]^m}\left|\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n} r_{i_1}(t_1)\cdots r_{i_m}(t_m)\,b_{i_1\ldots i_m}^{(k)}\right|^p dt_1\ldots dt_m\right)^{1/p}.
\end{aligned}$$

By the induction hypothesis,

$$\left(\int_{[0,1]^m}\left|\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n} r_{i_1}(t_1)\cdots r_{i_m}(t_m)\,b_{i_1\ldots i_m}^{(k)}\right|^p dt_1\ldots dt_m\right)^{1/p}\le B_p^m\left(\sum_{i_1=1}^{n}\cdots\sum_{i_{m+1}=1}^{n}\left|b_{i_1\ldots,i_m}^{(k)}\right|^2\right)^{1/2}$$

and

$$\begin{aligned}
&\left(\int_{[0,1]^{m+1}}\left|\sum_{i_1=1}^{n}\cdots\sum_{i_{m+1}=1}^{n} r_{i_1}(t_1)\cdots r_{i_{m+1}}(t_{m+1})\,a_{i_1\ldots i_{m+1}}\right|^p dt_1\ldots dt_{m+1}\right)^{1/p}\\
&\le\frac{1}{2^{n/p}}B_p^m\left(\frac{1}{2^n}\sum_{k=1}^{2^n}\left(\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n}\left|b_{i_1\ldots i_m}^{(k)}\right|^2\right)^{p/2}\right)^{1/p}
\end{aligned}$$

Applying Minkowski inequality followed by the induction hypothesis, we conclude that

$$\begin{aligned}
&\frac{1}{2^{n/p}}B_p^m\left(\frac{1}{2^n}\sum_{k=1}^{2^n}\left(\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n}\left|b_{i_1\ldots i_m}^{(k)}\right|^2\right)^{p/2}\right)^{1/p}\\
&\le\frac{1}{2^{n/p}}\left(\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n}\left(\sum_{k=1}^{2^n}\left|b_{i_1\ldots i_m}^{(k)}\right|^p\right)^{\frac{2}{p}}\right)^{\frac{1}{2}}\\
&=\left(\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n}\left(\frac{1}{2^n}\sum_{k=1}^{2^n}\left|\sum_{i_{m+1}=1}^{n} r_{i_{m+1}}\left(t_{m+1}^{(k)}\right)a_{i_1\ldots i_{m+1}}\right|^p\right)^{\frac{2}{p}}\right)^{\frac{1}{2}}\\
&=\left(\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n}\left(\int_0^1\left|\sum_{i_{m+1}=1}^{n} r_{i_{m+1}}(t_{m+1})\,a_{i_1\ldots i_{m+1}}\right|^p dt_{m+1}\right)^{\frac{2}{p}}\right)^{\frac{1}{2}}\\
&\le B_p\left(\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n}\sum_{i_{m+1}=1}^{n}\left|a_{i_1\ldots i_{m+1}}\right|^2\right)^{1/2},
\end{aligned}$$

and the proof of the multiple version of Khinchin inequality is complete. □

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