# Nikishin systems on star-like sets: limiting functions in ratio asymptotics

A. López-García

University of Central Florida

Joint work with G. López Lagomasino

OPSFA, Hagenberg, Austria, July 25, 2019

# Multiorthogonal polynomials

Multiorthogonal polynomials are polynomials of a single complex variable that satisfy orthogonality conditions with respect to a collection of $p \geq 2$ measures.

# Multiorthogonal polynomials

Multiorthogonal polynomials are polynomials of a single complex variable that satisfy orthogonality conditions with respect to a collection of $p \geq 2$ measures.

$\mathbf{s} = (s_0, \ldots, s_{p-1})$: system of $p$ complex measures with compact support.

# Multiorthogonal polynomials

Multiorthogonal polynomials are polynomials of a single complex variable that satisfy orthogonality conditions with respect to a collection of $p \geq 2$ measures.

$\mathbf{s} = (s_0, \ldots, s_{p-1})$: system of $p$ complex measures with compact support.

$\mathbf{n} = (n_0, \ldots, n_{p-1})$: multi-index with $n_j \in \mathbb{Z}_{\geq 0}$ for all $0 \leq j \leq p-1$.

# Multiorthogonal polynomials

Multiorthogonal polynomials are polynomials of a single complex variable that satisfy orthogonality conditions with respect to a collection of $p \geq 2$ measures.

$\mathbf{s} = (s_0, \ldots, s_{p-1})$: system of $p$ complex measures with compact support.

$\mathbf{n} = (n_0, \ldots, n_{p-1})$: multi-index with $n_j \in \mathbb{Z}_{\geq 0}$ for all $0 \leq j \leq p-1$.

$P_{\mathbf{n}}(z)$ is multiorthogonal with respect to $\mathbf{s}$ and $\mathbf{n}$ if it is a non-zero polynomial of degree at most $|\mathbf{n}| = n_0 + \cdots + n_{p-1}$, satisfying

$$\int P_{\mathbf{n}}(z)\, z^j\, ds_0(z) = 0, \quad 0 \leq j \leq n_0 - 1,$$

$$\vdots$$

$$\int P_{\mathbf{n}}(z)\, z^j\, ds_{p-1}(z) = 0, \quad 0 \leq j \leq n_{p-1} - 1.$$

# Multiorthogonal polynomials

Multiorthogonal polynomials are polynomials of a single complex variable that satisfy orthogonality conditions with respect to a collection of $p \geq 2$ measures.

$\mathbf{s} = (s_0, \ldots, s_{p-1})$: system of $p$ complex measures with compact support.

$\mathbf{n} = (n_0, \ldots, n_{p-1})$: multi-index with $n_j \in \mathbb{Z}_{\geq 0}$ for all $0 \leq j \leq p-1$.

$P_{\mathbf{n}}(z)$ is multiorthogonal with respect to $\mathbf{s}$ and $\mathbf{n}$ if it is a non-zero polynomial of degree at most $|\mathbf{n}| = n_0 + \cdots + n_{p-1}$, satisfying

$$\int P_{\mathbf{n}}(z)\, z^j\, ds_0(z) = 0, \quad 0 \leq j \leq n_0 - 1,$$

$$\vdots$$

$$\int P_{\mathbf{n}}(z)\, z^j\, ds_{p-1}(z) = 0, \quad 0 \leq j \leq n_{p-1} - 1.$$

Finding $P_{\mathbf{n}}$ amounts to solve a homogeneous linear system of $|\mathbf{n}|$ equations with $|\mathbf{n}| + 1$ unknowns (the coefficients of $P_{\mathbf{n}}$), so a solution always exists.

If $\deg(P_{\mathbf{n}}) = |\mathbf{n}|$ for any solution, then $\mathbf{n}$ is called normal. In this case, the subspace of solutions has dimension 1.

# Nikishin systems on star-like sets

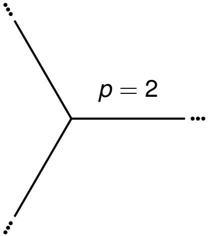
Nikishin systems of measures on the real line where introduced by E.M. Nikishin in 1980. In this talk we consider Nikishin systems on a star.

# Nikishin systems on star-like sets

Nikishin systems of measures on the real line where introduced by E.M. Nikishin in 1980. In this talk we consider Nikishin systems on a star.

Let $p \geq 1$, and let

$$S_+ = \{z \in \mathbb{C} : z^{p+1} \in [0, +\infty)\}.$$




A Nikishin system $\mathbf{s} = (s_0, \ldots, s_{p-1})$ on $S_+$ is a system of complex measures with common support on $S_+$, constructed as follows.

# Nikishin systems on star-like sets (cont.)

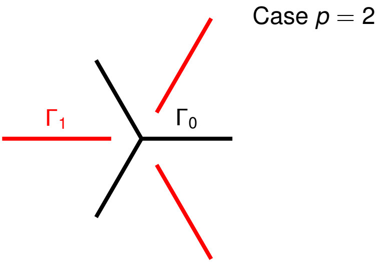
Let $(\Gamma_0, \Gamma_1, \ldots, \Gamma_{p-1})$ be a system of sets given by

$$\Gamma_j := T^{-1}(\Delta_j), \qquad j = 0, \ldots, p-1,$$

where $T(z) = z^{p+1}$ and $(\Delta_0, \Delta_1, \ldots, \Delta_{p-1})$ are compact intervals such that

$$\Delta_j \subset \begin{cases} [0, +\infty) & \text{if } j \text{ is even,} \\ (-\infty, 0] & \text{if } j \text{ is odd.} \end{cases}$$

We also assume that $\Delta_j \cap \Delta_{j+1} = \emptyset$ for all $j = 0, \ldots, p-2$.



# Nikishin systems on star-like sets (cont.)

Let $(\sigma_0, \ldots, \sigma_{p-1})$ be a system of measures supported on $(\Gamma_0, \ldots, \Gamma_{p-1})$, respectively, such that each $\sigma_j$ is positive, rotationally invariant, with infinitely many points in its support.

# Nikishin systems on star-like sets (cont.)

Let $(\sigma_0, \ldots, \sigma_{p-1})$ be a system of measures supported on $(\Gamma_0, \ldots, \Gamma_{p-1})$, respectively, such that each $\sigma_j$ is positive, rotationally invariant, with infinitely many points in its support.

Let

$$\widehat{\tau}(z) = \int \frac{d\tau(t)}{z-t}$$

denote the Stieltjes transform of a measure $\tau$.

# Nikishin systems on star-like sets (cont.)

Let $(\sigma_0, \dots, \sigma_{p-1})$ be a system of measures supported on $(\Gamma_0, \dots, \Gamma_{p-1})$, respectively, such that each $\sigma_j$ is positive, rotationally invariant, with infinitely many points in its support.

Let

$$\widehat{\tau}(z) = \int \frac{d\tau(t)}{z-t}$$

denote the Stieltjes transform of a measure $\tau$.

Suppose that $\tau_1, \tau_2, \dots, \tau_k$ are arbitrary complex measures with compact support, such that $\mathrm{supp}(\tau_j) \cap \mathrm{supp}(\tau_{j+1}) = \emptyset$ for all $j = 1, \dots, k-1$.

# Nikishin systems on star-like sets (cont.)

Let $(\sigma_0, \ldots, \sigma_{p-1})$ be a system of measures supported on $(\Gamma_0, \ldots, \Gamma_{p-1})$, respectively, such that each $\sigma_j$ is positive, rotationally invariant, with infinitely many points in its support.

Let

$$\widehat{\tau}(z) = \int \frac{d\tau(t)}{z - t}$$

denote the Stieltjes transform of a measure $\tau$.

Suppose that $\tau_1, \tau_2, \ldots, \tau_k$ are arbitrary complex measures with compact support, such that $\mathrm{supp}(\tau_j) \cap \mathrm{supp}(\tau_{j+1}) = \emptyset$ for all $j = 1, \ldots, k-1$.

Inductively, one defines:

$$\begin{aligned}
\langle \tau_1 \rangle &:= \tau_1 \\
\langle \tau_1, \tau_2 \rangle &:= \widehat{\tau}_2\, \tau_1 \\
\langle \tau_1, \tau_2, \tau_3 \rangle &:= \langle \tau_1, \langle \tau_2, \tau_3 \rangle \rangle \\
&\vdots \\
\langle \tau_1, \tau_2, \ldots, \tau_k \rangle &:= \langle \tau_1, \langle \tau_2, \ldots, \tau_k \rangle \rangle
\end{aligned}$$

# Nikishin systems on star-like sets (cont.)

$(s_0, \ldots, s_{p-1}) = \mathcal{N}(\sigma_0, \ldots, \sigma_{p-1})$ is the **Nikishin system** generated by $(\sigma_0, \ldots, \sigma_{p-1})$, if

$$s_j = \langle \sigma_0, \ldots, \sigma_j \rangle, \quad \text{for all } 0 \leq j \leq p-1.$$

# Nikishin systems on star-like sets (cont.)

$(s_0, \ldots, s_{p-1}) = \mathcal{N}(\sigma_0, \ldots, \sigma_{p-1})$ is the **Nikishin system** generated by $(\sigma_0, \ldots, \sigma_{p-1})$, if

$$s_j = \langle \sigma_0, \ldots, \sigma_j \rangle, \quad \text{for all } 0 \leq j \leq p-1.$$

Let $\sigma_j^*$ be the push-forward of the measure $\sigma_j$ on $\Gamma_j$ under the transformation $T(z) = z^{p+1}$, that is, $\sigma_j^*$ is the measure on $\Delta_j$ such that

$$\sigma_j^*(E) = \sigma_j(\{z \in \mathbb{C} : T(z) \in E\}), \qquad E \subset \Delta_j.$$

# Multiorthogonal polynomials and functions of the second kind

Let $(s_0, \ldots, s_{p-1}) = \mathcal{N}(\sigma_0, \ldots, \sigma_{p-1})$.

# Multiorthogonal polynomials and functions of the second kind

Let $(s_0, \ldots, s_{p-1}) = \mathcal{N}(\sigma_0, \ldots, \sigma_{p-1})$.

**Definition of multiple orthogonal polynomials**

Let $(Q_n)_{n=0}^{\infty}$ be the sequence of **monic** polynomials of lowest degree that satisfy the multiple orthogonality conditions

$$\int_{\Gamma_0} Q_n(z)\, z^l\, ds_j(z) = 0, \qquad l = 0, \ldots, \left\lfloor \frac{n-j-1}{p} \right\rfloor,$$

for each $j = 0, \ldots, p-1$.

In terms of the notation used before, we are considering here multi-indices $\mathbf{n} = (n_0, \ldots, n_{p-1})$ such that

$$n_0 \geq n_1 \geq n_2 \geq \cdots \geq n_{p-1} \geq n_0 - 1,$$

so we identify $\mathbf{n}$ with $|\mathbf{n}|$ and write $Q_{|\mathbf{n}|}$ instead of $Q_{\mathbf{n}}$.

# Multiorthogonal polynomials and functions of the second kind

Let $(s_0, \ldots, s_{p-1}) = \mathcal{N}(\sigma_0, \ldots, \sigma_{p-1})$.

**Definition of multiple orthogonal polynomials**

Let $(Q_n)_{n=0}^{\infty}$ be the sequence of **monic** polynomials of lowest degree that satisfy the multiple orthogonality conditions

$$\int_{\Gamma_0} Q_n(z)\, z^l\, ds_j(z) = 0, \qquad l = 0, \ldots, \left\lfloor \frac{n-j-1}{p} \right\rfloor,$$

for each $j = 0, \ldots, p-1$.

In terms of the notation used before, we are considering here multi-indices $\mathbf{n} = (n_0, \ldots, n_{p-1})$ such that

$$n_0 \geq n_1 \geq n_2 \geq \cdots \geq n_{p-1} \geq n_0 - 1,$$

so we identify $\mathbf{n}$ with $|\mathbf{n}|$ and write $Q_{|\mathbf{n}|}$ instead of $Q_{\mathbf{n}}$.

**Definition of functions of the second kind**

Set $\Psi_{n,0} := Q_n$, and let

$$\Psi_{n,k}(z) := \int_{\Gamma_{k-1}} \frac{\Psi_{n,k-1}(t)}{z-t}\, d\sigma_{k-1}(t), \qquad k = 1, \ldots, p.$$

# Algebraic properties

**Theorem (López-García, Miña-Díaz)**

*We have*

1) $Q_n$ *has maximal degree* $n$.
2) *If* $n \equiv \ell \mod (p+1)$, $0 \leq \ell \leq p$, *then* $Q_n(z) = z^{\ell} q_n(z^{p+1})$, *and* $q_n$ *has exactly* $\frac{n-\ell}{p+1}$ *simple zeros in the interior of* $\Delta_0$.
3) *The zeros of* $q_n$ *and* $q_{n+1}$ *interlace on* $\Delta_0$.
4) *The sequences* $(Q_n(z))_{n=0}^{\infty}$, $(\Psi_{n,k}(z))_{n=0}^{\infty}$, $1 \leq k \leq p$, *satisfy a linear difference equation of the form*
$$y_{n+1} = z y_n - a_n \, y_{n-p}, \qquad n \geq p, \tag{1}$$
*where* $a_n > 0$ *for all* $n \geq p$. *These* $p+1$ *sequences form a basis for the space of solutions of* (1).

# Ratio asymptotics

From now on, we assume that the Nikishin system satisfies the following property (**P**):

**For each $0 \leq j \leq p-1$, the measure $\sigma_j^*$ has positive Radon-Nikodym derivative a.e. on $\Delta_j$.**

# Ratio asymptotics

From now on, we assume that the Nikishin system satisfies the following property (**P**):

**For each $0 \leq j \leq p-1$, the measure $\sigma_j^*$ has positive Radon-Nikodym derivative a.e. on $\Delta_j$.**

Assuming (**P**), in a joint work with G. López Lagomasino, the following was proved:

1) For each fixed $0 \leq \rho \leq p(p+1)-1$, the following limits hold, uniformly on compact subsets of the indicated regions:

$$\lim_{n\to\infty} \frac{Q_{np(p+1)+\rho+1}(z)}{Q_{np(p+1)+\rho}(z)} \qquad z \in \mathbb{C} \setminus (\Gamma_0 \cup \{0\}),$$

$$\lim_{n\to\infty} \frac{\Psi_{np(p+1)+\rho+1,k}(z)}{\Psi_{np(p+1)+\rho,k}(z)} \qquad z \in \mathbb{C} \setminus (\Gamma_{k-1} \cup \Gamma_k \cup \{0\}), \quad 1 \leq k \leq p,$$

where $\Gamma_p = \emptyset$.

# Ratio asymptotics

From now on, we assume that the Nikishin system satisfies the following property (**P**):

**For each $0 \leq j \leq p-1$, the measure $\sigma_j^*$ has positive Radon-Nikodym derivative a.e. on $\Delta_j$.**

Assuming (**P**), in a joint work with G. López Lagomasino, the following was proved:

1) For each fixed $0 \leq \rho \leq p(p+1)-1$, the following limits hold, uniformly on compact subsets of the indicated regions:

$$\lim_{n\to\infty} \frac{Q_{np(p+1)+\rho+1}(z)}{Q_{np(p+1)+\rho}(z)} \qquad z \in \mathbb{C} \setminus (\Gamma_0 \cup \{0\}),$$

$$\lim_{n\to\infty} \frac{\Psi_{np(p+1)+\rho+1,k}(z)}{\Psi_{np(p+1)+\rho,k}(z)} \qquad z \in \mathbb{C} \setminus (\Gamma_{k-1} \cup \Gamma_k \cup \{0\}), \quad 1 \leq k \leq p,$$

where $\Gamma_p = \emptyset$.

2) For any fixed $0 \leq \rho \leq p(p+1)-1$,

$$\lim_{n\to\infty} a_{np(p+1)+\rho} = a^{(\rho)}.$$

# Ratio asymptotics

From now on, we assume that the Nikishin system satisfies the following property (**P**):

**For each** $0 \le j \le p-1$**, the measure** $\sigma_j^*$ **has positive Radon-Nikodym derivative a.e. on** $\Delta_j$**.**

Assuming (**P**), in a joint work with G. López Lagomasino, the following was proved:

1) For each fixed $0 \le \rho \le p(p+1)-1$, the following limits hold, uniformly on compact subsets of the indicated regions:

$$\lim_{n\to\infty} \frac{Q_{np(p+1)+\rho+1}(z)}{Q_{np(p+1)+\rho}(z)} \qquad z \in \mathbb{C} \setminus (\Gamma_0 \cup \{0\}),$$

$$\lim_{n\to\infty} \frac{\Psi_{np(p+1)+\rho+1,k}(z)}{\Psi_{np(p+1)+\rho,k}(z)} \qquad z \in \mathbb{C} \setminus (\Gamma_{k-1} \cup \Gamma_k \cup \{0\}), \quad 1 \le k \le p,$$

where $\Gamma_p = \emptyset$.

2) For any fixed $0 \le \rho \le p(p+1)-1$,

$$\lim_{n\to\infty} a_{np(p+1)+\rho} = a^{(\rho)}.$$

We describe these limits in terms of certain algebraic functions defined on a Riemann surface.

# Riemann surface of genus zero and conformal mappings

Let $\mathcal{R}$ denote the compact Riemann surface

$$\mathcal{R} = \overline{\bigcup_{k=0}^{p} \mathcal{R}_k}$$

formed by the $p+1$ consecutively "glued" sheets

$$\mathcal{R}_0 := \overline{\mathbb{C}} \setminus \Delta_0, \qquad \mathcal{R}_k := \overline{\mathbb{C}} \setminus (\Delta_{k-1} \cup \Delta_k), \quad k = 1, \ldots, p-1, \qquad \mathcal{R}_p := \overline{\mathbb{C}} \setminus \Delta_{p-1},$$

where the upper and lower banks of the common slits of two neighboring sheets are identified. This surface has genus zero.

# Riemann surface of genus zero and conformal mappings

Let $\mathcal{R}$ denote the compact Riemann surface

$$\mathcal{R} = \overline{\bigcup_{k=0}^{p} \mathcal{R}_k}$$

formed by the $p+1$ consecutively "glued" sheets

$$\mathcal{R}_0 := \overline{\mathbb{C}} \setminus \Delta_0, \qquad \mathcal{R}_k := \overline{\mathbb{C}} \setminus (\Delta_{k-1} \cup \Delta_k), \quad k = 1, \ldots, p-1, \qquad \mathcal{R}_p := \overline{\mathbb{C}} \setminus \Delta_{p-1},$$

where the upper and lower banks of the common slits of two neighboring sheets are identified. This surface has genus zero.

Given $l \in \{1, \ldots, p\}$, let $\varphi^{(l)} : \mathcal{R} \longrightarrow \overline{\mathbb{C}}$ denote a conformal mapping whose divisor consists of a simple zero at the point $\infty^{(0)} \in \mathcal{R}_0$ and a simple pole at the point $\infty^{(l)} \in \mathcal{R}_l$. For each $k = 0, \ldots, p$, let

$$\varphi_k^{(l)} := \varphi^{(l)}|_{\mathcal{R}_k}.$$

We normalize $\varphi^{(l)}$ so that

$$\prod_{k=0}^{p} \varphi_k^{(l)} \equiv \pm 1, \qquad \omega_l := \lim_{z \to \infty} z \varphi_0^{(l)}(z) > 0.$$

# Asymptotic formulae

## Theorem (López-García, López Lagomasino)

*Assume that (**P**) holds. The following formulas hold, uniformly on compact subsets of the indicated regions:*

1) *For each fixed* $0 \leq \rho \leq p(p+1)-1$,

$$\lim_{n\to\infty} \frac{Q_{np(p+1)+\rho+1}(z)}{Q_{np(p+1)+\rho}(z)} = \frac{z}{1 + a^{(\rho)}\,\omega_l^{-1}\,\varphi_0^{(l)}(z^{p+1})}, \qquad z \in \mathbb{C} \setminus (\Gamma_0 \cup \{0\}),$$

*where* $l = l(\rho)$ *is the integer satisfying the conditions* $1 \leq l \leq p$ *and* $l - 1 \equiv \rho \mod p$. *Convergence takes place in* $\mathbb{C} \setminus \Gamma_0$ *if* $\rho \not\equiv p \mod (p+1)$.

# Asymptotic formulae

## Theorem (López-García, López Lagomasino)

*Assume that (**P**) holds. The following formulas hold, uniformly on compact subsets of the indicated regions:*

1) *For each fixed* $0 \le \rho \le p(p+1)-1$,

$$\lim_{n\to\infty} \frac{Q_{np(p+1)+\rho+1}(z)}{Q_{np(p+1)+\rho}(z)} = \frac{z}{1 + a^{(\rho)}\, \omega_l^{-1}\, \varphi_0^{(l)}(z^{p+1})}, \qquad z \in \mathbb{C} \setminus (\Gamma_0 \cup \{0\}),$$

*where* $l = l(\rho)$ *is the integer satisfying the conditions* $1 \le l \le p$ *and* $l - 1 \equiv \rho \mod p$. *Convergence takes place in* $\mathbb{C} \setminus \Gamma_0$ *if* $\rho \not\equiv p \mod (p+1)$.

2) *For each fixed* $0 \le \rho \le p(p+1)-1$ *and* $1 \le k \le p$,

$$\lim_{n\to\infty} \frac{\Psi_{np(p+1)+\rho+1,k}(z)}{\Psi_{np(p+1)+\rho,k}(z)} = \frac{z}{1 + a^{(\rho)}\, \omega_l^{-1}\, \varphi_k^{(l)}(z^{p+1})}, \qquad z \in \mathbb{C} \setminus (\Gamma_{k-1} \cup \Gamma_k \cup \{0\}),$$

*with* $l = l(\rho)$ *as in 1), and* $\Gamma_p = \emptyset$.

# Asymptotic formulae

## Theorem (López-García, López Lagomasino)

*Assume that (**P**) holds. The following formulas hold, uniformly on compact subsets of the indicated regions:*

1) *For each fixed* $0 \leq \rho \leq p(p+1)-1$,

$$\lim_{n\to\infty} \frac{Q_{np(p+1)+\rho+1}(z)}{Q_{np(p+1)+\rho}(z)} = \frac{z}{1 + a^{(\rho)}\,\omega_l^{-1}\,\varphi_0^{(l)}(z^{p+1})}, \qquad z \in \mathbb{C} \setminus (\Gamma_0 \cup \{0\}),$$

*where* $l = l(\rho)$ *is the integer satisfying the conditions* $1 \leq l \leq p$ *and* $l - 1 \equiv \rho \mod p$. *Convergence takes place in* $\mathbb{C} \setminus \Gamma_0$ *if* $\rho \not\equiv p \mod (p+1)$.

2) *For each fixed* $0 \leq \rho \leq p(p+1)-1$ *and* $1 \leq k \leq p$,

$$\lim_{n\to\infty} \frac{\Psi_{np(p+1)+\rho+1,k}(z)}{\Psi_{np(p+1)+\rho,k}(z)} = \frac{z}{1 + a^{(\rho)}\,\omega_l^{-1}\,\varphi_k^{(l)}(z^{p+1})}, \qquad z \in \mathbb{C} \setminus (\Gamma_{k-1} \cup \Gamma_k \cup \{0\}),$$

*with* $l = l(\rho)$ *as in 1), and* $\Gamma_p = \emptyset$.

We extend the sequence $(a^{(\rho)})_{\rho=0}^{p(p+1)-1}$, periodically in $\mathbb{Z}$ with period $p(p+1)$, so that

$$a^{(\rho)} = a^{(\rho+p(p+1))}, \qquad \text{for all } \rho \in \mathbb{Z}.$$

# The asymptotic values $a^{(\rho)}$

## Theorem (López-García, López Lagomasino)

*The following properties stated in 1)–4) below hold for each* $0 \leq \rho \leq p(p+1)-1$*:*

1) $a^{(\rho)} > 0$.

# The asymptotic values $a^{(\rho)}$

## Theorem (López-García, López Lagomasino)

*The following properties stated in 1)–4) below hold for each* $0 \leq \rho \leq p(p+1)-1$*:*

1) $a^{(\rho)} > 0$.
2) *The set of p values* $\{a^{(\rho+m(p+1))}\}_{m=0}^{p-1}$ *is formed by distinct quantities.*

# The asymptotic values $a^{(\rho)}$

## Theorem (López-García, López Lagomasino)

*The following properties stated in 1)–4) below hold for each* $0 \leq \rho \leq p(p+1)-1$*:*

1) $a^{(\rho)} > 0$.
2) *The set of p values* $\{a^{(\rho+m(p+1))}\}_{m=0}^{p-1}$ *is formed by distinct quantities.*
3) *The following relation holds:*

$$\sum_{i=\rho}^{\rho+p-1} a^{(i)} = \sum_{i=\rho+p+1}^{\rho+2p} a^{(i)}.$$

# The asymptotic values $a^{(\rho)}$

## Theorem (López-García, López Lagomasino)

*The following properties stated in 1)–4) below hold for each* $0 \leq \rho \leq p(p+1)-1$*:*

1) $a^{(\rho)} > 0$.
2) *The set of p values* $\{a^{(\rho+m(p+1))}\}_{m=0}^{p-1}$ *is formed by distinct quantities.*
3) *The following relation holds:*

$$\sum_{i=\rho}^{\rho+p-1} a^{(i)} = \sum_{i=\rho+p+1}^{\rho+2p} a^{(i)}.$$

4) *We have*

$$a^{(\rho)} = -\frac{\omega_l}{\varphi_k^{(l)}(0)} \tag{2}$$

*where* $(k,l) = (k(\rho), l(\rho))$ *is the unique pair of integers satisfying the conditions* $0 \leq k \leq p$*,* $\rho \equiv k-1 \mod (p+1)$*, and* $1 \leq l \leq p$*,* $\rho \equiv l-1 \mod p$*.*

# The asymptotic values $a^{(\rho)}$

## Theorem (López-García, López Lagomasino)

*The following properties stated in 1)–4) below hold for each* $0 \leq \rho \leq p(p+1)-1$*:*

1) $a^{(\rho)} > 0$.
2) *The set of p values* $\{a^{(\rho+m(p+1))}\}_{m=0}^{p-1}$ *is formed by distinct quantities.*
3) *The following relation holds:*

$$\sum_{i=\rho}^{\rho+p-1} a^{(i)} = \sum_{i=\rho+p+1}^{\rho+2p} a^{(i)}.$$

4) *We have*

$$a^{(\rho)} = -\frac{\omega_l}{\varphi_k^{(l)}(0)} \qquad (2)$$

*where* $(k,l) = (k(\rho), l(\rho))$ *is the unique pair of integers satisfying the conditions* $0 \leq k \leq p$, $\rho \equiv k-1 \mod (p+1)$, *and* $1 \leq l \leq p$, $\rho \equiv l-1 \mod p$.

5) *Assume that* $0 \in \Delta_k$ *for some* $0 \leq k \leq p-1$*. Then, for any* $0 \leq \rho \leq p(p+1)-1$ *such that* $\rho \equiv k-1 \mod (p+1)$, *we have* $a^{(\rho-p)} = a^{(\rho)}$*. If* $0 \notin \Delta_k$ *for all* $0 \leq k \leq p-1$, *then for any* $0 \leq \rho \leq p(p+1)-1$, *the set of* $p+1$ *values* $\{a^{(\rho+mp)}\}_{m=0}^{p}$ *is formed by distinct quantities.*

# A conformal mapping

The function $\eta^{(\rho)} : \mathcal{R} \longrightarrow \overline{\mathbb{C}}$ defined by

$$\eta^{(\rho)}(z) = \frac{1}{1 + a^{(\rho)}\, \omega_{l(\rho)}^{-1}\, \varphi^{(l(\rho))}(z)}$$

is conformal since it is the composition of $\varphi^{(l(\rho))}$ with the Möbius transformation $w \mapsto (1 + a^{(\rho)}\, \omega_{l(\rho)}^{-1}\, w)^{-1}$.

# A conformal mapping

The function $\eta^{(\rho)} : \mathcal{R} \longrightarrow \overline{\mathbb{C}}$ defined by

$$\eta^{(\rho)}(z) = \frac{1}{1 + a^{(\rho)}\, \omega_{l(\rho)}^{-1}\, \varphi^{(l(\rho))}(z)}$$

is conformal since it is the composition of $\varphi^{(l(\rho))}$ with the Möbius transformation $w \mapsto (1 + a^{(\rho)}\, \omega_{l(\rho)}^{-1}\, w)^{-1}$.

As a consequence of (2) and the definition of $\varphi^{(l(\rho))}$, the function $\eta^{(\rho)} : \mathcal{R} \longrightarrow \overline{\mathbb{C}}$ is characterized as the unique conformal mapping with a simple zero at $\infty^{(l(\rho))}$, a simple pole at $0 \in \mathcal{R}_{k(\rho)}$, and satisfying $\eta^{(\rho)}(\infty^{(0)}) = 1$.

# A conformal mapping

The function $\eta^{(\rho)} : \mathcal{R} \longrightarrow \overline{\mathbb{C}}$ defined by

$$\eta^{(\rho)}(z) = \frac{1}{1 + a^{(\rho)}\, \omega_{l(\rho)}^{-1}\, \varphi^{(l(\rho))}(z)}$$

is conformal since it is the composition of $\varphi^{(l(\rho))}$ with the Möbius transformation $w \mapsto (1 + a^{(\rho)}\, \omega_{l(\rho)}^{-1}\, w)^{-1}$.

As a consequence of (2) and the definition of $\varphi^{(l(\rho))}$, the function $\eta^{(\rho)} : \mathcal{R} \longrightarrow \overline{\mathbb{C}}$ is characterized as the unique conformal mapping with a simple zero at $\infty^{(l(\rho))}$, a simple pole at $0 \in \mathcal{R}_{k(\rho)}$, and satisfying $\eta^{(\rho)}(\infty^{(0)}) = 1$.

Then, the asymptotic formulas take the simpler form

$$\lim_{n\to\infty} \frac{Q_{np(p+1)+\rho+1}(z)}{Q_{np(p+1)+\rho}(z)} = z\eta_0^{(\rho)}(z^{p+1}),$$

$$\lim_{n\to\infty} \frac{\Psi_{np(p+1)+\rho+1,k}(z)}{\Psi_{np(p+1)+\rho,k}(z)} = z\eta_k^{(\rho)}(z^{p+1}), \quad 1 \leq k \leq p,$$

where $\eta_k^{(\rho)} = \eta^{(\rho)}|_{\mathcal{R}_k}$.

## Main ideas in the proof

The proof is based on the simultaneous analysis of $p$ sequences of ratios

$$\left\{ \frac{P_{n+1,k}(z)}{P_{n,k}(z)} \right\}_{n=0}^{\infty} \qquad k = 0, \ldots, p-1,$$

constructed out of the polynomials $Q_n$ and the functions of the second kind $\Psi_{n,k}$.

# Main ideas in the proof

The proof is based on the simultaneous analysis of $p$ sequences of ratios

$$\left\{ \frac{P_{n+1,k}(z)}{P_{n,k}(z)} \right\}_{n=0}^{\infty} \qquad k = 0, \ldots, p-1,$$

constructed out of the polynomials $Q_n$ and the functions of the second kind $\Psi_{n,k}$. By definition, $P_{n,0} = q_n$ is the monic polynomial in the relation

$$Q_n(z) = z^{\ell} q_n(z^{p+1}), \qquad n \equiv \ell \mod (p+1), \quad 0 \leq \ell \leq p.$$

# Main ideas in the proof

The proof is based on the simultaneous analysis of $p$ sequences of ratios

$$\left\{ \frac{P_{n+1,k}(z)}{P_{n,k}(z)} \right\}_{n=0}^{\infty} \qquad k = 0, \ldots, p-1,$$

constructed out of the polynomials $Q_n$ and the functions of the second kind $\Psi_{n,k}$.

By definition, $P_{n,0} = q_n$ is the monic polynomial in the relation

$$Q_n(z) = z^{\ell} q_n(z^{p+1}), \qquad n \equiv \ell \mod (p+1), \quad 0 \leq \ell \leq p.$$

For each $1 \leq k \leq p-1$, by definition $P_{n,k}$ is the monic polynomial whose zeros are the zeros in $\mathrm{int}(\Delta_k)$ of the function $\psi_{n,k} \in \mathcal{H}(\mathbb{C} \setminus \Delta_{k-1})$ given by the relation

$$\Psi_{n,k}(z) = z^{\ell-k} \psi_{n,k}(z^{p+1}).$$

In a previous work, we proved under the hypothesis (**P**) the existence of the following limits, for each fixed $0 \leq \rho \leq p(p+1) - 1$ and $0 \leq k \leq p - 1$,

$$\lim_{\lambda \to \infty} \frac{P_{\lambda p(p+1)+\rho+1,k}(z)}{P_{\lambda p(p+1)+\rho,k}(z)} = \widetilde{F}_k^{(\rho)}(z), \qquad z \in \mathbb{C} \setminus \Delta_k,$$

where $\widetilde{F}_k^{(\rho)}$ and $1/\widetilde{F}_k^{(\rho)}$ are analytic in $\mathbb{C} \setminus \Delta_k$.

In a previous work, we proved under the hypothesis (**P**) the existence of the following limits, for each fixed $0 \leq \rho \leq p(p+1)-1$ and $0 \leq k \leq p-1$,

$$\lim_{\lambda \to \infty} \frac{P_{\lambda p(p+1)+\rho+1,k}(z)}{P_{\lambda p(p+1)+\rho,k}(z)} = \widetilde{F}_k^{(\rho)}(z), \qquad z \in \mathbb{C} \setminus \Delta_k,$$

where $\widetilde{F}_k^{(\rho)}$ and $1/\widetilde{F}_k^{(\rho)}$ are analytic in $\mathbb{C} \setminus \Delta_k$.

**Key fact**: $\psi_{n,k}$ satisfies the following orthogonality conditions with respect to **varying measures**:

If $k \geq \ell$, then

$$\int_{\Delta_k} \psi_{n,k}(t)\, t^l \, \frac{d\sigma_k^*(t)}{P_{n,k+1}(t)} = 0, \qquad l = 0, \ldots, \deg(P_{n,k}) - 1,$$

and if $k < \ell$, then

$$\int_{\Delta_k} \psi_{n,k}(t)\, t^l \, \frac{t\, d\sigma_k^*(t)}{P_{n,k+1}(t)} = 0, \qquad l = 0, \ldots, \deg(P_{n,k}) - 1,$$

where $n \equiv \ell \mod (p+1)$, $0 \leq \ell \leq p$.

**Key fact**: Let $0 \le \rho \le p(p+1)-1$ be fixed, and let $0 \le \ell \le p$ be the remainder in the division of $\rho$ by $p+1$. There exist positive constants $c_k^{(\rho)}$ so that the collection of functions $F_k^{(\rho)}(z) = c_k^{(\rho)} \widetilde{F}_k^{(\rho)}(z), 0 \le k \le p-1$ satisfies a system of boundary value equations:

1) If $0 \le \ell \le p-1$, the system is

$$\frac{|F_k^{(\rho)}(x)|^2}{|F_{k-1}^{(\rho)}(x)||F_{k+1}^{(\rho)}(x)|} = 1, \quad x \in \Delta_k, \quad 0 \le k \le p-1, \quad k \ne \ell, \ell+1,$$

$$\frac{|F_k^{(\rho)}(x)|^2 \, |x|}{|F_{k-1}^{(\rho)}(x)||F_{k+1}^{(\rho)}(x)|} = 1, \quad x \in \Delta_k \setminus \{0\}, \quad k = \ell,$$

$$\frac{|F_k^{(\rho)}(x)|^2}{|x||F_{k-1}^{(\rho)}(x)||F_{k+1}^{(\rho)}(x)|} = 1, \quad x \in \Delta_k \setminus \{0\}, \quad k = \ell+1.$$

(The last equation is dropped if $\ell = p-1$.)

**Key fact**: Let $0 \leq \rho \leq p(p+1)-1$ be fixed, and let $0 \leq \ell \leq p$ be the remainder in the division of $\rho$ by $p+1$. There exist positive constants $c_k^{(\rho)}$ so that the collection of functions $F_k^{(\rho)}(z) = c_k^{(\rho)} \widetilde{F}_k^{(\rho)}(z), 0 \leq k \leq p-1$ satisfies a system of boundary value equations:

1) If $0 \leq \ell \leq p-1$, the system is

$$\frac{|F_k^{(\rho)}(x)|^2}{|F_{k-1}^{(\rho)}(x)||F_{k+1}^{(\rho)}(x)|} = 1, \quad x \in \Delta_k, \quad 0 \leq k \leq p-1, \quad k \neq \ell, \ell+1,$$

$$\frac{|F_k^{(\rho)}(x)|^2 \, |x|}{|F_{k-1}^{(\rho)}(x)||F_{k+1}^{(\rho)}(x)|} = 1, \quad x \in \Delta_k \setminus \{0\}, \quad k = \ell,$$

$$\frac{|F_k^{(\rho)}(x)|^2}{|x||F_{k-1}^{(\rho)}(x)||F_{k+1}^{(\rho)}(x)|} = 1, \quad x \in \Delta_k \setminus \{0\}, \quad k = \ell+1.$$

(The last equation is dropped if $\ell = p-1$.)

2) For $\ell = p$, the system is

$$\frac{|F_0^{(\rho)}(x)|^2}{|x||F_1^{(\rho)}(x)|} = 1, \quad x \in \Delta_0 \setminus \{0\},$$

$$\frac{|F_k^{(\rho)}(x)|^2}{|F_{k-1}^{(\rho)}(x)||F_{k+1}^{(\rho)}(x)|} = 1, \quad x \in \Delta_k, \quad 1 \leq k \leq p-1.$$

In the above equations we use the convention $F_{-1}^{(\rho)} \equiv F_p^{(\rho)} \equiv 1$.

We consider the products

$$f_k^{(\rho)} := \prod_{j=0}^{p} F_k^{(\rho+j)}.$$

Then

1) $|f_k^{(\rho)}|$ has continuous and non-vanishing boundary values on $\Delta_k$ and

$$\frac{|f_k^{(\rho)}(x)|^2}{|f_{k-1}^{(\rho)}(x)||f_{k+1}^{(\rho)}(x)|} = 1, \qquad \text{for all } x \in \Delta_k, \qquad 0 \leq k \leq p-1,$$

   so the possible singularities at 0 present in the system of boundary value equations are now eliminated.

2) If $l - 1 \equiv \rho \mod p$, $1 \leq l \leq p$, then

$$f_k^{(\rho)}(z) = \begin{cases} c_{k,\rho}\, z + O(1), & 0 \leq k \leq l-1, \\ c_{k,\rho} + O(z^{-1}), & l \leq k \leq p-1, \end{cases}$$

   where $c_{k,\rho} > 0$ for all $0 \leq k \leq p-1$.

We consider the products

$$f_k^{(\rho)} := \prod_{j=0}^{p} F_k^{(\rho+j)}.$$

Then

1) $|f_k^{(\rho)}|$ has continuous and non-vanishing boundary values on $\Delta_k$ and

$$\frac{|f_k^{(\rho)}(x)|^2}{|f_{k-1}^{(\rho)}(x)||f_{k+1}^{(\rho)}(x)|} = 1, \qquad \text{for all } x \in \Delta_k, \qquad 0 \le k \le p-1,$$

so the possible singularities at 0 present in the system of boundary value equations are now eliminated.

2) If $l - 1 \equiv \rho \mod p$, $1 \le l \le p$, then

$$f_k^{(\rho)}(z) = \begin{cases} c_{k,\rho}\, z + O(1), & 0 \le k \le l-1, \\ c_{k,\rho} + O(z^{-1}), & l \le k \le p-1, \end{cases}$$

where $c_{k,\rho} > 0$ for all $0 \le k \le p-1$.

This implies by a result of Aptekarev-López-Rocha the **fundamental relation**

$$f_k^{(\rho)} = \prod_{j=0}^{p} F_k^{(\rho+j)} = \operatorname{sg}\left(\prod_{\nu=k+1}^{p} \varphi_\nu^{(l)}(\infty)\right) \prod_{\nu=k+1}^{p} \varphi_\nu^{(l)}.$$

In particular,

$$f_k^{(\rho)} = f_k^{(\rho+p)} \qquad \text{for any } \rho. \tag{3}$$

In particular,

$$f_k^{(\rho)} = f_k^{(\rho+p)} \qquad \text{for any } \rho. \tag{3}$$

Since

$$\widetilde{F}_0^{(\rho)}(z) = \begin{cases} 1 - a^{(\rho)} z^{-1} + O(z^{-2}), & \text{if } \rho \not\equiv p \mod (p+1), \\ z - a^{(\rho)} + O(z^{-1}), & \text{if } \rho \equiv p \mod (p+1), \end{cases}$$

equation (3) for $k = 0$ immediately gives

$$\sum_{i=\rho}^{\rho+p-1} a^{(i)} = \sum_{i=\rho+p+1}^{\rho+2p} a^{(i)}.$$

We have

$$\widetilde{F}_k^{(\rho)}(z) = \lim_{\lambda\to\infty} \frac{P_{\lambda p(p+1)+\rho+1,k}(z)}{P_{\lambda p(p+1)+\rho,k}(z)} = \begin{cases} \prod_{j=0}^{k} \widetilde{\eta}_j^{(\rho)}(z), & \text{if } \ 0 \leq k < k(\rho), \\ z\prod_{j=0}^{k} \widetilde{\eta}_j^{(\rho)}(z), & \text{if } \ k(\rho) \leq k \leq p-1, \end{cases}$$

where

$$\eta_j^{(\rho)}(z) = \frac{1}{1 + a^{(\rho)}\, \omega_{l(\rho)}^{-1}\, \varphi_j^{(l(\rho))}(z)},$$

$\widetilde{\eta}_j^{(\rho)}$ is the normalization at $\infty$ of $\eta_j^{(\rho)}$, and $(k(\rho), l(\rho))$ is the pair of integers satisfying

$$k(\rho) - 1 \equiv \rho \mod p, \quad 0 \leq k(\rho) \leq p, \qquad l(\rho) - 1 \equiv \rho \mod (p+1), \quad 1 \leq l(\rho) \leq p.$$

Proof of

$$\widetilde{F}_0^{(\rho)}(z) = \frac{z}{1 + a^{(\rho)}\,\omega_l^{-1}\,\varphi_0^{(l(\rho))}(z)}$$

for $\rho \equiv p \mod (p+1)$.

Proof of

$$\widetilde{F}_0^{(\rho)}(z) = \frac{z}{1 + a^{(\rho)}\,\omega_l^{-1}\,\varphi_0^{(l(\rho))}(z)}$$

for $\rho \equiv p \mod (p+1)$.

The recurrence relation gives

$$a^{(\rho)} = (z - \widetilde{F}_0^{(\rho)}(z)) \prod_{i=\rho-p}^{\rho-1} \widetilde{F}_0^{(i)}(z)$$

so

$$\widetilde{F}_0^{(\rho)}(z) = z - \frac{a^{(\rho)}}{\prod_{i=\rho-p}^{\rho-1} \widetilde{F}_0^{(i)}(z)} = z - \frac{a^{(\rho)}\widetilde{F}_0^{(\rho)}(z)}{\prod_{i=\rho-p}^{\rho} \widetilde{F}_0^{(i)}(z)} = z - \frac{a^{(\rho)}\widetilde{F}_0^{(\rho)}(z)}{\widetilde{f}_0^{(\rho)}(z)}$$

therefore

$$\widetilde{F}_0^{(\rho)}(z) = \frac{z}{1 + \frac{a^{(\rho)}}{\widetilde{f}_0^{(\rho)}(z)}} = \frac{z}{1 + a^{(\rho)}\,\widetilde{\varphi}_0^{(l(\rho))}(z)}.$$