ON ZOLEZZI'S THEOREM FOR INFINITE MEASURE SPACES

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Abstract. In this paper we discuss the infinite measure counterpart of Zolezzi's Theorem for infinite measure spaces. For a measure space with infinite measure, (Ω, Σ, μ), we construct a sequence in $L^\infty(\mu)$, with uniformly control upon its support measure, that does not converge in $L^p(\mu)$, for all $1 \le p < \infty$, however does converge weakly in $L^\infty(\mu)$.

1. Introduction

Let (Ω, Σ, μ) be a positive, finite measure space, i.e. $0 < \mu(\Omega) < +\infty$. A important result proven originally by Tullio Zolezzi in [9] asserts that any weakly convergent sequence in $L^\infty(\Omega, \Sigma, \mu)$ converges strongly in $L^p(\Omega, \Sigma, \mu)$ for $1 \le p < \infty$; see [5] for another proof.

Zolezzi's theorem finds important applications, for instance in fixed point theory [1], in weakly differentiable maps [2], in free boundary problems [4], as well as in infinite-dimensional control theory [7] and convergence of solutions of diffusive PDEs [8], just to cite a few.

As explained in [9], such a interesting result should be understood as a manifestation of how large the dual space of $L^\infty(\Omega, \Sigma, \mu)$ is. Indeed, a classical result, see for instance [3] or [6], yields a representation of $L^\infty(\Omega, \Sigma, \mu)^*$ as the set of all finitely additive finite signed measures defined on Σ, which are absolutely continuous with respect to μ. Such a vector space, endowed with the total variation norm, is isometric to $L^\infty(\Omega, \Sigma, \mu)^*$, equipped with its natural norm.

The infinite measure counterpart of Zolezzi's Theorem, that is the case when $\mu(\Omega) = +\infty$, is in principle a bit trickier. To begin with, the space $L^\infty(\Omega, \Sigma, \mu)$ is not embedded into $L^p(\Omega, \Sigma, \mu)$. A way to overcome this is to restrict the analysis to weakly convergent sequences in $L^\infty(\Omega, \Sigma, \mu)$ that are also bounded in $L^1(\Omega, \Sigma, \mu)$.

It turns out though that this is not quite the right assumption yet. Namely, it is easy to construct a sequence $(u_n)_n$ in $L^\infty(\Omega, \Sigma, \mu) \cap L^1(\Omega, \Sigma, \mu)$, converging *strongly* to zero with respect to the L^∞-norm; however $\|u_n\|_{L^1} = 1$, for all n. For that, one simply considers a nested sequence of finite measure sets Ω_n, with $\mu(\Omega_n) \nearrow +\infty$, as $n \to \infty$. The sequence $(u_n)_n$ defined as

$$u_n(x) = \frac{1}{\mu(\Omega_n)} \chi_{\Omega_n},$$

where χ_A denotes the indicator function of the set A, satisfies the properties required.

In the lights of the example above described, the natural question is whether Zolezzi's Theorem would have an infinite measure counterpart, under the assumption of uniform control of the support measure of the sequence. Note such a constrain, along with weak convergence in $L^\infty(\Omega, \Sigma, \mu)$, implies uniform boundedness in $L^1(\Omega, \Sigma, \mu)$.

In this article we discuss such this follow up question reminiscing Zolezzi's theorem. Our main theorem is the following:

Theorem 1. *Let (Ω, Σ, μ) be a positive, regular Radon measure space, with $\mu(\Omega) = +\infty$. There exists a sequence $(\varphi_n)_n \subset L^\infty(\Omega, \Sigma, \mu) \cap L^1(\Omega, \Sigma, \mu)$, with uniform control of their measure supports, i.e. verifying*

$$\mu(\mathit{Supp}\ \varphi_n) < C,$$

for all n, that converges weakly to zero in $L^\infty(\Omega, \Sigma, \mu)$; however it does not converge strongly to zero in any $L^p(\Omega, \Sigma, \mu)$, $1 \le p < \infty$.

Obviously, it follows from Theorem 1 that given any fixed function $f \in L^\infty$, with bounded measure support, one can find a sequence $(f_n)_n$, with uniform control of their measure supports, that converges weakly to f in $L^\infty(\Omega, \Sigma, \mu)$, but it does not converge strongly in any $L^p(\Omega, \Sigma, \mu)$, $1 \le p < \infty$.

We conclude the introduction by commenting that the main difficulty in the proof of Theorem 1 is to prove that the constructed sequence $(\varphi_n)_n$ does indeed converge weakly to zero in L^∞, as no useful representation theorem is in principle available. This will be attained by an indirect argument, which seems to be interesting by its own.

2. Proof

Let (Ω, Σ, μ) be a positive, regular Radon measure space, with $\mu(\Omega) = +\infty$. There exists a measurable subset $\Omega_1 \subset \Omega$ such that $1 \le \mu(\Omega_1) < 2$.

Next we look at Ω_1^C, i.e. the complement of the set Ω_1, and repeat the argument. That is, since $\mu(\Omega \setminus \Omega_1) = +\infty$, there exists a measurable subset $\Omega_2 \subset \Omega_1^C$ such that $1 \le \mu(\Omega_2) < 2$. Arguing recursively, we construct a sequence of disjoint measurable sets $\Omega_1, \Omega_2, \cdots$, satisfying:

$$1 \le \mu(\Omega_n) < 2,$$

for all $n \in \mathbb{N}$. We define

$$\varphi_n(x) = \chi_{\Omega_n}(x).$$

Clearly we have

$$1 \leq \|\varphi_n\|_p < 2,$$

for all $1 \leq p < \infty$.

We now claim that $(\varphi_n)_n$ converges weakly to zero in $L^\infty(\Omega, \Sigma, \mu)$. Indeed, let $\Psi \in L^\infty(\Omega, \Sigma, \mu)^*$ be an arbitrary element of the dual space. We have to show that

$$\Psi(\varphi_n) \to 0,$$

in $\mathbb{R}$ (or in $\mathbb{C}$ if we consider complex space).

We shall prove this indirectly. For $k \in \mathbb{N}$ fixed, consider:

$$\Phi_k(x) = \varphi_1(x) + \varphi_2(x) + \cdots + \varphi_k(x).$$

By construction, $\varphi_i(x) \cdot \varphi_j(x) = 0$, and thus

$$\|\Phi_k\|_{L^\infty(\Omega,\Sigma,\mu)} = 1.$$

Hence, we can estimate:

$$|\Psi(\Phi_k(x))| \leq \|\Psi\|_{L^\infty(\Omega,\Sigma,\mu)^*}, \tag{1}$$

independently of k. On the other hand, by linearity we have:

$$\Psi(\Phi_k(x)) = \Psi(\varphi_1(x)) + \Psi(\varphi_2(x)) + \cdots + \Psi(\varphi_k(x)). \tag{2}$$

Combining (1) and (2) we conclude

$$(\Psi(\varphi_k(x)))_{k\in\mathbb{N}} \in \ell_1,$$

and thus, in particular,

$$\lim_{n\to\infty} \Psi(\varphi_n) = 0,$$

as required. The proof of Theorem 1 is complete. □

Remark 1. *If the measure space (Ω, Σ, μ) has more structure, say it is the Euclidean Space $\mathbb{R}^d$ endowed with the Lebesgue measure, then we can refine our construction as to have a sequence $(\varphi_n)_n \subset C_c^\infty(\mathbb{R}^d)$ converging weakly to zero in $L^\infty(\mathbb{R}^d)$, but failing to converge strongly in $L^p(\mathbb{R}^d)$ for all $1 \leq p < \infty$. Indeed, let φ_0 be a smooth positive function defined on $\mathbb{R}^d$, supported in B_1. Fix a unitary vector $\mu \in \mathbb{S}^{d-1}$ and define*

$$\varphi_n(x) := \varphi(x + 2n\nu).$$

It is not hard to verify that such a sequence has all the properties needed in the proof of Theorem 1, and thus a similar reasoning implies φ_n is weakly null in $L^\infty(\mathbb{R}^d)$; however $\int_{\mathbb{R}^d} |\varphi_n| dx \geq c_0 > 0$.

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