Random polynomials generated by a three-term recurrence relation

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Joint work with

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IX Jaén Conference on Approximation Theory
Úbeda, Spain, July 8-13, 2018

Assumptions

μ: probability measure on $(0, +\infty)$ with

$$m_k := \int x^k \, d\mu(x) < \infty, \qquad \text{for all } k \in \mathbb{Z}_{\geq 0}.$$

$(a_n)_{n\in\mathbb{Z}}$: sequence of independent, identically distributed random variables with distribution μ, taking values in $(0, +\infty)$.

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$(a_n)_{n\in\mathbb{Z}}$: sequence of independent, identically distributed random variables with distribution μ, taking values in $(0,+\infty)$.

This means that each $a_n : \Omega \longrightarrow (0,+\infty)$ is a measurable function defined on a common measure space $(\Omega, \Sigma, \mathbb{P})$,

$$\mathbb{P}(a_n \in S) = \mu(S), \qquad \text{for every Borel set } S \subset (0,+\infty),$$

and

$$\mathbb{P}(a_{n_i} \in S_i \text{ for all } 1 \leq i \leq l) = \prod_{i=1}^{l} \mathbb{P}(a_{n_i} \in S_i)$$

for distinct indices $n_1, \ldots, n_l \in \mathbb{Z}$, and $S_i \subset (0,+\infty)$, $i = 1, \ldots, l$.

The random polynomials

Consider the sequence $(P_n)_{n=0}^{\infty}$ of polynomials generated by

$$P_{n+1}(z) = zP_n(z) - a_n P_{n-1}(z), \qquad n \geq 1,$$

with

$$P_\ell(z) = z^\ell, \qquad \ell = 0, 1.$$

The first few are:

$$\begin{aligned} P_0(z) &= 1, \\ P_1(z) &= z, \\ P_2(z) &= z^2 - a_1, \\ P_3(z) &= z^3 - (a_1 + a_2)z, \\ P_4(z) &= z^4 - (a_1 + a_2 + a_3)z^2 + a_1 a_2 \end{aligned}$$

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For each realization of the random variables $(a_n)_{n\in\mathbb{Z}}$, the sequence $(P_n)_{n=0}^{\infty}$ is a sequence of monic orthogonal polynomials on the real line.

Consider

$$H = \begin{pmatrix} 0 & 1 & & & & \\ a_1 & 0 & 1 & & & \\ & a_2 & 0 & 1 & & \\ & & a_3 & 0 & 1 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}$$

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and let H_n denote its principal $n \times n$ truncation.

Then

$$P_n(z) = \det(zI_n - H_n).$$

The **zeros** of P_n are real and simple, denoted

$$\lambda_1^{(n)} < \lambda_2^{(n)} < \cdots < \lambda_n^{(n)}$$

Two random discrete measures

Fix $n \geq 1$. Let σ_n be the normalized zero counting measure for P_n:

$$\sigma_n := \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j^{(n)}}.$$

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Let τ_n be the spectral measure associated with H_n. This is the measure on $[\lambda_1^{(n)}, \lambda_n^{(n)}]$ with moments given by

$$\int x^k \, d\tau_n(x) = \langle H_n^k e_1, e_1 \rangle = H_n^k(1,1), \qquad k \in \mathbb{Z}_{\geq 0}.$$

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Then

$$\tau_n = \sum_{j=1}^{n} q_{j,n}^2 \, \delta_{\lambda_j^{(n)}}, \qquad \sum_{j=1}^{n} q_{j,n}^2 = 1.$$

The coefficients $q_{j,n}^2$ are the Christoffel numbers (appearing in the Gauss-Jacobi quadrature formula).

Statement of problem

We investigate the relation between μ and the asymptotic behavior of the average measures $\mathbb{E}\sigma_n$, $\mathbb{E}\tau_n$, which can be defined via duality by

$$\int f \, d\mathbb{E}\sigma_n = \mathbb{E}\left(\int f \, d\sigma_n\right)$$

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Our approach is the classical **moment method**, which consists of expressing the moments of σ_n and τ_n

$$\int x^k\, d\sigma_n(x) = \frac{1}{n}\mathrm{Tr}(H_n^k), \qquad k \geq 0$$
$$\int x^k\, d\tau_n(x) = H_n^k(1,1), \qquad k \geq 0$$

combinatorially, in this case in terms of certain classes of lattice paths.

Lattice paths

A *path* $\gamma = e_1 e_2 \cdots e_k$ is a finite, connected union of segments of the form

$$e_j : (j-1, i_{j-1}) \to (j, i_j), \quad |i_j - i_{j-1}| = 1, \quad \text{for all } j = 1, \ldots, k,$$

where the *heights* $i_0, i_1, \ldots, i_k$ are integers and $i_0 = i_k$. We say that γ has *length* k.

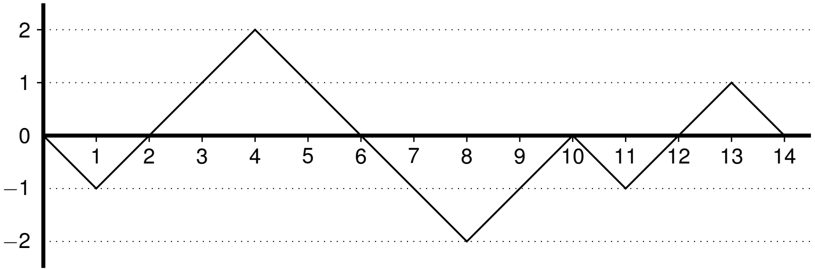

Figure: Example of a path of length 14 with initial and final height 0.

The *weight* of the edge e_j is

$$w(e_j) := \begin{cases} 1 & \text{if } i_j - i_{j-1} = 1 \\ a_{i_j} & \text{if } i_j - i_{j-1} = -1. \end{cases}$$

The *weight* of a path γ is

$$w(\gamma) := \prod_{e \subset \gamma} w(e),$$

the product taken over all edges of γ.

Let

$$\mathcal{P}(n,k,i) = \{\gamma = e_1 e_2 \cdots e_k | 1 \leq i_j \leq n \text{ for all } j = 0, \ldots, k \text{ and } i_0 = i_k = i\}.$$

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$$\mathrm{Tr}(H_n^k) = \sum_{i=1}^{n} \sum_{\gamma \in \mathcal{P}(n,k,i)} w(\gamma)$$

$$H_n^k(1,1) = \sum_{\gamma \in \mathcal{P}(n,k,1)} w(\gamma)$$

Dyck paths and generalized Dyck paths

A *Dyck path* of length $2n$ is a path with heights $(i_0, i_1, \ldots, i_{2n})$ satisfying

1) $i_0 = i_{2n} = 0$.
2) $i_j \geq 0$ for all $j = 0, \ldots, 2n$.

We use $\mathcal{D}_n$ to denote the set of all such paths.

$$\mathrm{card}(\mathcal{D}_n) = \frac{1}{n+1}\binom{2n}{n}, \qquad n \geq 0. \quad \text{(Catalan numbers)}$$

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Let

$$\overline{\mathcal{D}}_n := \{\overline{\gamma} | \gamma \in \mathcal{D}_n\}$$

Weight polynomials

To make the formulas symmetric, we rename the random variables with negative index:

$$b_n := a_{-n-1}, \qquad n \geq 0.$$

Weight polynomials

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We define three sequences $(W_n)_{n=0}^{\infty}$, $(A_n)_{n=0}^{\infty}$, $(B_n)_{n=0}^{\infty}$ of weight polynomials:

$$W_n := \sum_{\gamma \in \mathcal{P}_n} w(\gamma) \qquad n \geq 0,$$

$$A_n := \sum_{\gamma \in \mathcal{D}_n} w(\gamma) \qquad n \geq 0,$$

$$B_n := \sum_{\gamma \in \overline{\mathcal{D}}_n} w(\gamma) \qquad n \geq 0,$$

where by definition $W_0 = A_0 = B_0 = 1$.

Note: In general, if $\mathcal{S} \subset \mathcal{P}_n$, then we call the expression $\sum_{\gamma \in \mathcal{S}} w(\gamma)$ the weight polynomial for $\mathcal{S}$.

$$
\begin{aligned}
W_n &= W_n(a_0, \ldots, a_{n-1}; b_0, \ldots, b_{n-1}),\\
A_n &= A_n(a_0, \ldots, a_{n-1}),\\
B_n &= B_n(b_0, \ldots, b_{n-1}).
\end{aligned}
$$

Some explicit expressions:

$$
\begin{aligned}
A_0 &= 1\\
A_1 &= a_0\\
A_2 &= a_0(a_0 + a_1)\\
A_3 &= a_0(a_0^2 + 2a_0a_1 + a_1^2 + a_1a_2)\\
W_0 &= 1\\
W_1 &= a_0 + b_0\\
W_2 &= a_0(a_0 + a_1) + 2a_0b_0 + b_0(b_0 + b_1)\\
W_3 &= a_0(a_0^2 + 2a_0a_1 + a_1^2 + a_1a_2) + a_0b_0(3a_0 + 3b_0 + 2a_1 + 2b_1)\\
&\quad + b_0(b_0^2 + 2b_0b_1 + b_1^2 + b_1b_2).
\end{aligned}
$$

Some simple properties, valid for every $n \geq 0$:

1) A_n, B_n and W_n are homogeneous polynomials of degree n.
2) $W_n(a_0, \ldots, a_{n-1}; b_0, \ldots, b_{n-1}) = W_n(b_0, \ldots, b_{n-1}; a_0, \ldots, a_{n-1})$.
3) $B_n = A_n(b_0, \ldots, b_{n-1})$.

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3) $B_n = A_n(b_0, \ldots, b_{n-1})$.

We also need the shifted polynomials: For each $k \geq 0$, $n \geq 0$, let

$$A_n^{(k)} := A_n(a_k, \ldots, a_{k+n-1})$$
$$B_n^{(k)} := B_n(b_k, \ldots, b_{k+n-1})$$

Note that $A_n = A_n^{(0)}$, $B_n = B_n^{(0)}$.

Formal Laurent series

We associate to the sequences of weight polynomials certain formal Laurent series in $\mathbb{C}((z^{-1}))$:

$$W(z) := \sum_{n=0}^{\infty} \frac{W_n}{z^{2n+1}}$$

$$A^{(k)}(z) := \sum_{n=0}^{\infty} \frac{A_n^{(k)}}{z^{2n+1}} \qquad k \geq 0$$

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$$B^{(k)}(z) := \sum_{n=0}^{\infty} \frac{B_n^{(k)}}{z^{2n+1}} \qquad k \geq 0$$

In the case $k = 0$ we write

$$A(z) := \sum_{n=0}^{\infty} \frac{A_n}{z^{2n+1}}$$

$$B(z) := \sum_{n=0}^{\infty} \frac{B_n}{z^{2n+1}}$$

Lemma

The following relations hold:

$$W(z) = \frac{1}{A(z)^{-1} + B(z)^{-1} - z}$$

and for each $k \geq 0$*,*

$$A^{(k)}(z) = \frac{1}{z - a_k\, A^{(k+1)}(z)}$$
$$B^{(k)}(z) = \frac{1}{z - b_k\, B^{(k+1)}(z)}$$

Flajolet's formula for A_n

For any integer $n \geq 1$, let

$$\mathrm{C}(n) := \{(n_0, \ldots, n_r) |\ n_0 + \cdots + n_r = n, n_j \in \mathbb{N} \text{ is an integer for all } 0 \leq j \leq r\},$$

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For example,

$$\mathrm{C}(4) = \{(4), (3,1), (1,3), (2,2), (2,1,1), (1,2,1), (1,1,2), (1,1,1,1)\}.$$

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The following formula is due to **P. Flajolet**, who calls the polynomials A_n *Stieltjes-Rogers* polynomials:

$$A_n = \sum_{(n_0,\ldots,n_r)\in\mathrm{C}(n)} \binom{n_0+n_1-1}{n_0-1} \cdots \binom{n_{r-1}+n_r-1}{n_{r-1}-1} a_0^{n_0} \cdots a_r^{n_r},$$

see P. Flajolet, *Combinatorial aspects of continued fractions*, Discr. Math. 32 (1980), 125–161.

To make the formulas compact, we introduce some more notations.

Given $n \in \mathbb{Z}_{\geq 0}$ and $\overline{n} \in \mathrm{C}(n)$, let

$$\rho_1(\overline{n}) := \begin{cases} \prod_{j=0}^{r-1} \binom{n_j+n_{j+1}-1}{n_j-1} & \text{if } \overline{n} = (n_0, \dots, n_r),\ r \geq 1 \\ 1 & \text{if } \overline{n} = (n),\ n \geq 1,\ \text{or } \overline{n} = e \end{cases}$$

and

$$a(\overline{n}) := \begin{cases} \prod_{j=0}^{r} a_j^{n_j} & \text{if } \overline{n} = (n_0, \dots, n_r),\ r \geq 0 \\ 1 & \text{if } \overline{n} = e \end{cases}$$

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The same way we define $b(\overline{n})$.

With these notations, Flajolet's formula is

$$A_n = \sum_{\overline{n} \in \mathrm{C}(n)} \rho_1(\overline{n})\, a(\overline{n}).$$

Formula for W_n

For $n \in \mathbb{Z}_{\geq 0}$, let

$$\widehat{\mathrm{C}}(n) := \bigcup_{j=0}^{n} \mathrm{C}(j) \times \mathrm{C}(n-j),$$

i.e., $\widehat{\mathrm{C}}(n)$ consists of all pairs $(\overline{p}, \overline{q})$ with $\overline{p} \in \mathrm{C}(j)$ and $\overline{q} \in \mathrm{C}(n-j)$ for some $0 \leq j \leq n$. Additionally, for $(\overline{p}, \overline{q}) \in \widehat{\mathrm{C}}(n)$ we define

$$\rho_2((\overline{p}, \overline{q})) := \begin{cases} \binom{n_0+n_0'}{n_0} \rho_1(\overline{p})\, \rho_1(\overline{q}) & \text{if } \overline{p} \neq e, \overline{q} \neq e, n_0 = \overline{p}(1), n_0' = \overline{q}(1) \\ \rho_1(\overline{p}) & \text{if } \overline{q} = e \\ \rho_1(\overline{q}) & \text{if } \overline{p} = e \end{cases}$$

Lemma

For every $n \geq 0$,

$$W_n = \sum_{(\overline{p}, \overline{q}) \in \widehat{\mathrm{C}}(n)} \rho_2((\overline{p}, \overline{q}))\, a(\overline{p})\, b(\overline{q}).$$

Taking expectation

We define

$$\alpha_n := \mathbb{E}(A_n), \qquad n \geq 0$$
$$\omega_n := \mathbb{E}(W_n), \qquad n \geq 0$$

and more generally,

$$\alpha_n^{(k)} := \mathbb{E}([A^k]_{k+2n}), \qquad k, n \in \mathbb{Z}_{\geq 0},$$

where $[A^k]_{k+2n}$ is the coefficient of $z^{-(k+2n)}$ in the series expansion of $A(z)^k$.
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Main question: How are the three sequences $(m_n)_{n=0}^{\infty}, (\alpha_n)_{n=0}^{\infty}, (\omega_n)_{n=0}^{\infty}$ related?

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Main question: How are the three sequences $(m_n)_{n=0}^{\infty}, (\alpha_n)_{n=0}^{\infty}, (\omega_n)_{n=0}^{\infty}$ related?

For a composition $\overline{n} = (n_0, \ldots, n_r)$, let

$$m(\overline{n}) := \prod_{j=0}^{r} m_{n_j} \qquad \alpha(\overline{n}) := \prod_{j=0}^{r} \alpha_{n_j} \qquad \omega(\overline{n}) := \prod_{j=0}^{r} \omega_{n_j}$$

Theorem

The following identities hold. For every $n \geq 0$,

$$\alpha_n = \sum_{\overline{n} \in \mathrm{C}(n)} \rho_1(\overline{n})\, m(\overline{n}),$$

$$\omega_n = \sum_{(\overline{p},\overline{q}) \in \widehat{\mathrm{C}}(n)} \rho_2(\overline{p},\overline{q})\, m(\overline{p})\, m(\overline{q}).$$

For all $k, n \geq 0$,

$$\alpha_n^{(k)} = \sum_{\overline{n} \in \mathrm{C}(n)} \binom{\overline{n}(1) + k - 1}{k-1} \rho_1(\overline{n})\, m(\overline{n}),$$

where $\overline{n}(1)$ denotes the first entry of $\overline{n}$.

Theorem

The following relations hold. For any $n \geq 0$,

$$\alpha_n^{(k)} = \sum_{j=0}^{n} \binom{j+k-1}{k-1} m_j \, \alpha_{n-j}^{(j)} \qquad k \geq 0$$

$$\omega_n = \sum_{j=0}^{n} \sum_{\ell=0}^{n-j} m_j \, \alpha_\ell^{(j)} \, \alpha_{n-j-\ell}^{(j+1)}$$

$$\omega_n = \sum_{j=0}^{n} \sum_{i=0}^{j} \sum_{\ell=0}^{n-j} \binom{j}{i} m_i \, m_{j-i} \, \alpha_\ell^{(i)} \, \alpha_{n-j-\ell}^{(j-i)}$$

Let

$$g_k(z) := \sum_{n=0}^{\infty} \frac{\alpha_n^{(k)}}{z^{2n+k}} \qquad k \geq 0$$

$$f(z) := \sum_{n=0}^{\infty} \frac{\omega_n}{z^{2n+1}}$$

Theorem (Cont.)

The previous relations are equivalent to the following:

$$g_k(z) = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} \frac{m_n\, g_n(z)}{z^{n+k}} \qquad k \geq 0$$

$$f(z) = \sum_{n=0}^{\infty} m_n\, g_n(z)\, g_{n+1}(z)$$

$$f(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{m_k\, m_{n-k}\, g_k(z)\, g_{n-k}(z)}{z^{n+1}}$$

Asymptotics of $\mathbb{E}\sigma_n$ and $\mathbb{E}\tau_n$

Theorem

Let $k \in \mathbb{Z}_{\geq 0}$ be fixed. Then

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}(\mathrm{Tr}(H_n^k)) = \begin{cases} 0, & \text{if } k \text{ is odd}, \\ \omega_{k/2}, & \text{if } k \text{ is even}, \end{cases} \tag{1}$$

$$\lim_{n\to\infty} \mathbb{E}(H_n^k(1,1)) = \begin{cases} 0, & \text{if } k \text{ is odd}, \\ \alpha_{k/2}, & \text{if } k \text{ is even}. \end{cases} \tag{2}$$

Assume there exist unique probability measures σ and τ on $\mathbb{R}$ with moments of all orders finite, such that

$$\int x^k \, d\sigma(x) = \text{RHS of (1)}, \qquad k \geq 0$$

$$\int x^k \, d\tau(x) = \text{RHS of (2)}, \qquad k \geq 0$$

Then $\mathbb{E}\sigma_n \xrightarrow{*} \sigma$ and $\mathbb{E}\tau_n \xrightarrow{*} \tau$.

Other relations and trees

$$
\begin{aligned}
m_1 &= \alpha_1 \\
m_2 &= \alpha_2 - \alpha_1^2 \\
m_3 &= \alpha_3 - 3\alpha_2\alpha_1 + 2\alpha_1^3 \\
m_4 &= \alpha_4 - 4\alpha_3\alpha_1 + 13\alpha_2\alpha_1^2 - 3\alpha_2^2 - 7\alpha_1^4 \\
m_5 &= \alpha_5 - 5\alpha_4\alpha_1 - 10\alpha_3\alpha_2 + 23\alpha_3\alpha_1^2 + 34\alpha_2^2\alpha_1 - 79\alpha_2\alpha_1^3 + 36\alpha_1^5
\end{aligned}
$$

Other relations and trees

$$m_1 = \alpha_1$$
$$m_2 = \alpha_2 - \alpha_1^2$$
$$m_3 = \alpha_3 - 3\alpha_2\alpha_1 + 2\alpha_1^3$$
$$m_4 = \alpha_4 - 4\alpha_3\alpha_1 + 13\alpha_2\alpha_1^2 - 3\alpha_2^2 - 7\alpha_1^4$$
$$m_5 = \alpha_5 - 5\alpha_4\alpha_1 - 10\alpha_3\alpha_2 + 23\alpha_3\alpha_1^2 + 34\alpha_2^2\alpha_1 - 79\alpha_2\alpha_1^3 + 36\alpha_1^5$$

To invert the relation

$$\alpha_n = \sum_{\overline{n}\in \mathrm{C}(n)} \rho_1(\overline{n})\, m(\overline{n}) = m_n + \sum_{\overline{n}\in \mathrm{C}(n)\setminus\{(n)\}} \rho_1(\overline{n})\, m(\overline{n}),$$

we apply repeatedly the relation

$$m_n = \alpha_n - \sum_{\overline{n}\in \mathrm{C}(n)\setminus\{(n)\}} \rho_1(\overline{n})\, m(\overline{n}).$$

This process is suitably expressed with the help of trees.

Let $n \geq 1$ and $\overline{n} = (n_0, \dots, n_r) \in \mathrm{C}(n)$. We define a class $\mathcal{T}_1(\overline{n})$ of **rooted leveled** trees associated with $\overline{n}$ as follows:

T1) Each vertex of the tree has a positive integer *value*. The root vertex has value n.

T2) The vertices of the tree are organized in $d+1$ disjoint levels $\ell = 0, \dots, d$, $d \geq 0$, where level 0 is formed solely by the root vertex, and level d consists of the vertices with values $n_0, n_1, \dots, n_r$, from left to right.

T3) For each $\ell = 0, \dots, d-1$, every vertex at level ℓ has at least one direct descendant at level $\ell + 1$, and there exists at least one vertex at level ℓ that has at least two direct descendants. For each $\ell = 0, \dots, d$, the sum of the values of the vertices at level ℓ is n. If $v_1, \dots, v_k$ are the direct descendants of a vertex v, then the sum of the values of $v_1, \dots, v_k$ is the value of v.

T4) If a vertex v has only one direct descendant v', then v' has only one direct descendant as well, unless v' is a vertex in the last level of the tree.

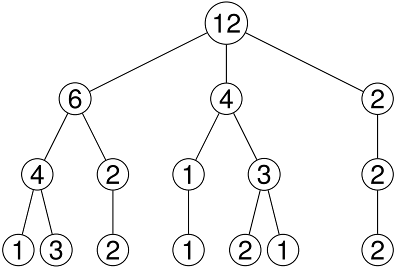

Figure: Tree in the class $\mathcal{T}_1(\overline{n})$, where $\overline{n} = (1, 3, 2, 1, 2, 1, 2) \in \mathrm{C}(12)$.

We define a weight for each tree $t \in \mathcal{T}_1(\overline{n})$. First, for any vertex v of t, let

$$\kappa_1(v) := \begin{cases} -\rho_1((\lambda_1, \ldots, \lambda_s)) & \text{if } v \text{ is multi-branching, and } \lambda_1, \ldots, \lambda_s,\ s \geq 2, \text{ are the values of the direct descendants of } v, \text{ from left to right,} \\ 1 & \text{otherwise.} \end{cases}$$

Then, for an admissible tree t we define

$$w_1(t) := \prod_{v \in \mathcal{V}(t)} \kappa_1(v), \qquad \mathcal{V}(t) : \text{set of vertices of } t.$$

Theorem

For each integer $n \geq 1$,

$$m_n = \sum_{\overline{n} \in \mathrm{C}(n)} \phi_1(\overline{n})\, \alpha(\overline{n}),$$

where

$$\phi_1(\overline{n}) := \sum_{t \in \mathcal{T}_1(\overline{n})} w_1(t).$$

Moreover, for each $n \geq 2$ we have

$$\sum_{\overline{n} \in \mathrm{C}(n)} \phi_1(\overline{n}) = 0.$$

The remaining relations between the sequences $(m_n)_{n=0}^{\infty}$, $(\alpha_n)_{n=0}^{\infty}$, $(\omega_n)_{n=0}^{\infty}$ are expressed in terms of certain classes of bi-colored trees.

References

For more details, see

1) P. Flajolet, *Combinatorial aspects of continued fractions*, Discr. Math. 32 (1980), 125–161.
2) A. López-García and V.A. Prokhorov, *On random polynomials generated by a symmetric three-term recurrence relation*, preprint arXiv:1804.03205.