ON REAL-VALUED HOMOGENEOUS POLYNOMIALS WITH MANY VARIABLES

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Abstract. We prove that the optimal constants in the classical Bohnenblust–Hille inequality for real-valued m-homogeneous polynomials, C_m, satisfy $\lim_{m\to\infty} C_m^{1/m} = 2$, independently of dimension.

1. Introduction

"*For polynomials in many variables, what are the estimates independent of the number of variables?*" This question is from the abstract of Aron-Beauzamy-Enflo's influential paper [1]. Nowadays it clearly constitutes one of main threads of research in the field.

In this framework of research, the classical Bohnenblust–Hille inequality is a cornerstone. It says that $2m/(m+1)$ is the smallest possible $p>0$ for which there is a constant $C_{p,m}\geq 1$, independent of the number of variables n, such that

$$|P_m|_p \leq C_{p,m} \sup_{\|(x_1,\ldots,x_n)\|_\infty \leq 1} |P_m(x_1,\ldots,x_n)|, \tag{1.1}$$

for all m-homogeneous polynomials $P_m(x_1,\ldots,x_n) = \sum_{|\alpha|=m} a_\alpha x^\alpha$ on $\mathbb{R}^n$. Here $|P_m|_p$ denotes the ℓ_p norm of the coefficients of P_m. It is worth noting that the real-valued form of the Bohnenblust–Hille inequality, i.e. (1.1), follows from its complex-valued version, viz. [5, page 452].

Throughout the paper we shall consider the real vector space $\mathbb{R}^n$ endowed with the supremum norm, that is ℓ_∞^n. Also, we denote simply by C_m the minimal constant $C_{2m/(m+1),m}$, corresponding to the case $p=2m/(m+1)$. As usual, we denote

$$\|P_m\| := \sup_{\|(x_1,\ldots,x_n)\|_\infty \leq 1} |P_m(x_1,\ldots,x_n)|.$$

In this note, we are interested in the asymptotic growth with respect to m of the supremum of the ℓ_p sums of the coefficients of real-valued m-homogeneous polynomials acting on $\mathbb{R}^n$. To put it in mathematical terms, for all $p\in(0,\infty]$ and all positive integers m, we investigate the asymptotic behavior of

$$\sup\left\{|P_m|_p : \|P_m\colon \mathbb{R}^n \to \mathbb{R}\| = 1 \text{ and } n\in\mathbb{N}\right\}.$$

Observe that, above, m is fixed, and the supremum is taken over both P_m and n. For the sake of simplicity we shall just write $\sup\left\{|P_m|_p : \|P_m\|=1\right\}$ instead of the above expression.

The main result we prove in this note reads as follows:

Theorem 1.1. *For all $p\in[2,\infty]$, there holds*

$$\lim_{m\to\infty}\left(\sup\left\{|P_m|_p : \|P_m\|=1\right\}\right)^{1/m} = 2, \tag{1.2}$$

and, moreover,

$$\lim_{m\to\infty}\left(\sup\left\{|P_m|_{\frac{2m}{m+1}} : \|P_m\|=1\right\}\right)^{1/m} = 2. \tag{1.3}$$

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In contrast, since $2m/(m+1)$ is the optimal (smallest) admissible value for p in (1.1), the corresponding limit in (1.2), when $p \in (0,2)$, is actually infinity. From (1.3) it readily follows that

$$\lim_{m\to\infty} C_m^{1/m} = 2,$$

for the constants C_m of the Bohnenblust–Hille inequality (1.1).

2. Preliminaries and literature review

The investigation of the constants of the real and complex Bohnenblust–Hille inequalities is a central topic of investigation, with applications in different fields of mathematics and applied sciences (see, for instance, [2, 3, 4] and the references therein).

In [3], it was shown that

$$\limsup_{m\to\infty} \left(\sup\left\{|P_m|_{\frac{2m}{m+1}} : \|P_m\| = 1\right\}\right)^{1/m} = 2 \tag{2.1}$$

and

$$\limsup_{m\to\infty} \left(\sup\left\{|P_m|_p : \|P_m\| = 1\right\}\right)^{1/m} = 2 \tag{2.2}$$

for all $p \in [2,\infty]$. The technique used by these authors combines an old result due to Visser [6] with the subexponentiality of the Bohnenblust–Hille inequality for complex m-homogeneous polynomials, viz. [2]. They then prove that for all $\varepsilon > 0$ there is a $\kappa > 0$ such that

$$\sup\left\{|P_m|_{\frac{2m}{m+1}} : \|P_m\| = 1\right\} \le \kappa\,(1+\varepsilon)^m\, 2^{m-1}.$$

In particular, this assures that

$$\limsup_{m\to\infty} \left(\sup\left\{|P_m|_{\frac{2m}{m+1}} : \|P_m\| = 1\right\}\right)^{1/m} \le 2.$$

To prove the converse inequality, the authors use the family of polynomials generated as

$$\begin{cases} P_2(x_1,x_2) = x_1^2 - x_2^2, \\ P_4(x_1,x_2,x_3,x_4) = \left(x_1^2-x_2^2\right)^2 - \left(x_3^2-x_4^2\right)^2, \\ P_8(x_1,\dots,x_8) = \left[\left(x_1^2-x_2^2\right)^2 - \left(x_3^2-x_4^2\right)^2\right]^2 - \left[\left(x_5^2-x_6^2\right)^2 - \left(x_7^2-x_8^2\right)^2\right]^2 \end{cases}$$

and so on. Clearly $\|(P_{2^m})^n\| = 1$ and, by induction, one can show that such $n2^m$-homogeneous polynomials $(P_{2^m})^n$ satisfy

$$|(P_{2^m})^n|_\infty \ge \left(\frac{2^n}{n+1}\right)^{2^m-1},$$

for all m, n. Since

$$|(P_{2^m})^n|_{2(n2^m)/(n2^m+1)}^{1/(n2^m)} \ge |(P_{2^m})^n|_\infty^{1/(n2^m)} \ge \left(\frac{2^n}{n+1}\right)^{\frac{2^m-1}{n2^m}} = \left(\frac{2}{\sqrt[n]{n+1}}\right)^{1-\frac{1}{2^m}}$$

and

$$\sup_{n,m}\left(\frac{2}{\sqrt[n]{n+1}}\right)^{1-\frac{1}{2^m}} = 2,$$

it follows that

$$\limsup_{m\to\infty} \left(\sup\left\{|P_m|_{\frac{2m}{m+1}} : \|P_m\| = 1\right\}\right)^{1/m} \ge \limsup_{m\to\infty} \left(\sup\left\{|P_m|_\infty : \|P_m\| = 1\right\}\right)^{1/m} \ge 2.$$

The question whether lim sup could be replaced by the (full) limit has been open since then. This type of question is often delicate and the main result we prove in this article settles this issue. Indeed we show that both in (2.1) and in (2.2) the result holds as a full limit, that is, we show that

$$\liminf_{m\to\infty} \left(\sup\left\{|P_m|_\infty : \|P_m\| = 1\right\}\right)^{1/m} \ge 2. \tag{2.3}$$

3. The proof

The next lemma is crucial for the proof of our main result.

Lemma 3.1. *Let $\varepsilon \in (0,1)$ and $N \in \mathbb{N}$ be given. There exists an $M \in \mathbb{N}$, depending only upon ε and N, such that whenever $m \geq M$, one can find an $n \in \mathbb{N}$, with $n > N$, satisfying*

$$(1-\varepsilon)\, m < n \cdot 2^N \leq m.$$

Proof. Let

$$M = \min\left\{L \in \mathbb{N} : L \geq \max\left\{2^N/\varepsilon, (N+1)\, 2^N\right\}\right\}.$$

Given $m \geq M$, let n and $0 \leq r < 2^N$ be respectively the quotient and the remainder of the Euclidean division of m by 2^N. By the definition of n and M, it is plain that $n > N$. Notice that

$$\varepsilon m \geq 2^N, \tag{3.1}$$

since $m \geq M \geq 2^N/\varepsilon$, and

$$m = n \cdot 2^N + r < n \cdot 2^N + 2^N = (n+1)\, 2^N. \tag{3.2}$$

Moreover, by (3.1) and (3.2), we have

$$n \cdot 2^N = m - r \leq m < (n+1)\, 2^N = n \cdot 2^N + 2^N \leq n \cdot 2^N + \varepsilon m. \tag{3.3}$$

Finally, from (3.3), we can deduce that

$$n \cdot 2^N \leq m \qquad \text{and} \qquad m < n \cdot 2^N + \varepsilon m$$

which ultimately yields

$$(1-\varepsilon)\, m < n \cdot 2^N \leq m,$$

as required. □

We are now in position to establish (2.3). Initially, for all $\delta \in (0,1)$, there exists a positive integer N such that

$$\left(\frac{2^n}{n+1}\right)^{\frac{2^m-1}{n2^m}} > 2 - \delta \tag{3.4}$$

for all $n, m \geq N$. It follows from Lemma 3.1 that, given $\varepsilon > 0$, there exists a positive integer M such that for all $r \geq M$, we can find $t_r = n_r 2^N$ with $n_r > N$ satisfying

$$(1-\varepsilon)\, r \leq t_r \leq r. \tag{3.5}$$

If $r \geq M$, let us consider

$$P_r\left(x_1, \ldots, x_{2^N+1}\right) = x_{2^N+1}^{r-t_r}\left(P_{2^N}\right)^{n_r}\left(x_1, \ldots, x_{2^N}\right).$$

Since

$$|P_r|_\infty = \left|\left(P_{2^N}\right)^{n_r}\right|_\infty \geq \left(\frac{2^{n_r}}{n_r+1}\right)^{2^N-1},$$

by (3.4) and (3.5), we have

$$|P_r|_\infty^{1/r} \geq \left(\frac{2^{n_r}}{n_r+1}\right)^{\frac{2^N-1}{r}} \geq \left[\left(\frac{2^{n_r}}{n_r+1}\right)^{\frac{2^N-1}{t_r}}\right]^{1-\varepsilon} = \left[\left(\frac{2^{n_r}}{n_r+1}\right)^{\frac{2^N-1}{n_r 2^N}}\right]^{1-\varepsilon} > (2-\delta)^{1-\varepsilon},$$

and this completes the proof of (2.3).

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