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Ratio asymptotic of Hermite–Padé orthogonal polynomials for Nikishin systems. II

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Abstract

We prove ratio asymptotic for sequences of multiple orthogonal polynomials with respect to a Nikishin system of measures $\mathcal{N}(\sigma_1, \ldots, \sigma_m)$ such that for each k, σ_k has constant sign on its support consisting on an interval $\tilde{\Delta}_k$, on which $|\sigma_k'| > 0$ almost everywhere, and a set without accumulation points in $\mathbb{R} \setminus \tilde{\Delta}_k$.

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1. Introduction

Let s be a finite positive Borel measure supported on a bounded interval Δ of the real line $\mathbb{R}$ such that $s' > 0$ almost everywhere on Δ and let $\{Q_n\}$, $n \in \mathbb{Z}_+$, be the corresponding sequence of monic orthogonal polynomials; that is, with leading coefficients equal to one. In a series of two papers (see [15] and [16]), E.A. Rakhmanov proved that under these conditions

$$\lim_{n \in \mathbb{Z}_+} \frac{Q_{n+1}(z)}{Q_n(z)} = \frac{\varphi(z)}{\varphi'(\infty)}, \qquad \mathcal{K} \subset \mathbb{C} \setminus \Delta \tag{1}$$

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(uniformly on each compact subset of $\mathbb{C} \setminus \Delta$), where $\varphi(z)$ denotes the conformal representation of $\overline{\mathbb{C}} \setminus \Delta$ onto $\{w: |w| > 1\}$ such that $\varphi(\infty) = \infty$ and $\varphi'(\infty) > 0$. This result attracted great attention because of its theoretical interest within the general theory of orthogonal polynomials and its applications to the theory of rational approximation of analytic functions. Simplified proofs of Rakhmanov's theorem may be found in [17] and [12].

This result has been extended in several directions. Orthogonal polynomials with respect to varying measures (depending on the degree of the polynomial) arise in the study of multi-point Padé approximation of Markov functions. In this context, in [10] and [11], an analogue of Rakhmanov's theorem for such sequences of orthogonal polynomials was proved. Recently, S.A. Denisov [4] (see also [13]) obtained a remarkable extension of Rakhmanov's result to the case when the support of s verifies $\operatorname{supp}(s) = \tilde{\Delta} \cup e \subset \mathbb{R}$, where $\tilde{\Delta}$ is a bounded interval, e is a set without accumulation points in $\mathbb{R} \setminus \tilde{\Delta}$, and $s' > 0$ a.e. on $\tilde{\Delta}$. A version for orthogonal polynomials with respect to varying Denisov type measures was given in [2].

Another direction of generalization is connected with multiple orthogonal polynomials. These are polynomials whose orthogonality relations are distributed between several measures. They appear as the common denominator of Hermite–Padé approximations of systems of Markov functions. An interesting class of such systems is formed by the so-called Nikishin systems of functions introduced in [14]. For Nikishin multiple orthogonal polynomials a version of Rakhmanov's theorem was proved in [1].

An elegant notation for Nikishin systems was proposed in [8]. Let σ_1, σ_2 be two finite Borel measures with constant sign, whose supports $\operatorname{supp}(\sigma_1)$, $\operatorname{supp}(\sigma_2)$ are contained in nonintersecting intervals of $\mathbb{R}$. Set

$$d\langle \sigma_1, \sigma_2 \rangle(x) = \int \frac{d\sigma_2(t)}{x-t}\, d\sigma_1(x) = \hat{\sigma}_2(x)\, d\sigma_1(x).$$

This expression defines a new measure with constant sign whose support coincides with that of σ_1. Whenever convenient, we use the differential notation of a measure.

Let $\Sigma = (\sigma_1, \ldots, \sigma_m)$ be a system of finite Borel measures on the real line with constant sign and compact support containing infinitely many points. Let $\operatorname{Co}(\operatorname{supp}(\sigma_k)) = \Delta_k$ denote the smallest interval which contains $\operatorname{supp}(\sigma_k)$. Assume that

$$\Delta_k \cap \Delta_{k+1} = \emptyset, \quad k = 1, \ldots, m-1.$$

By definition, $S = (s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$, where

$$s_1 = \sigma_1, \quad s_2 = \langle \sigma_1, \sigma_2 \rangle, \quad \ldots, \quad s_m = \big\langle \sigma_1, \langle \sigma_2, \ldots, \sigma_m \rangle \big\rangle \tag{2}$$

is called the *Nikishin system* of measures generated by Σ. The system $(\hat{s}_1, \ldots, \hat{s}_m)$ of Cauchy transforms of a Nikishin system of measures gives a Nikishin system of functions.

Fix a multi-index $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m$. The polynomial $Q_{\mathbf{n}}(x)$ is called an $\mathbf{n}$th *multiple orthogonal polynomial* with respect to S if it is not identically equal to zero, $\deg Q_{\mathbf{n}} \leqslant |\mathbf{n}| = n_1 + \cdots + n_m$, and

$$\int Q_{\mathbf{n}}(x) x^{\nu}\, ds_k(x) = 0, \quad \nu = 0, \ldots, n_k - 1,\ k = 1, \ldots, m. \tag{3}$$

In the sequel, we assume that $Q_{\mathbf{n}}$ is monic.

If (3) implies that $\deg Q_{\mathbf{n}} = |\mathbf{n}|$, the multi-index $\mathbf{n}$ is said to be **normal** and the corresponding monic multiple orthogonal polynomial is uniquely determined. In addition, if the zeros of $Q_{\mathbf{n}}$ are simple and lie in the interior of $\mathrm{Co}(\mathrm{supp}(\sigma_1))$ the multi-index is said to be **strongly normal**. (In relation to intervals of the real line the interior refers to the Euclidean topology of $\mathbb{R}$.) For Nikishin systems with $m = 1, 2, 3$, all multi-indices are strongly normal (see [5]). An open question is whether or not this is true for all $m \in \mathbb{N}$. The best result when $m \geqslant 4$ is that all

$$\mathbf{n} \in \mathbb{Z}_+^m(*) = \left\{\mathbf{n} \in \mathbb{Z}_+^m\colon\ \nexists 1 \leqslant i < j < k \leqslant m,\ \text{with } n_i < n_j < n_k\right\}$$

are strongly normal (see [6]).

In [1], a Rakhmanov type theorem was proved for Nikishin systems such that $|\sigma_k'| > 0$ a.e. on $\mathrm{Co}(\mathrm{supp}(\sigma_k))$, $k = 1, \ldots, m$, and sequences of multi-indices contained in

$$\mathbb{Z}_+^m(\circledast) = \left\{\mathbf{n} \in \mathbb{Z}_+^m\colon\ 1 \leqslant i < j \leqslant m \Rightarrow n_j \leqslant n_i + 1\right\}.$$

It is easy to see that $\mathbb{Z}_+^m(\circledast) \subset \mathbb{Z}_+^m(*)$. Here, we assume that $\mathrm{supp}(\sigma_k) = \tilde{\Delta}_k \cup e_k$, $k = 1, \ldots, m$, where $\tilde{\Delta}_k$ is a bounded interval of the real line, $|\sigma_k'| > 0$ a.e. on $\tilde{\Delta}_k$, e_k is a set without accumulation points in $\mathbb{R} \setminus \tilde{\Delta}_k$, and the sequence of multi-indices on which the limit is taken is in $\mathbb{Z}_+^m(*)$.

The proof of Theorem 1.1 below uses the construction of so-called second type functions. This construction depends on the relative value of the components of the multi-indices in $\mathbb{Z}_+^m(*)$ under consideration. A crucial step in our study consists in proving an interlacing property for the zeros of the second type functions corresponding to "consecutive" multi-indices (see Lemma 3.2). For this purpose, we need to be sure that the second type functions are built using the same procedure. To distinguish different classes of multi-indices which respond for the same construction of second type functions, we introduce the following definition.

Definition 1.1. Suppose that $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m$. Let $\tau_{\mathbf{n}}$ denote the permutation of $\{1, 2, \ldots, m\}$ given by

$$\tau_{\mathbf{n}}(i) = j \quad \text{if} \quad \begin{cases} n_j > n_k & \text{for } k < j,\ k \notin \{\tau_{\mathbf{n}}(1), \ldots, \tau_{\mathbf{n}}(i-1)\}, \\ n_j \geqslant n_k & \text{for } k > j,\ k \notin \{\tau_{\mathbf{n}}(1), \ldots, \tau_{\mathbf{n}}(i-1)\}. \end{cases}$$

In words, $\tau_{\mathbf{n}}(1)$ is the subindex of the first component of $\mathbf{n}$ (from left to right) which is greater or equal than the rest, $\tau_{\mathbf{n}}(2)$ is the subindex of the first component which is second largest, and so forth. For example, if $n_1 \geqslant \cdots \geqslant n_m$ then $\tau_{\mathbf{n}}$ is the identity.

Let τ denote a permutation of $\{1, 2, \ldots, m\}$. Set

$$\mathbb{Z}_+^m(*, \tau) = \left\{\mathbf{n} \in \mathbb{Z}_+^m(*)\colon\ \tau_{\mathbf{n}} = \tau\right\}.$$

Let $\mathbf{n} \in \mathbb{Z}_+^m$ and $l \in \{1, \ldots, m\}$. Define

$$\mathbf{n}_l := (n_1, \ldots, n_{l-1}, n_l + 1, n_{l+1}, \ldots, n_m).$$

Consider the $(m+1)$-sheeted Riemann surface

$$\mathcal{R} = \overline{\bigcup_{k=0}^{m} \mathcal{R}_k},$$

formed by the consecutively "glued" sheets

$$\mathcal{R}_0 := \overline{\mathbb{C}} \setminus \tilde{\Delta}_1, \qquad \mathcal{R}_k := \overline{\mathbb{C}} \setminus (\tilde{\Delta}_k \cup \tilde{\Delta}_{k+1}), \quad k = 1, \dots, m-1, \qquad \mathcal{R}_m = \overline{\mathbb{C}} \setminus \tilde{\Delta}_m,$$

where the upper and lower banks of the slits of two neighboring sheets are identified. Fix $l \in \{1, \dots, m\}$. There exists a conformal representation $G^{(l)}$ of $\mathcal{R}$ onto $\overline{\mathbb{C}}$ such that

$$G^{(l)}(z) = z + \mathcal{O}(1), \quad z \to \infty^{(0)}, \qquad G^{(l)}(z) = C/z + \mathcal{O}\big(1/z^2\big), \quad z \to \infty^{(l)}.$$

By $G_k^{(l)}$ we denote the branch of $G^{(l)}$ on $\mathcal{R}_k$.

Theorem 1.1. *Let $S = \mathcal{N}(\sigma_1, \dots, \sigma_m)$ be a Nikishin system with* $\operatorname{supp}(\sigma_k) = \tilde{\Delta}_k \cup e_k$*, $k = 1, \dots, m$, where $\tilde{\Delta}_k$ is a bounded interval of the real line, $|\sigma_k'| > 0$ a.e. on $\tilde{\Delta}_k$, and e_k is a set without accumulation points in $\mathbb{R} \setminus \tilde{\Delta}_k$. Let $\Lambda \subset \mathbb{Z}_+^m(*)$ be an infinite sequence of distinct multi-indices with the property that* $\max_{\mathbf{n} \in \Lambda}(\max_{k=1,\dots,m} mn_k - |\mathbf{n}|) < \infty$*. Let us assume that there exists $l \in \{1, \dots, m\}$ and a fixed permutation τ of $\{1, \dots, m\}$ such that for all $\mathbf{n} \in \Lambda$ we have that $\mathbf{n}, \mathbf{n}_l \in \mathbb{Z}_+^m(*, \tau)$. Then,*

$$\lim_{\mathbf{n} \in \Lambda} \frac{Q_{\mathbf{n}_l}(z)}{Q_{\mathbf{n}}(z)} = G_0^{(\tau^{-1}(l))}(z), \qquad \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_1). \tag{4}$$

When $m = 1$ this result reduces to Denisov's version of Rakhmanov's theorem. The proof of Theorem 1.1 follows the guidelines employed in [1] but it is technically more complicated because of the more general assumptions on the measures and the sequence of multi-indices.

Let $\mathbf{1} = (1, \dots, 1)$. An immediate consequence of Theorem 1.1 is

Corollary 1.1. *Let $S = \mathcal{N}(\sigma_1, \dots, \sigma_m)$ be a Nikishin system with* $\operatorname{supp}(\sigma_k) = \tilde{\Delta}_k \cup e_k$*, $k = 1, \dots, m$, where $\tilde{\Delta}_k$ is a bounded interval of the real line, $|\sigma_k'| > 0$ a.e. on $\tilde{\Delta}_k$, and e_k is a set without accumulation points in $\mathbb{R} \setminus \tilde{\Delta}_k$. Let $\Lambda \subset \mathbb{Z}_+^m(*)$ be an infinite sequence of distinct multi-indices with the property* $\max_{\mathbf{n} \in \Lambda}(\max_{k=1,\dots,m} mn_k - |\mathbf{n}|) < \infty$*. Then,*

$$\lim_{\mathbf{n} \in \Lambda} \frac{Q_{\mathbf{n}+\mathbf{1}}(z)}{Q_{\mathbf{n}}(z)} = \prod_{l=1}^{m} G_0^{(l)}(z), \qquad \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_1). \tag{5}$$

The paper is organized as follows. In Section 2 we introduce and study an auxiliary system of second type functions. An interlacing property for the zeros of the polynomials $Q_{\mathbf{n}}$ and of the second type functions is proved in Section 3. Using the interlacing property of zeros and results on ratio and relative asymptotic of polynomials orthogonal with respect to varying measures, in Section 4 a system of boundary value problems is derived which implies the existence of limit in (4). Actually, a more general result is proved which also contains the ratio asymptotic of the second type functions.

2. Functions of second type and orthogonality properties

Fix $\mathbf{n} = (n_1, \dots, n_m) \in \mathbb{Z}_+^m(*)$ and consider $Q_{\mathbf{n}}$ the $\mathbf{n}$th multi-orthogonal polynomial with respect to a Nikishin system $S = \mathcal{N}(\Sigma)$, $\Sigma = (\sigma_1, \dots, \sigma_m)$. For short, in the sequel we denote $\Delta_k = \operatorname{Co}(\operatorname{supp}(\sigma_k))$, $k = 1, \dots, m$. Inductively, we define functions of second type $\Psi_{\mathbf{n},k}$, $k = 0, 1, \dots, m$, systems of measures $\Sigma^k = (\sigma_{k+1}^k, \dots, \sigma_m^k)$, $k = 0, 1, \dots, m-1$,

$\mathrm{Co}(\mathrm{supp}(\sigma_j^k)) \subset \Delta_j$, which generate Nikishin systems, and multi-indices $\mathbf{n}^k \in \mathbb{Z}_+^{m-k}(*)$, $k = 0, \ldots, m-1$. Take $\Psi_{\mathbf{n},0} = Q_{\mathbf{n}}$, $\mathbf{n}^0 = \mathbf{n}$, and $\Sigma^0 = \Sigma$.

Suppose that $\mathbf{n}^k = (n_{k+1}^k, \ldots, n_m^k)$, $\Sigma^k = (\sigma_{k+1}^k, \ldots, \sigma_m^k)$ and $\Psi_{\mathbf{n},k}$ have already been defined, where $0 \leqslant k \leqslant m-2$. Let

$$\mathbf{n}^{k+1} = \left(n_{k+2}^{k+1}, \ldots, n_m^{k+1}\right) \in \mathbb{Z}_+^{m-k-1}(*)$$

be the multi-index obtained deleting from $\mathbf{n}^k$ the first component $n_{r_k}^k$ which verifies

$$n_{r_k}^k = \max\left\{n_j^k \colon k+1 \leqslant j \leqslant m\right\}.$$

The components of $\mathbf{n}^{k+1}$ and $\mathbf{n}^k$ are related as follows:

$$n_{k+1}^k = n_{k+2}^{k+1}, \quad \ldots, \quad n_{r_k-1}^k = n_{r_k}^{k+1}, \quad n_{r_k+1}^k = n_{r_k+1}^{k+1}, \quad \ldots, \quad n_m^k = n_m^{k+1}.$$

Denote

$$\Psi_{\mathbf{n},k+1}(z) = \int_{\Delta_{k+1}} \frac{\Psi_{\mathbf{n},k}(x)}{z-x}\, ds_{r_k}^k(x), \tag{6}$$

where $s_{r_k}^k = \langle \sigma_{k+1}^k, \ldots, \sigma_{r_k}^k \rangle$ is the corresponding component of the Nikishin system $S^k = \mathcal{N}(\Sigma^k) = (s_{k+1}^k, \ldots, s_m^k)$.

In order to define Σ^{k+1} we introduce the following notation. Set

$$s_{i,j}^k = \left\langle \sigma_i^k, \ldots, \sigma_j^k \right\rangle, \quad k+1 \leqslant i \leqslant j \leqslant m,$$

where $\sigma_i^k \in \Sigma^k$. In page 390 of [9] it is proved that there exists a finite measure $\tau_{i,j}^k$ with constant sign such that

$$\mathrm{Co}\left(\mathrm{supp}\left(\tau_{i,j}^k\right)\right) \subset \mathrm{Co}\left(\mathrm{supp}\left(s_{i,j}^k\right)\right),$$
$$\frac{1}{\hat{s}_{i,j}^k(z)} = l_{i,j}^k(z) + \hat{\tau}_{i,j}^k(z)$$

where $l_{i,j}^k$ is a certain polynomial of degree 1. That $\mathrm{Co}(\mathrm{supp}(s_{i,j}^k)) \subset \Delta_i$ easily follows by induction. We wish to remark that the continuous part of $\mathrm{supp}(s_{i,j}^k)$ and $\mathrm{supp}(\tau_{i,j}^k)$ coincide, but not their isolated parts. In fact, zeros of $\hat{s}_{i,j}^k$ on Δ_i (there is one such zero between two consecutive mass points of $s_{i,j}^k$) become poles of $\hat{\tau}_{i,j}^k$ (mass points of $\tau_{i,j}^k$).

Suppose that $r_k = k+1$. In this case, we take

$$\Sigma^{k+1} = \left(\sigma_{k+2}^k, \ldots, \sigma_m^k\right) = \left(\sigma_{k+2}^{k+1}, \ldots, \sigma_m^{k+1}\right)$$

deleting the first measure of Σ^k. If $r_k \geqslant k+2$, then Σ^{k+1} is defined by

$$\left(\tau_{k+2,r_k}^k, \hat{s}_{k+2,r_k}^k d\tau_{k+3,r_k}^k, \ldots, \hat{s}_{r_k-1,r_k}^k d\tau_{r_k,r_k}^k, \hat{s}_{r_k,r_k}^k d\sigma_{r_k+1}^k, \sigma_{r_k+2}^k, \ldots, \sigma_m^k\right),$$

Table 1
$m = 2$

$m = 2$	k	r_{k-1}	$\Psi_{\mathbf{n},k}$	Σ^k	$\mathbf{n}^k$
$n_1 \geqslant n_2$	1	1	$\mathcal{C}(Q_{\mathbf{n}}; \sigma_1)$	(σ_2)	(n_2)
$n_1 < n_2$	1	2	$\mathcal{C}(Q_{\mathbf{n}}; \langle \sigma_1, \sigma_2 \rangle)$	(τ_2)	(n_1)

Table 2
$m = 3$

$m = 3$	k	r_{k-1}	$\Psi_{\mathbf{n},k}$	Σ^k	$\mathbf{n}^k$
$n_1 \geqslant n_2 \geqslant n_3$	1	1	$\mathcal{C}(Q_{\mathbf{n}}; \sigma_1)$	(σ_2, σ_3)	(n_2, n_3)
	2	2	$\mathcal{C}(\Psi_{\mathbf{n},1}; \sigma_2)$	(σ_3)	(n_3)
$n_1 \geqslant n_3 > n_2$	1	1	$\mathcal{C}(Q_{\mathbf{n}}; \sigma_1)$	(σ_2, σ_3)	(n_2, n_3)
	2	3	$\mathcal{C}(\Psi_{\mathbf{n},1}; \langle \sigma_2, \sigma_3 \rangle)$	(τ_3)	(n_2)
$n_2 > n_1 \geqslant n_3$	1	2	$\mathcal{C}(Q_{\mathbf{n}}; \langle \sigma_1, \sigma_2 \rangle)$	$(\tau_2, \langle \sigma_3, \sigma_2 \rangle)$	(n_1, n_3)
	2	2	$\mathcal{C}(\Psi_{\mathbf{n},1}; \tau_2)$,	$(\langle \sigma_3, \sigma_2 \rangle)$	(n_3)
$n_2 \geqslant n_3 > n_1$	1	2	$\mathcal{C}(Q_{\mathbf{n}}; \langle \sigma_1, \sigma_2 \rangle)$	$(\tau_2, \langle \sigma_3, \sigma_2 \rangle)$	(n_1, n_3)
	2	3	$\mathcal{C}(\Psi_{\mathbf{n},1}; \langle \tau_2, \sigma_3, \sigma_2 \rangle)$	$(\tau_{3,2})$	(n_1)
$n_3 > n_1 \geqslant n_2$	1	3	$\mathcal{C}(Q_{\mathbf{n}}; \langle \sigma_1, \sigma_2, \sigma_3 \rangle)$	$(\tau_{2,3}, \langle \tau_3, \sigma_2, \sigma_3 \rangle)$	(n_1, n_2)
	2	2	$\mathcal{C}(\Psi_{\mathbf{n},1}; \tau_{2,3})$	$(\langle \tau_3, \sigma_2, \sigma_3 \rangle)$	(n_2)

where $\mathrm{Co}(\mathrm{supp}(\sigma_j^{k+1})) \subset \Delta_j$, $j = k+2, \ldots, m$. Any two consecutive measures in the system Σ^{k+1} are supported on disjoint intervals; therefore, Σ^{k+1} generates a Nikishin system. To conclude we define

$$\Psi_{\mathbf{n},m}(z) = \int_{\Delta_m} \frac{\Psi_{\mathbf{n},m-1}(x)}{z - x} \, ds_m^{m-1}(x).$$

If $n_1 \geqslant \cdots \geqslant n_m$, we have that $\mathbf{n}^k = (n_{k+1}, \ldots, n_m)$, $\Sigma^k = (\sigma_{k+1}, \ldots, \sigma_m)$ and $\Psi_{\mathbf{n},k}(z) = \int_{\Delta_k} \frac{\Psi_{\mathbf{n},k-1}(x)}{z-x} \, d\sigma_k(x)$, $k = 1, \ldots, m$. Basically, this is the situation considered in [1].

To fix ideas let us turn our attention to the cases $m = 2$ and $m = 3$. We denote by $\mathcal{C}(f; \mu)$ the Cauchy transform of $f \, d\mu$; that is,

$$\mathcal{C}(f; \mu)(z) = \int \frac{f(x)}{z - x} \, d\mu(x).$$

In Tables 1 and 2, we omit the line corresponding to $k = 0$ because by definition $\Sigma^0 = \Sigma$, $\Psi_{\mathbf{n},0} = Q_{\mathbf{n}}$ and $\mathbf{n}^0 = \mathbf{n}$.

In Theorem 2 of [6] it was proved that the functions $\Psi_{\mathbf{n},k}$ verify the following orthogonality relations. For each $k = 0, 1, \ldots, m-1$,

$$\int_{\Delta_{k+1}} x^\nu \Psi_{\mathbf{n},k}(x) \, ds_i^k(x) = 0, \quad \nu = 0, 1, \ldots, n_i^k - 1, \; i = k+1, \ldots, m, \tag{7}$$

where $s_i^k = \langle \sigma_{k+1}^k, \ldots, \sigma_i^k \rangle$.

We wish to underline that since $\mathbb{Z}_+^2(*) = \mathbb{Z}_+^2$, all multi-indices with two components have associated functions of second type. However, for $m = 3$ the case $n_1 < n_2 < n_3$ has not been considered (see Table 2). The rest of this section will be devoted to the construction of certain functions $\Psi_{\mathbf{n},k}$ for this case and to the proof of the orthogonality relations they satisfy. We use the following auxiliary result.

Lemma 2.1. *Let* $s_{3,2} = \langle \sigma_3, \sigma_2 \rangle$. *Then*

$$\int_{\Delta_2} \frac{\hat{s}_{3,2}(x)}{\hat{\sigma}_3(x)} \frac{d\tau_{2,3}(x)}{(z-x)} + C_1 = \frac{\hat{\sigma}_2(z)}{\hat{s}_{2,3}(z)}, \quad z \in \mathbb{C} \setminus \operatorname{supp}(\sigma_2), \tag{8}$$

where $C_1 = \sigma_2(\Delta_2)/s_{2,3}(\Delta_2)$.

Proof. We employ two useful relations. The first one is

$$\hat{\sigma}_2(\zeta)\hat{\sigma}_3(\zeta) = \hat{s}_{2,3}(\zeta) + \hat{s}_{3,2}(\zeta), \quad \zeta \in \mathbb{C} \setminus \big(\operatorname{supp}(\sigma_2) \cup \operatorname{supp}(\sigma_3)\big). \tag{9}$$

The proof is straightforward and may be found in Lemma 4 of [5]. The second one was mentioned above and states that there exists a polynomial $l_{2,3}$ of degree 1 and a measure $\tau_{2,3}$ such that

$$\frac{1}{\hat{s}_{2,3}(z)} = \hat{\tau}_{2,3}(z) + l_{2,3}(z), \quad z \in \mathbb{C} \setminus \operatorname{supp}(\sigma_2). \tag{10}$$

Notice that

$$\frac{\hat{\sigma}_2(z)}{\hat{s}_{2,3}(z)} - C_1 = \mathcal{O}\left(\frac{1}{z}\right) \in \mathcal{H}(\overline{\mathbb{C}} \setminus \Delta_2).$$

Let Γ be a positively oriented smooth closed Jordan curve such that Δ_2 and $\{z\} \cup \Delta_3$ lie on the bounded and unbounded connected components, respectively, of $\mathbb{C} \setminus \Gamma$. By Cauchy's integral formula, we have

$$\frac{\hat{\sigma}_2(z)}{\hat{s}_{2,3}(z)} - C_1 = \frac{1}{2\pi i} \int_{\Gamma} \left(\frac{\hat{\sigma}_2(\zeta)}{\hat{s}_{2,3}(\zeta)} - C_1 \right) \frac{d\zeta}{z-\zeta} = \frac{1}{2\pi i} \int_{\Gamma} \frac{\hat{\sigma}_2(\zeta)}{\hat{s}_{2,3}(\zeta)} \frac{d\zeta}{z-\zeta}.$$

Multiply and divide the expression under the last integral sign by $\hat{\sigma}_3$ and use (9) to obtain

$$\frac{\hat{\sigma}_2(z)}{\hat{s}_{2,3}(z)} - C_1 = \frac{1}{2\pi i} \int_{\Gamma} \frac{\hat{s}_{2,3}(\zeta) + \hat{s}_{3,2}(\zeta)}{\hat{\sigma}_3(\zeta)\hat{s}_{2,3}(\zeta)} \frac{d\zeta}{z-\zeta} = \frac{1}{2\pi i} \int_{\Gamma} \frac{\hat{s}_{3,2}(\zeta)}{\hat{\sigma}_3(\zeta)\hat{s}_{2,3}(\zeta)} \frac{d\zeta}{z-\zeta}.$$

Taking account of (10) it follows that

$$\frac{\hat{\sigma}_2(z)}{\hat{s}_{2,3}(z)} - C_1 = \frac{1}{2\pi i} \int_{\Gamma} \frac{\hat{s}_{3,2}(\zeta)}{\hat{\sigma}_3(\zeta)} \frac{(\hat{\tau}_{2,3}(\zeta) + l_{2,3}(\zeta))\, d\zeta}{z-\zeta} = \frac{1}{2\pi i} \int_{\Gamma} \frac{\hat{s}_{3,2}(\zeta)}{\hat{\sigma}_3(\zeta)} \frac{\hat{\tau}_{2,3}(\zeta)\, d\zeta}{z-\zeta}.$$

Now, substitute $\hat{\tau}_{2,3}(\zeta)$ by its integral expression and use the Fubini and Cauchy theorems to obtain

$$\frac{\hat{\sigma}_2(z)}{\hat{s}_{2,3}(z)} - C_1 = \int \frac{1}{2\pi i} \int_{\Gamma} \frac{\hat{s}_{3,2}(\zeta)}{\hat{\sigma}_3(\zeta)(z-\zeta)} \frac{d\zeta}{\zeta - x} d\tau_{2,3}(x) = \int \frac{\hat{s}_{3,2}(x)}{\hat{\sigma}_3(x)} \frac{d\tau_{2,3}(x)}{z-x},$$

which is what we set out to prove. $\square$

We are ready to define the functions of second type and to prove the orthogonality properties they verify for multi-indices with 3 components not in $\mathbb{Z}_+^3(*)$ (with $n_1 < n_2 < n_3$).

Lemma 2.2. *Fix $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{Z}_+^3$ where $n_1 < n_2 < n_3$ and consider $Q_{\mathbf{n}}$ the $\mathbf{n}$th orthogonal polynomial associated to a Nikishin system $S = (s_1, s_2, s_3) = \mathcal{N}(\sigma_1, \sigma_2, \sigma_3)$. Set $\Psi_{\mathbf{n},0} = Q_{\mathbf{n}}$,*

$$\Psi_{\mathbf{n},1}(z) = \int_{\Delta_1} \frac{Q_{\mathbf{n}}(x)}{z-x} ds_{1,3}(x), \tag{11}$$

$$\Psi_{\mathbf{n},2}(z) = \int_{\Delta_2} \frac{\Psi_{\mathbf{n},1}(x)}{z-x} \frac{\hat{s}_{3,2}(x)}{\hat{\sigma}_3(x)} d\tau_{2,3}(x). \tag{12}$$

Then

$$\int_{\Delta_1} t^\nu \Psi_{\mathbf{n},0}(t)\, ds_{1,j}(t) = 0, \quad 0 \leqslant \nu \leqslant n_j - 1,\ 1 \leqslant j \leqslant 3, \tag{13}$$

$$\int_{\Delta_2} t^\nu \Psi_{\mathbf{n},1}(t)\, d\tau_{2,3}(t) = 0, \quad 0 \leqslant \nu \leqslant n_1 - 1, \tag{14}$$

$$\int_{\Delta_2} t^\nu \Psi_{\mathbf{n},1}(t) \frac{\hat{s}_{3,2}(t)}{\hat{\sigma}_3(t)}\, d\tau_{2,3}(t) = 0, \quad 0 \leqslant \nu \leqslant n_2 - 1, \tag{15}$$

$$\int_{\Delta_3} t^\nu \Psi_{\mathbf{n},2}(t) \frac{\hat{s}_{2,3}(t)}{\hat{\sigma}_2(t)}\, d\tau_{3,2}(t) = 0, \quad 0 \leqslant \nu \leqslant n_1 - 1. \tag{16}$$

Remark 2.1. The measure $\hat{s}_{3,2}\, d\tau_{2,3}/\hat{\sigma}_3$ supported on Δ_2 cannot be written in the form $\langle \tau_{2,3}, \mu \rangle$ for some measure μ supported on Δ_3, so there is no Σ^1 and S^1 in this case.

Proof. The relations (13) follow directly from the definition of $Q_{\mathbf{n}}$. Let us justify (14) and (15).

For $0 \leqslant \nu \leqslant n_1 - 1 (\leqslant n_3 - 3)$, applying Fubini's theorem,

$$\begin{aligned}
\int_{\Delta_2} t^\nu \Psi_{\mathbf{n},1}(t)\, d\tau_{2,3}(t) &= \int_{\Delta_2} t^\nu \int_{\Delta_1} \frac{Q_{\mathbf{n}}(x)}{t-x} ds_{1,3}(x)\, d\tau_{2,3}(t) \\
&= \int_{\Delta_1} Q_{\mathbf{n}}(x) \int_{\Delta_2} \frac{t^\nu - x^\nu + x^\nu}{t-x} d\tau_{2,3}(t)\, ds_{1,3}(x) \\
&= \int_{\Delta_1} Q_{\mathbf{n}}(x) p_\nu(x)\, ds_{1,3}(x) - \int_{\Delta_1} x^\nu Q_{\mathbf{n}}(x) \hat{\tau}_{2,3}(x)\, ds_{1,3}(x),
\end{aligned}$$

where $p_\nu(x) = \int_{\Delta_2} \frac{t^\nu - x^\nu}{t-x}\, d\tau_{2,3}(t)$ is a polynomial of degree at most $n_1 - 2$. Since $ds_{1,3}(x) = \hat{s}_{2,3}(x)\, d\sigma_1(x)$ and $\hat{\tau}_{2,3}(x)\hat{s}_{2,3}(x) = 1 - l_{2,3}(x)\hat{s}_{2,3}(x)$, the measure $\hat{\tau}_{2,3}(x)\, ds_{1,3}(x)$ is equal to $d\sigma_1(x) - l_{2,3}(x)\, ds_{1,3}(x)$. Therefore, applying (13) both integrals vanish and we obtain (14). Actually, we only needed that $n_1 \leqslant n_3 - 1$.

If $0 \leqslant \nu \leqslant n_2 - 1$ $(\leqslant n_3 - 2)$,

$$\begin{aligned}
\int_{\Delta_2} t^\nu \Psi_{\mathbf{n},1}(t) \frac{\hat{s}_{3,2}(t)}{\hat{\sigma}_3(t)}\, d\tau_{2,3}(t) &= \int_{\Delta_2} t^\nu \frac{\hat{s}_{3,2}(t)}{\hat{\sigma}_3(t)} \int_{\Delta_1} \frac{Q_{\mathbf{n}}(x)}{t-x}\, ds_{1,3}(x)\, d\tau_{2,3}(t) \\
&= \int_{\Delta_1} Q_{\mathbf{n}}(x) \int_{\Delta_2} \frac{t^\nu - x^\nu + x^\nu}{t-x} \frac{\hat{s}_{3,2}(t)}{\hat{\sigma}_3(t)}\, d\tau_{2,3}(t)\, ds_{1,3}(x) \\
&= \int_{\Delta_1} Q_{\mathbf{n}}(x) x^\nu \int_{\Delta_2} \frac{\hat{s}_{3,2}(t)}{\hat{\sigma}_3(t)} \frac{d\tau_{2,3}(t)}{t-x}\, ds_{1,3}(x).
\end{aligned}$$

By Lemma 2.1, the last expression is equal to

$$\begin{aligned}
&C_1 \int_{\Delta_1} Q_{\mathbf{n}}(x) x^\nu\, ds_{1,3}(x) - \int_{\Delta_1} Q_{\mathbf{n}}(x) x^\nu \frac{\hat{\sigma}_2(x)}{\hat{s}_{2,3}(x)}\, ds_{1,3}(x) \\
&\quad = - \int_{\Delta_1} Q_{\mathbf{n}}(x) x^\nu\, ds_{1,2}(x) = 0
\end{aligned}$$

taking into account that $ds_{1,3}(x) = \hat{s}_{2,3}(x)\, d\sigma_1(x)$ and (13). This proves (15). It would have been sufficient to require $n_2 \leqslant n_3$.

Let us prove (16). Take $0 \leqslant \nu \leqslant n_1 - 1$, we have

$$\begin{aligned}
\int_{\Delta_3} t^\nu \Psi_{\mathbf{n},2}(t) \frac{\hat{s}_{2,3}(t)}{\hat{\sigma}_2(t)}\, d\tau_{3,2}(t) &= \int_{\Delta_3} t^\nu \int_{\Delta_2} \frac{\Psi_{\mathbf{n},1}(x)}{t-x} \frac{\hat{s}_{3,2}(x)}{\hat{\sigma}_3(x)}\, d\tau_{2,3}(x) \frac{\hat{s}_{2,3}(t)}{\hat{\sigma}_2(t)}\, d\tau_{3,2}(t) \\
&= \int_{\Delta_2} \Psi_{\mathbf{n},1}(x) \frac{\hat{s}_{3,2}(x)}{\hat{\sigma}_3(x)} \int_{\Delta_3} \frac{t^\nu - x^\nu + x^\nu}{t-x} \frac{\hat{s}_{2,3}(t)}{\hat{\sigma}_2(t)}\, d\tau_{3,2}(t)\, d\tau_{2,3}(x) \\
&= \int_{\Delta_2} p_\nu(x) \Psi_{\mathbf{n},1}(x) \frac{\hat{s}_{3,2}(x)}{\hat{\sigma}_3(x)}\, d\tau_{2,3}(x) \\
&\quad + \int_{\Delta_2} \frac{\Psi_{\mathbf{n},1}(x) x^\nu \hat{s}_{3,2}(x)}{\hat{\sigma}_3(x)} \int_{\Delta_3} \frac{\hat{s}_{2,3}(t)}{\hat{\sigma}_2(t)} \frac{d\tau_{3,2}(t)}{t-x}\, d\tau_{2,3}(x)
\end{aligned}$$

where $p_\nu(x)$ is the polynomial defined by

$$\int_{\Delta_3} \frac{t^\nu - x^\nu}{t-x} \frac{\hat{s}_{2,3}(t)}{\hat{\sigma}_2(t)}\, d\tau_{3,2}(t),$$

of degree $\leqslant n_1 - 2$. Applying (15), the first integral after the last equality equals zero since $n_1 < n_2$ (though $n_1 \leqslant n_2 + 1$ would have been sufficient). If we interchange the sub-indices 2 and 3 in Lemma 2.1, we obtain

$$\int_{\Delta_3} \frac{\hat{s}_{2,3}(t)}{\hat{\sigma}_2(t)} \frac{d\tau_{3,2}(t)}{t-x} = -\frac{\hat{\sigma}_3(x)}{\hat{s}_{3,2}(t)} + C_2, \tag{17}$$

where $C_2 = \sigma_3(\Delta_3)/s_{3,2}(\Delta_3)$. Therefore, using (17), (15) and (14), it follows that

$$\begin{aligned}
&\int_{\Delta_2} \frac{\Psi_{\mathbf{n},1}(x)x^\nu \hat{s}_{3,2}(t)}{\hat{\sigma}_3(x)} \int_{\Delta_3} \frac{\hat{s}_{2,3}(t)}{\hat{\sigma}_2(t)} \frac{d\tau_{3,2}(t)}{t-x}\, d\tau_{2,3}(x) \\
&\quad = \int_{\Delta_2} \Psi_{\mathbf{n},1}(x)x^\nu \frac{\hat{s}_{3,2}(t)}{\hat{\sigma}_3(x)} \left(C_2 - \frac{\hat{\sigma}_3(x)}{\hat{s}_{3,2}(t)}\right) d\tau_{2,3}(x) = 0,
\end{aligned}$$

since $n_1 \leqslant n_2$. This completes the proof. □

3. Interlacing property of zeros and varying measures

As we have pointed out, from the definition $\mathbb{Z}_+^m(*) = \mathbb{Z}_+^m$, $m = 1, 2$. We have introduced adequate functions of second type also when $m = 3$ and $n_1 < n_2 < n_3$ which were the only multi-indices initially not in $\mathbb{Z}_+^3(*)$. To unify notation, in the rest of the paper we will consider that $\mathbb{Z}_+^3(*) = \mathbb{Z}_+^3$.

In this section, we show that for $\mathbf{n} \in \mathbb{Z}_+^m(*)$, $m \in \mathbb{N}$, the functions $\Psi_{\mathbf{n},k}$, $k = 0, \ldots, m-1$, have exactly $|\mathbf{n}^k|$ simple zeros in the interior of Δ_{k+1} and no other zeros on $\mathbb{C} \setminus \Delta_k$. The zeros of "consecutive" $\Psi_{\mathbf{n},k}$ satisfy an interlacing property. These properties are proved in Lemma 3.2 below which complements Theorem 2.1 (see also Lemma 2.1) in [1] and substantially enlarges the class of multi-indices for which it is applicable. The concept of AT system is crucial in its proof.

Definition 3.1. Let $(\omega_1, \omega_2, \ldots, \omega_m)$ be a collection of functions which are analytic on a neighborhood of an interval Δ. We say that it forms an AT-system for the multi-index $\mathbf{n} = (n_1, n_2, \ldots, n_m)$ on Δ if whenever one chooses polynomials $P_{n_1}, \ldots, P_{n_m}$ with $\deg(P_{n_j}) \leqslant n_j - 1$, not all identically equal to zero, the function

$$P_{n_1}(x)w_1(x) + \cdots + P_{n_m}(x)w_m(x)$$

has at most $|\mathbf{n}| - 1$ zeros on Δ, counting multiplicities. $(\omega_1, \ldots, \omega_m)$ is an AT-system on Δ if it is an AT-system on that interval for all $\mathbf{n} \in \mathbb{Z}_+^m$.

Theorem 1 of [5] (for $m = 3$) and Theorem 1 of [6] prove the following.

Lemma 3.1. *Let $(s_1, \ldots, s_{m-1}) = \mathcal{N}(\sigma_1, \ldots, \sigma_{m-1})$, $m \geqslant 2$, be a Nikishin system of $m-1$ measures. Then $(1, \hat{s}_1, \ldots, \hat{s}_{m-1})$ forms an AT system on any interval Δ disjoint from Δ_1 with respect to any $\mathbf{n} \in \mathbb{Z}_+^m(*)$.*

Recall that $\mathbf{n}_l$ denotes the multi-index obtained adding 1 to the lth component of $\mathbf{n}$.

Lemma 3.2. *Let $S = \mathcal{N}(\sigma_1, \dots, \sigma_m)$ be a Nikishin system. Let $\mathbf{n} \in \mathbb{Z}_+^m(*)$, $m \in \mathbb{N}$, then for each $k = 0, \dots, m-1$, the function $\Psi_{\mathbf{n},k}$ has exactly $|\mathbf{n}^k|$ simple zeros in the interior of Δ_{k+1} and no other zeros on $\mathbb{C} \setminus \Delta_k$. Let I denote the closure of any one of the connected components of $\Delta_{k+1} \setminus \operatorname{supp}(\sigma_{k+1}^k)$, then $\Psi_{\mathbf{n},k}$ has at most one simple zero on I. Assume that $l \in \{1, 2, \dots, m\}$ is such that $\mathbf{n}, \mathbf{n}_l \in \mathbb{Z}_+^m(*, \tau)$ for a fixed permutation τ. Then, for each $k \in \{0, \dots, m-1\}$ between two consecutive zeros of $\Psi_{\mathbf{n}_l,k}$ lies exactly one zero of $\Psi_{\mathbf{n},k}$ and vice versa* (*that is, the zeros of $\Psi_{\mathbf{n}_l,k}$ and $\Psi_{\mathbf{n},k}$ on Δ_{k+1} interlace*).

Proof. Assume that $\mathbf{n}, \mathbf{n}_l \in \mathbb{Z}_+^m(*, \tau)$. We claim that for any real constants A, B, $|A| + |B| > 0$, and $k \in \{0, 1, \dots, m-1\}$, the function

$$G_{\mathbf{n},k}(x) = A\Psi_{\mathbf{n},k}(x) + B\Psi_{\mathbf{n}_l,k}(x)$$

has at most $|\mathbf{n}^k| + 1$ zeros in $\mathbb{C} \setminus \Delta_k$ (counting multiplicities) and at least $|\mathbf{n}^k|$ simple zeros in the interior of Δ_{k+1} ($\Delta_0 = \emptyset$). We prove this by induction on k.

Let $k = 0$. The polynomial $G_{\mathbf{n},0} = A\Psi_{\mathbf{n},0} + B\Psi_{\mathbf{n}_l,0}$ is not identically equal to zero, and $|\mathbf{n}| \leqslant \deg(G_{\mathbf{n},0}) \leqslant |\mathbf{n}| + 1$. Therefore, $G_{\mathbf{n},0}$ has at most $|\mathbf{n}| + 1$ zeros in $\mathbb{C}$. Let h_j, $j = 1, \dots, m$, denote polynomials, where $\deg(h_j) \leqslant n_j - 1$. According to (7),

$$\int_{\Delta_1} G_{\mathbf{n},0}(x) \sum_{j=1}^m h_j(x)\hat{s}_{2,j}(x)\, d\sigma_1(x) = 0 \tag{18}$$

($\hat{s}_{2,1} \equiv 1$).

In the sequel, we call change knot a point on the real line where a function changes its sign. Notice that for each $k \in \{0, \dots, m-1\}$, $G_{\mathbf{n},k}$ is a real function when restricted to the real line. Assume that $G_{\mathbf{n},0}$ has $N \leqslant |\mathbf{n}| - 1$ change knots in the interior of Δ_1. We can find polynomials h_j, $j = 1, \dots, m$, $\deg(h_j) \leqslant n_j - 1$, such that $\sum_{j=1}^m h_j \hat{s}_{2,j}$ has a simple zero at each change knot of $G_{\mathbf{n},0}$ on Δ_1 and a zero of order $|\mathbf{n}| - 1 - N$ at one of the extreme points of Δ_1. By Lemma 3.1, $(1, \hat{s}_{2,2}, \dots, \hat{s}_{2,m})$ forms an AT system with respect to $\mathbf{n} \in \mathbb{Z}_+^m(*)$; therefore, $\sum_{j=1}^m h_j \hat{s}_{2,j}$ can have no other zero on Δ_1, but this contradicts (18) since $G_{\mathbf{n},0} \sum_{j=1}^m h_j \hat{s}_{2,j}$ would have a constant sign on Δ_1 (and $\operatorname{supp}(\sigma_1)$ contains infinitely many points). Therefore, $G_{\mathbf{n},0}$ has at least $|\mathbf{n}|$ change knots in the interior of Δ_1. Consequently, all the zeros of $G_{\mathbf{n},0}$ are simple and lie on $\mathbb{R}$ as claimed.

Assume that for each $k \in \{0, \dots, \kappa - 1\}$, $1 \leqslant \kappa \leqslant m - 1$, the claim is satisfied whereas it is violated when $k = \kappa$. Let h_j denote polynomials such that $\deg(h_j) \leqslant n_j^\kappa - 1$, $\kappa + 1 \leqslant j \leqslant m$. Using (7) or (13)–(16) according to the situation (to simplify the writing we use the notation of (7) but the arguments are the same when $m = 3$ and $n_1 < n_2 < n_3$; in particular, in this case, $ds_{r_0}^0 = ds_{1,3}$, $ds_{r_1}^1 = \hat{s}_{3,2}\, d\tau_{2,3}/\hat{\sigma}_3$ and $ds_{r_2}^2 = \hat{s}_{2,3}\, d\tau_{3,2}/\hat{\sigma}_2$)

$$\int_{\Delta_{\kappa+1}} G_{\mathbf{n},\kappa}(x) \sum_{j=\kappa+1}^m h_j(x)\hat{s}_{\kappa+2,j}^\kappa(x)\, d\sigma_{\kappa+1}^\kappa(x) = 0 \tag{19}$$

($\hat{s}^{\kappa}_{\kappa+2,\kappa+1} \equiv 1$). Arguing as above, since $(1, \hat{s}^{\kappa}_{\kappa+2,\kappa+2}, \ldots, \hat{s}^{\kappa}_{\kappa+2,m})$ forms an AT system with respect to $\mathbf{n}^{\kappa} \in \mathbb{Z}_+^{m-\kappa}(*)$, we conclude that $G_{\mathbf{n},\kappa}$ has at least $|\mathbf{n}^{\kappa}|$ change knots in the interior of $\Delta_{\kappa+1}$.

Let us suppose that $G_{\mathbf{n},\kappa}$ has at least $|\mathbf{n}^{\kappa}| + 2$ zeros in $\mathbb{C} \setminus \Delta_{\kappa}$ and let $W_{\mathbf{n},\kappa}$ be the monic polynomial whose zeros are those points (counting multiplicities). The complex zeros of $G_{\mathbf{n},\kappa}$ (if any) must appear in conjugate pairs since $G_{\mathbf{n},\kappa}(\bar{z}) = \overline{G_{\mathbf{n},\kappa}(z)}$; therefore, the coefficients of $W_{\mathbf{n},\kappa}$ are real numbers. On the other hand, from (7) ((13) or (15) when necessary)

$$0 = \int_{\Delta_{\kappa}} G_{\mathbf{n},\kappa-1}(x) \frac{z^{n^{\kappa-1}_{r_{\kappa-1}}} - x^{n^{\kappa-1}_{r_{\kappa-1}}}}{z - x}\, ds^{\kappa-1}_{r_{\kappa-1}}(x).$$

Therefore,

$$G_{\mathbf{n},\kappa}(z) = \frac{1}{z^{n^{\kappa-1}_{r_{\kappa-1}}}} \int_{\Delta_{\kappa}} \frac{x^{n^{\kappa-1}_{r_{\kappa-1}}} G_{\mathbf{n},\kappa-1}(x)}{z - x}\, ds^{\kappa-1}_{r_{\kappa-1}}(x) = \mathcal{O}\left(\frac{1}{z^{n^{\kappa-1}_{r_{\kappa-1}}+1}}\right), \quad z \to \infty,$$

and taking into consideration the degree of $W_{\mathbf{n},\kappa}$, we obtain

$$\frac{z^j G_{\mathbf{n},\kappa}}{W_{\mathbf{n},\kappa}} = \mathcal{O}\left(\frac{1}{z^2}\right) \in H(\mathbb{C} \setminus \Delta_{\kappa}), \quad j = 0, \ldots, \left|\mathbf{n}^{\kappa-1}\right| + 1.$$

Let Γ be a closed Jordan curve which surrounds Δ_{κ} and such that all the zeros of $W_{\mathbf{n},\kappa}$ lie in the exterior of Γ. Using Cauchy's theorem, the integral expression for $G_{\mathbf{n},\kappa}$, Fubini's theorem, and Cauchy's integral formula, for each $j = 0, \ldots, |\mathbf{n}^{\kappa-1}| + 1$, we have

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \int_{\Gamma} \frac{z^j G_{\mathbf{n},\kappa}(z)}{W_{\mathbf{n},\kappa}(z)}\, dz = \frac{1}{2\pi i} \int_{\Gamma} \frac{z^j}{W_{\mathbf{n},\kappa}(z)} \int_{\Delta_{\kappa}} \frac{G_{\mathbf{n},\kappa-1}(x)}{z - x}\, ds^{\kappa-1}_{r_{\kappa-1}}(x)\, dz \\ &= \int_{\Delta_{\kappa}} \frac{x^j G_{\mathbf{n},\kappa-1}(x)}{W_{\mathbf{n},\kappa}(x)}\, ds^{\kappa-1}_{r_{\kappa-1}}(x), \end{aligned}$$

which implies that $G_{\mathbf{n},\kappa-1}$ has at least $|\mathbf{n}^{\kappa-1}| + 2$ change knots in the interior of Δ_{κ}. This contradicts our induction hypothesis since this function can have at most $|\mathbf{n}^{\kappa-1}| + 1$ zeros in $\mathbb{C} \setminus \Delta_{\kappa-1} \supset \Delta_{\kappa}$. Hence $G_{\mathbf{n},\kappa}$ has at most $|\mathbf{n}^{\kappa}| + 1$ zeros in $\mathbb{C} \setminus \Delta_{\kappa}$ as claimed.

Taking $B = 0$ the assumption $\mathbf{n}_l \in \mathbb{Z}_+^m(*, \tau)$ is not required, and the arguments above lead to the proof that $\Psi_{\mathbf{n},k}$ has at most $|\mathbf{n}^k|$ zeros on $\mathbb{C} \setminus \Delta_k$ since $Q_{\mathbf{n}} = \Psi_{\mathbf{n},0}$ has at most $|\mathbf{n}|$ zeros on $\mathbb{C}$. Consequently, the zeros of $\Psi_{\mathbf{n},k}$ in $\mathbb{C} \setminus \Delta_k$ are exactly the $|\mathbf{n}^k|$ simple ones it has in the interior of Δ_{k+1}.

Let I be the closure of a connected component of $\Delta_{k+1} \setminus \operatorname{supp}(\sigma^k_{k+1})$ and let us assume that I contains two consecutive simple zeros x_1, x_2 of $\Psi_{\mathbf{n},k}$. Taking $B = 0$ and $A = 1$, we can rewrite (19) as follows

$$\int_{\Delta_{k+1}} \frac{\Psi_{\mathbf{n},k}(x)}{(x - x_1)(x - x_2)} \sum_{j=k+1}^{m} h_j(x) \hat{s}^k_{k+2,j}(x)(x - x_1)(x - x_2)\, d\sigma^k_{k+1}(x) = 0, \tag{20}$$

where $\deg(h_j) \leqslant n_j^k - 1$, $j = k+1, \ldots, m$. The measure $(x - x_1)(x - x_2)\, d\sigma_{k+1}^k(x)$ has a constant sign on Δ_{k+1} and $\Psi_{\mathbf{n},k}(x)/(x - x_1)(x - x_2)$ has $|\mathbf{n}^k| - 2$ change knots on Δ_{k+1}. Using again Lemma 3.1, we can construct appropriate polynomials h_j to contradict (20). Therefore, I contains at most one zero of $\Psi_{\mathbf{n},k}$.

Fix $y \in \mathbb{R} \setminus \Delta_k$ and $k \in \{0, 1, \ldots, m-1\}$. It cannot occur that $\Psi_{\mathbf{n}_l,k}(y) = \Psi_{\mathbf{n},k}(y) = 0$. If this was so, y would have to be a simple zero of $\Psi_{\mathbf{n}_l,k}$ and $\Psi_{\mathbf{n},k}$. Therefore, $(\Psi_{\mathbf{n}_l,k})'(y) \neq 0 \neq (\Psi_{\mathbf{n},k})'(y)$. Taking $A = 1$, $B = -\Psi'_{\mathbf{n},k}(y)/\Psi'_{\mathbf{n}_l,k}(y)$, we find that

$$G_{\mathbf{n},k}(y) = (A\Psi_{\mathbf{n},k} + B\Psi_{\mathbf{n}_l,k})(y) = (G_{\mathbf{n},k})'(y) = 0,$$

which means that $G_{\mathbf{n},k}$ has at least a double zero at y against what we proved before.

Now, taking $A = \Psi_{\mathbf{n}_l,k}(y)$, $B = -\Psi_{\mathbf{n},k}(y)$, we have that $|A| + |B| > 0$. Since

$$\Psi_{\mathbf{n}_l,k}(y)\Psi_{\mathbf{n},k}(y) - \Psi_{\mathbf{n},k}(y)\Psi_{\mathbf{n}_l,k}(y) = 0,$$

and the zeros on $\mathbb{R} \setminus \Delta_k$ of $\Psi_{\mathbf{n}_l,k}(y)\Psi_{\mathbf{n},k}(x) - \Psi_{\mathbf{n},k}(y)\Psi_{\mathbf{n}_l,k}(x)$ with respect to x are simple, using again what we proved above, it follows that

$$\Psi_{\mathbf{n}_l,k}(y)\Psi'_{\mathbf{n},k}(y) - \Psi_{\mathbf{n},k}(y)\Psi'_{\mathbf{n}_l,k}(y) \neq 0.$$

But $\Psi_{\mathbf{n}_l,k}(y)\Psi'_{\mathbf{n},k}(y) - \Psi_{\mathbf{n},k}(y)\Psi'_{\mathbf{n}_l,k}(y)$ is a continuous real function on $\mathbb{R} \setminus \Delta_k$ so it must have constant sign on each one of the intervals forming $\mathbb{R} \setminus \Delta_k$; in particular, its sign on Δ_{k+1} is constant.

We know that $\Psi_{\mathbf{n}_l,k}$ has at least $|\mathbf{n}^k|$ simple zeros in the interior of Δ_{k+1}. Evaluating $\Psi_{\mathbf{n}_l,k}(y)\Psi'_{\mathbf{n},k}(y) - \Psi_{\mathbf{n},k}(y)\Psi'_{\mathbf{n}_l,k}(y)$ at two consecutive zeros of $\Psi_{\mathbf{n}_l,k}$, since the sign of $\Psi'_{\mathbf{n}_l,k}$ at these two points changes the sign of $\Psi_{\mathbf{n},k}$ must also change. Using Bolzano's theorem we find that there must be an intermediate zero of $\Psi_{\mathbf{n},k}$. Analogously, one proves that between two consecutive zeros of $\Psi_{\mathbf{n},k}$ on Δ_{k+1} there is one of $\Psi_{\mathbf{n}_l,k}$. Thus, the interlacing property has been proved. □

Let $Q_{\mathbf{n},k+1}$, $k = 0, \ldots, m-1$, denote the monic polynomial whose zeros are equal to those of $\Psi_{\mathbf{n},k}$ on Δ_{k+1}. From (7) ((13), (15), or (16) when necessary)

$$0 = \int_{\Delta_{k+1}} \Psi_{\mathbf{n},k}(x) \frac{z^{n_{r_k}^k} - x^{n_{r_k}^k}}{z - x}\, ds_{r_k}^k(x).$$

(Recall that when $m = 3$ and $n_1 < n_2 < n_3$, we take $ds_{r_0}^0 = ds_{1,3}$, $ds_{r_1}^1 = \hat{s}_{3,2}\, d\tau_{2,3}/\hat{\sigma}_3$ and $ds_{r_2}^2 = \hat{s}_{2,3}\, d\tau_{3,2}/\hat{\sigma}_2$.) Therefore,

$$\Psi_{\mathbf{n},k+1}(z) = \frac{1}{z^{n_{r_k}^k}} \int_{\Delta_{k+1}} \frac{x^{n_{r_k}^k}\Psi_{\mathbf{n},k}(x)}{z - x}\, ds_{r_k}^k(x) = \mathcal{O}\left(\frac{1}{z^{n_{r_k}^k + 1}}\right), \quad z \to \infty,$$

and taking into consideration the degree of $Q_{\mathbf{n},k+2}$ (by definition $Q_{\mathbf{n},m+1} \equiv 1$), we obtain

$$\frac{z^j \Psi_{\mathbf{n},k+1}}{Q_{\mathbf{n},k+2}} = \mathcal{O}\left(\frac{1}{z^2}\right) \in H(\mathbb{C} \setminus \Delta_{k+1}), \quad j = 0, \ldots, \left|\mathbf{n}^k\right| - 1.$$

Let Γ be a closed Jordan curve which surrounds Δ_{k+1} such that all the zeros of $Q_{\mathbf{n},k+2}$ lie in the exterior of Γ. Using Cauchy's theorem, the integral expression for $\Psi_{\mathbf{n},k+1}$, Fubini's theorem, and Cauchy's integral formula, for each $j = 0, \ldots, |\mathbf{n}^k| - 1$ (we also define $Q_{\mathbf{n},0} \equiv 1$), we have

$$0 = \frac{1}{2\pi i} \int_{\Gamma} \frac{z^j \Psi_{\mathbf{n},k+1}(z)}{Q_{\mathbf{n},k+2}(z)}\, dz = \frac{1}{2\pi i} \int_{\Gamma} \frac{z^j}{Q_{\mathbf{n},k+2}(z)} \int_{\Delta_{k+1}} \frac{\Psi_{\mathbf{n},k}(x)}{z - x}\, ds_{r_k}^k(x)\, dz$$

$$= \int_{\Delta_{k+1}} x^j Q_{\mathbf{n},k+1}(x) \frac{H_{\mathbf{n},k+1}(x)\, ds_{r_k}^k(x)}{Q_{\mathbf{n},k}(x) Q_{\mathbf{n},k+2}(x)}, \quad k = 0, \ldots, m-1, \tag{21}$$

where

$$H_{\mathbf{n},k+1} = \frac{Q_{\mathbf{n},k} \Psi_{\mathbf{n},k}}{Q_{\mathbf{n},k+1}}, \quad k = 0, \ldots, m,$$

has constant sign on Δ_{k+1}.

This last relation implies that

$$\int_{\Delta_{k+1}} \frac{(Q(z) - Q(x))}{z - x} Q_{\mathbf{n},k+1}(x) \frac{H_{\mathbf{n},k+1}(x)\, ds_{r_k}^k(x)}{Q_{\mathbf{n},k}(x) Q_{\mathbf{n},k+2}(x)} = 0,$$

where Q is any polynomial of degree $\leqslant |\mathbf{n}^k|$. If we use this formula with $Q = Q_{\mathbf{n},k+1}$ and $Q = Q_{\mathbf{n},k+2}$, respectively, we obtain

$$\int_{\Delta_{k+1}} \frac{Q_{\mathbf{n},k+1}(x)}{z - x} \frac{H_{\mathbf{n},k+1}(x)\, ds_{r_k}^k(x)}{Q_{\mathbf{n},k}(x) Q_{\mathbf{n},k+2}(x)}$$

$$= \frac{1}{Q_{\mathbf{n},k+1}(z)} \int_{\Delta_{k+1}} \frac{Q_{\mathbf{n},k+1}^2(x)}{z - x} \frac{H_{\mathbf{n},k+1}(x)\, ds_{r_k}^k(x)}{Q_{\mathbf{n},k}(x) Q_{\mathbf{n},k+2}(x)}$$

and

$$\int_{\Delta_{k+1}} \frac{Q_{\mathbf{n},k+1}(x)}{z - x} \frac{H_{\mathbf{n},k+1}(x)\, ds_{r_k}^k(x)}{Q_{\mathbf{n},k}(x) Q_{\mathbf{n},k+2}(x)}$$

$$= \frac{1}{Q_{\mathbf{n},k+2}(z)} \int_{\Delta_{k+1}} \frac{\Psi_{\mathbf{n},k}(x)\, ds_{r_k}^k(x)}{z - x}.$$

Equating these two relations and using the definition of $\Psi_{\mathbf{n},k+1}$ and $H_{\mathbf{n},k+2}$, we obtain

$$H_{\mathbf{n},k+2}(z) = \int_{\Delta_{k+1}} \frac{Q_{\mathbf{n},k+1}^2(x)}{z - x} \frac{H_{\mathbf{n},k+1}(x)\, ds_{r_k}^k(x)}{Q_{\mathbf{n},k}(x) Q_{\mathbf{n},k+2}(x)}, \quad k = 0, \ldots, m-1. \tag{22}$$

Notice that from the definition $H_{\mathbf{n},1} \equiv 1$.

For each $k = 1, \dots, m$, set

$$K_{\mathbf{n},k}^{-2} = \int_{\Delta_k} Q_{\mathbf{n},k}^2(x) \left| \frac{Q_{\mathbf{n},k-1}(x)\Psi_{\mathbf{n},k-1}(x)}{Q_{\mathbf{n},k}(x)} \right| \frac{d|s_{r_{k-1}}^{k-1}|(x)}{|Q_{\mathbf{n},k-1}(x)Q_{\mathbf{n},k+1}(x)|}, \tag{23}$$

where $|s|$ denotes the total variation of the measures s. Take

$$K_{\mathbf{n},0} = 1, \qquad \kappa_{\mathbf{n},k} = \frac{K_{\mathbf{n},k}}{K_{\mathbf{n},k-1}}, \quad k = 1, \dots, m.$$

Define

$$q_{\mathbf{n},k} = \kappa_{\mathbf{n},k} Q_{\mathbf{n},k}, \qquad h_{\mathbf{n},k} = K_{\mathbf{n},k-1}^2 H_{\mathbf{n},k}, \tag{24}$$

and

$$d\rho_{\mathbf{n},k}(x) = \frac{h_{\mathbf{n},k}(x)\, ds_{r_{k-1}}^{k-1}(x)}{Q_{\mathbf{n},k-1}(x)Q_{\mathbf{n},k+1}(x)}. \tag{25}$$

Notice that the measure $\rho_{\mathbf{n},k}$ has constant sign on Δ_k. Let $\varepsilon_{\mathbf{n},k}$ be the sign of $\rho_{\mathbf{n},k}$. From (21) and the notation introduced above, we obtain

$$\int_{\Delta_k} x^\nu q_{\mathbf{n},k}(x)\, d|\rho_{\mathbf{n},k}|(x) = 0, \qquad \nu = 0, \dots, \left|\mathbf{n}^{k-1}\right| - 1,\ k = 1, \dots, m, \tag{26}$$

and $q_{n,k}$ is orthonormal with respect to the varying measure $|\rho_{\mathbf{n},k}|$. On the other hand, using (22) it follows that

$$h_{\mathbf{n},k+1}(z) = \varepsilon_{\mathbf{n},k} \int_{\Delta_k} \frac{q_{\mathbf{n},k}^2(x)}{z-x}\, d|\rho_{\mathbf{n},k}|(x), \quad k = 1, \dots, m. \tag{27}$$

Lemma 3.3. *Let $S = \mathcal{N}(\sigma_1, \dots, \sigma_m)$ be a Nikishin system such that* $\operatorname{supp}(\sigma_k) = \tilde{\Delta}_k \cup e_k$, *$k = 1, \dots, m$, where $\tilde{\Delta}_k$ is a bounded interval of the real line, $|\sigma_k'| > 0$ a.e. on $\tilde{\Delta}_k$, and e_k is a set without accumulation points in $\mathbb{R} \setminus \tilde{\Delta}_k$. Let $\Lambda \subset \mathbb{Z}_+^m(*)$ be an infinite sequence of distinct multi-indices with the property that* $\max_{\mathbf{n}\in\Lambda}(\max_{k=1,\dots,m} mn_k - |\mathbf{n}|) < \infty$. *For any continuous function f on* $\operatorname{supp}(\sigma_k^{k-1})$

$$\lim_{\mathbf{n}\in\Lambda} \int_{\Delta_k} f(x) q_{\mathbf{n},k}^2(x)\, d|\rho_{\mathbf{n},k}|(x) = \frac{1}{\pi} \int_{\tilde{\Delta}_k} f(x) \frac{dx}{\sqrt{(b_k - x)(x - a_k)}}, \tag{28}$$

where $\tilde{\Delta}_k = [a_k, b_k]$. In particular,

$$\lim_{\mathbf{n}\in\Lambda} \varepsilon_{\mathbf{n},k} h_{\mathbf{n},k+1}(z) = \frac{1}{\sqrt{(z-b_k)(z-a_k)}}, \qquad \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}\bigl(\sigma_k^{k-1}\bigr), \tag{29}$$

where $\sqrt{(z-b_k)(z-a_k)} > 0$ *if* $z > 0$. *Consequently, for* $k = 1, \dots, m$, *each point of* $\operatorname{supp}(\sigma_k^{k-1}) \setminus \tilde{\Delta}_k$, *is a limit of zeros of* $\{Q_{\mathbf{n},k}\}$, $\mathbf{n} \in \Lambda$.

Proof. We will proof this by induction on k. For $k = 1$, using Corollary 3 in [2], it follows that

$$\lim_{\mathbf{n}\in\Lambda} \int_{\Delta_1} f(x) q_{\mathbf{n},1}^2(x) \frac{d|s_{r_0}^0|(x)}{|Q_{\mathbf{n},2}(x)|} = \frac{1}{\pi} \int_{\tilde{\Delta}_1} f(x) \frac{dx}{\sqrt{(b_1-x)(x-a_1)}},$$

where f is continuous on $\operatorname{supp}(\sigma_1)$. Take $f(x) = (z-x)^{-1}$ where $z \in \mathbb{C} \setminus \operatorname{supp}(\sigma_1)$. According to (27) and the previous limit one obtains that

$$\lim_{\mathbf{n}\in\Lambda} \varepsilon_{\mathbf{n},1} h_{\mathbf{n},2}(z) = \frac{1}{\sqrt{(z-b_1)(z-a_1)}} =: h_2(z),$$

pointwise on $\mathbb{C} \setminus \operatorname{supp}(\sigma_1)$. Since

$$\left| \int_{\Delta_1} \frac{q_{\mathbf{n},1}^2(x)}{z-x} \frac{d|s_{r_0}^0|(x)}{|Q_{\mathbf{n},2}(x)|} \right| \leqslant \frac{1}{d(\mathcal{K}, \operatorname{supp}(\sigma_1))}, \quad z \in \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_1),$$

where $d(\mathcal{K}, \operatorname{supp}(\sigma_1))$ denotes the distance between the two compact sets, the sequence $\{h_{\mathbf{n},2}\}$, $\mathbf{n} \in \Lambda$, is uniformly bounded on compact subsets of $\mathbb{C} \setminus \operatorname{supp}(\sigma_1)$ and (29) follows for $k = 1$.

Let $\zeta \in \operatorname{supp}(\sigma_1) \setminus \tilde{\Delta}_1$. Take $r > 0$ sufficiently small so that the circle $C_r = \{z\colon |z-\zeta| = r\}$ surrounds no other point of $\operatorname{supp}(\sigma_1) \setminus \tilde{\Delta}_1$ and contains no zero of $q_{\mathbf{n},1}$, $\mathbf{n} \in \Lambda$. From (29) for $k = 1$

$$\lim_{\mathbf{n}\in\Lambda} \frac{1}{2\pi i} \int_{C_r} \frac{\varepsilon_{\mathbf{n},1} h'_{\mathbf{n},2}(z)}{\varepsilon_{\mathbf{n},1} h_{\mathbf{n},2}(z)} dz = \frac{1}{2\pi i} \int_{C_r} \frac{h'_2(z)}{h_2(z)} dz = 0.$$

From the definition, $\Psi_{\mathbf{n},1}$, $\mathbf{n} \in \Lambda$, has either a simple pole at ζ or $Q_{\mathbf{n},1}$ has a zero at ζ. In the second case there is nothing to prove. Let us restrict our attention to those $\mathbf{n} \in \Lambda$ such that $\Psi_{\mathbf{n},1}$, $\mathbf{n} \in \Lambda$, has a simple pole at ζ. Then, $h_{\mathbf{n},2} = K_{\mathbf{n},1}^2 Q_{\mathbf{n},1} \Psi_{\mathbf{n},1} / Q_{\mathbf{n},2}$ also has a simple pole at ζ. Using the argument principle, it follows that for all sufficiently large $|\mathbf{n}|$, $\mathbf{n} \in \Lambda$, $Q_{\mathbf{n},1}$ must have a simple zero inside C_r. The parameter r can be taken arbitrarily small; therefore, the last statement of the lemma readily follows and the basis of induction is fulfilled.

Let us assume that the lemma is satisfied for $k \in \{1, \dots, \kappa - 1\}$, $1 \leqslant \kappa \leqslant m$, and let us prove that it is also true for κ. From (29) applied to $\kappa - 1$, we have that

$$\lim_{\mathbf{n}\in\Lambda} \left| h_{\mathbf{n},\kappa}(x) \right| = \frac{1}{\sqrt{|(x-b_{\kappa-1})(x-a_{\kappa-1})|}},$$

uniformly on $\Delta_\kappa \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_{\kappa-1}^{\kappa-2})$. It follows that $\{|h_{\mathbf{n},\kappa}| d|s_{r_{\kappa-1}}^{\kappa-1}|\}$, $\mathbf{n} \in \Lambda$, is a sequence of Denisov type measures according to Definition 3 in [2] and $(\{|h_{\mathbf{n},\kappa}| d|s_{r_{\kappa-1}}^{\kappa-1}|\}, \{|Q_{\mathbf{n},\kappa-1} Q_{\mathbf{n},\kappa+1}|\}, l)$, $\mathbf{n} \in \Lambda$, is strongly admissible as in Definition 2 of [2] for each $l \in \mathbb{Z}$ (see paragraph just after both definitions in the referred paper). Therefore, we can apply Corollary 3 in [2] of which (28) is a particular case. In the proof of Corollary 3 of [2] (see also Theorem 9 in [3]) it is required that $\deg(Q_{\mathbf{n},k-1}Q_{\mathbf{n},k+1}) - 2\deg(Q_{\mathbf{n},k}) \leqslant C$ where $C \geqslant 0$ is a constant. For $k = 1$ this is trivially true (with $C = 0$). Since we apply an induction procedure on k, in order that this requirement be fulfilled for all $k \in \{1, \ldots, m\}$ we impose that $\max_{\mathbf{n}\in\Lambda}(\max_{k=1,\ldots,m} mn_k - |\mathbf{n}|) < \infty$. From (28), (29) and the rest of the statements of the lemma immediately follow just as in the case when $k = 1$. With this we conclude the proof. □

Remark 3.1. The last statement of Lemma 3.3 concerning the convergence of the zeros of $Q_{\mathbf{n},1}$ outside $\tilde{\Delta}_1$ to the mass points of σ_1 on $\operatorname{supp}(\sigma_1) \setminus \tilde{\Delta}_1$ can be proved without the assumption that $|\sigma_k'| > 0$ a.e. on $\tilde{\Delta}_k$, $k = 1, \ldots, m$. This is an easy consequence of Theorem 1 in [7]. From the proof of Lemma 3.3 it also follows that if we only have $|\sigma_k'| > 0$ a.e. on $\tilde{\Delta}_k$, $k = 1, \ldots, m'$, $m' \leqslant m$, then (28)–(29) are satisfied for $k = 1, \ldots, m'$ and the statement concerning the zeros holds for $k = 1, \ldots, m' + 1$.

Lemma 3.4. *Let $S = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ be a Nikishin system such that $\operatorname{supp}(\sigma_k) = \tilde{\Delta}_k \cup e_k$, $k = 1, \ldots, m$, where $\tilde{\Delta}_k$ is a bounded interval of the real line, $|\sigma_k'| > 0$ a.e. on $\tilde{\Delta}_k$, and e_k is a set without accumulation points in $\mathbb{R} \setminus \tilde{\Delta}_k$. Let $\Lambda \subset \mathbb{Z}_+^m(*)$ be an infinite sequence of distinct multi-indices with the property that $\max_{\mathbf{n}\in\Lambda}(\max_{k=1,\ldots,m} mn_k - |\mathbf{n}|) < \infty$. Let us assume that there exists $l \in \{1, \ldots, m\}$ and a fixed permutation τ of $\{1, \ldots, m\}$ such that for all $\mathbf{n} \in \Lambda$ we have that $\mathbf{n}, \mathbf{n}_l \in \mathbb{Z}_+^m(*, \tau)$. Then, for each $k = 1, \ldots, m$, and each compact set $\mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_k^{k-1})$ there exist positive constants $C_{k,1}(\mathcal{K}), C_{k,2}(\mathcal{K})$ such that*

$$C_{k,1}(\mathcal{K}) \leqslant \inf_{z\in\mathcal{K}} \left| \frac{Q_{\mathbf{n}_l,k}(z)}{Q_{\mathbf{n},k}(z)} \right| \leqslant \sup_{z\in\mathcal{K}} \left| \frac{Q_{\mathbf{n}_l,k}(z)}{Q_{\mathbf{n},k}(z)} \right| \leqslant C_{k,2}(\mathcal{K}),$$

for all sufficiently large $|\mathbf{n}|$, $\mathbf{n} \in \Lambda$.

Proof. The uniform bound from above and below on each fixed compact subset $\mathcal{K} \subset \mathbb{C} \setminus \Delta_k$ (for all $\mathbf{n} \in \Lambda$) is a direct consequence of the interlacing property of the zeros of $Q_{\mathbf{n}_l,k}$ and $Q_{\mathbf{n},k}$. In fact, comparing distances to $z \in \mathcal{K}$ of consecutive interlacing zeros, it is easy to verify that

$$d_1 \leqslant \inf_{z\in\mathcal{K}} \left| \frac{Q_{\mathbf{n}_l,k}(z)}{Q_{\mathbf{n},k}(z)} \right| \leqslant \sup_{z\in\mathcal{K}} \left| \frac{Q_{\mathbf{n}_l,k}(z)}{Q_{\mathbf{n},k}(z)} \right| \leqslant \frac{d_2^2}{d_1},$$

where d_2 denotes the diameter of $\mathcal{K} \cup \Delta_k$ and d_1 denotes the distance between $\mathcal{K}$ and Δ_k. It is not needed that $|\sigma_k'| > 0$ or $\max_{\mathbf{n}\in\Lambda}(\max_{k=1,\ldots,m} mn_k - |\mathbf{n}|) < \infty$.

These restrictions come in so as to guarantee that the zeros of the polynomials $Q_{\mathbf{n}_l,k}$ and $Q_{\mathbf{n},k}$ lying in $\Delta_k \setminus \operatorname{supp}(\sigma_k^{k-1})$ converge to the mass points as Lemma 3.3 asserts in this case. Then, we can allow $\mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_k^{k-1})$. Let $\mathcal{K}$ be such. Notice that $\mathcal{K}$ can intersect at most a finite number $I_0, \ldots, I_M$ ($I_0 = \emptyset$) of the open intervals forming $\Delta_k \setminus \operatorname{supp}(\sigma_k^{k-1})$. The polynomials $Q_{\mathbf{n}_l,k}$ and $Q_{\mathbf{n},k}$ can have at most one zero in each of those intervals. Consequently, for all $|\mathbf{n}|$, $\mathbf{n} \in \Lambda$, sufficiently large, the zeros of $Q_{\mathbf{n}_l,k}$ and $Q_{\mathbf{n},k}$ lie at a positive distance ε from $\mathcal{K}$. Now, it is easy to show that

$$\tilde{d}_1 \left(\frac{\varepsilon}{d_2} \right)^M \leqslant \inf_{z\in\mathcal{K}} \left| \frac{Q_{\mathbf{n}_l,k}(z)}{Q_{\mathbf{n},k}(z)} \right| \leqslant \sup_{z\in\mathcal{K}} \left| \frac{Q_{\mathbf{n}_l,k}(z)}{Q_{\mathbf{n},k}(z)} \right| \leqslant \frac{\tilde{d}_2^2}{\tilde{d}_1} \left(\frac{d_2}{\varepsilon} \right)^M,$$

where $\tilde{d}_2$ denotes the diameter of $\mathcal{K} \cup (\Delta_k \setminus \bigcup_{j=0}^{M} I_j)$ and $\tilde{d}_1$ the distance between $\mathcal{K}$ and $\Delta_k \setminus \bigcup_{j=0}^{M} I_j$. $\square$

4. Proof of main results

In this final section, $S = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ is a Nikishin system with $\operatorname{supp}(\sigma_k) = \tilde{\Delta}_k \cup e_k$, $k = 1, \ldots, m$, where $\tilde{\Delta}_k$ is a bounded interval of the real line, $|\sigma_k'| > 0$ a.e. on $\tilde{\Delta}_k$, and e_k is a set without accumulation points in $\mathbb{R} \setminus \tilde{\Delta}_k$. Let $\Lambda \subset \mathbb{Z}_+^m(*)$ be a sequence of distinct multi-indices. Let us assume that there exists $l \in \{1, \ldots, m\}$ and a fixed permutation τ of $\{1, \ldots, m\}$ such that for all $\mathbf{n} \in \Lambda$ we have that $\mathbf{n}, \mathbf{n}_l \in \mathbb{Z}_+^m(*, \tau)$. From Lemma 3.4 we know that the sequences

$$\{Q_{\mathbf{n}_l,k}/Q_{\mathbf{n},k}\}_{\mathbf{n}\in\Lambda}, \quad k = 1, \ldots, m,$$

are uniformly bounded on each compact subset of $\mathbb{C} \setminus \operatorname{supp}(\sigma_k^{k-1})$ for all sufficiently large $|\mathbf{n}|$. By Montel's theorem, there exists a subsequence of multi-indices $\Lambda' \subset \Lambda$ and a collection of functions $\tilde{F}_k^l$, holomorphic in $\mathbb{C} \setminus \operatorname{supp}(\sigma_k^{k-1})$, respectively, such that

$$\lim_{\mathbf{n}\in\Lambda'} \frac{Q_{\mathbf{n}_l,k}(z)}{Q_{\mathbf{n},k}(z)} = \tilde{F}_k^{(l)}(z), \qquad \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}\bigl(\sigma_k^{k-1}\bigr), \quad k = 1, \ldots, m. \tag{30}$$

In principle, the functions $\tilde{F}_k^{(l)}$ may depend on Λ'. We shall see that this is not the case and, therefore, the limit in (30) holds for $\mathbf{n} \in \Lambda$. First, let us obtain some general information on the functions $\tilde{F}_k^{(l)}$.

The points in $\operatorname{supp}(\sigma_k^{k-1}) \setminus \tilde{\Delta}_k$ are isolated singularities of $\tilde{F}_k^{(l)}$. Let $\zeta \in \operatorname{supp}(\sigma_k^{k-1}) \setminus \tilde{\Delta}_k$. By Lemma 3.3 each such point is a limit of zeros of $Q_{\mathbf{n},k}$ and $Q_{\mathbf{n}_l,k}$ as $|\mathbf{n}| \to \infty$, $\mathbf{n} \in \Lambda$, and in a sufficiently small neighborhood of them, for each $\mathbf{n} \in \Lambda$, there can be at most one such zero of these polynomials (so there is exactly one, for all sufficiently large $|\mathbf{n}|$). Let $\lim_{\mathbf{n}\in\Lambda} \zeta_{\mathbf{n}} = \zeta$ where $Q_{\mathbf{n},k}(\zeta_{\mathbf{n}}) = 0$. From (30)

$$\lim_{\mathbf{n}\in\Lambda'} \frac{(z-\zeta_{\mathbf{n}})Q_{\mathbf{n}_l,k}(z)}{Q_{\mathbf{n},k}(z)} = (z-\zeta)\tilde{F}_k^{(l)}(z), \qquad \mathcal{K} \subset \bigl(\mathbb{C} \setminus \operatorname{supp}\bigl(\sigma_k^{k-1}\bigr)\bigr) \cup \{\zeta\},$$

and $(z-\zeta)\tilde{F}_k^{(l)}(z)$ is analytic in a neighborhood of ζ. Hence ζ is not an essential singularity of $\tilde{F}_k^{(l)}$. Taking into consideration that $Q_{\mathbf{n}_l,k}, \mathbf{n} \in \Lambda$ also has a sequence of zeros converging to ζ, from the argument principle it follows that ζ is a removable singularity of $\tilde{F}_k^{(l)}$ which is not a zero. By Lemma 3.4 we also know that the sequence of functions $|Q_{\mathbf{n}_l,k}/Q_{\mathbf{n},k}|$, $\mathbf{n} \in \Lambda$, is uniformly bounded from below by a positive constant for all sufficiently large $|\mathbf{n}|$. Therefore, in $\mathbb{C} \setminus \operatorname{supp}(\sigma_k^{k-1})$ the function $\tilde{F}_k^{(l)}$ is also different from zero. According to the definition of $Q_{\mathbf{n},k}$ and $Q_{\mathbf{n}_l,k}$ and Lemma 3.2, for $k = 1, \ldots, \tau^{-1}(l)$, we have that $\deg Q_{\mathbf{n}_l,k} = |\mathbf{n}_l^{k-1}| = |\mathbf{n}^{k-1}| + 1 = \deg Q_{\mathbf{n},k} + 1$ whereas, for $k = \tau^{-1}(l) + 1, \ldots, m$, we obtain that $\deg Q_{\mathbf{n}_l,k} = |\mathbf{n}_l^{k-1}| = |\mathbf{n}^{k-1}| = \deg Q_{\mathbf{n},k}$. Consequently, for $k = 1, \ldots, \tau^{-1}(l)$, the function $\tilde{F}_k^{(l)}$ has a simple pole at infinity and $(\tilde{F}_k^{(l)})'(\infty) = 1$, whereas, for $k = \tau^{-1}(l) + 1, \ldots, m$, it is analytic at infinity and $\tilde{F}_k^{(l)}(\infty) = 1$.

Now let us prove that the functions $\tilde{F}_k^{(l)}$ satisfy a system of boundary value problems.

Lemma 4.1. *Let $S=\mathcal{N}(\sigma_1,\ldots,\sigma_m)$ be a Nikishin system with* $\operatorname{supp}(\sigma_k)=\tilde{\Delta}_k\cup e_k$, $k=1,\ldots,m$, *where $\tilde{\Delta}_k$ is a bounded interval of the real line, $|\sigma_k'|>0$ a.e. on $\tilde{\Delta}_k$, and e_k is a set without accumulation points in $\mathbb{R}\setminus\tilde{\Delta}_k$. Let $\Lambda\subset\mathbb{Z}_+^m(*)$ be a sequence of distinct multi-indices such that $\max_{\mathbf{n}\in\Lambda}(\max_{k=1,\ldots,m} mn_k-|\mathbf{n}|)<\infty$. Let us assume that there exists $l\in\{1,\ldots,m\}$ and a fixed permutation τ of $\{1,\ldots,m\}$ such that for all $\mathbf{n}\in\Lambda$ we have that $\mathbf{n},\mathbf{n}_l\in\mathbb{Z}_+^m(*,\tau)$. Take $\Lambda'\subset\Lambda$ such that* (30) *holds. Then, there exists a normalization $F_k^{(l)}$, $k=1,\ldots,m$, by positive constants, of the functions $\tilde{F}_k^{(l)}$, $k=1,\ldots,m$, given in* (30)*, which verifies the system of boundary value problems*

$$
\begin{aligned}
&1) \qquad F_k^{(l)},\ 1/F_k^{(l)}\in H(\mathbb{C}\setminus\tilde{\Delta}_k),\\
&2) \qquad \big(F_k^{(l)}\big)'(\infty)>0,\quad k=1,\ldots,\tau^{-1}(l),\\
&2') \qquad F_k^{(l)}(\infty)>0,\quad k=\tau^{-1}(l)+1,\ldots,m,\\
&3) \qquad \big|F_k^{(l)}(x)\big|^2\frac{1}{|(F_{k-1}^{(l)}F_{k+1}^{(l)})(x)|}=1,\ x\in\tilde{\Delta}_k,
\end{aligned}
\tag{31}
$$

where $F_0^{(l)}\equiv F_{m+1}^{(l)}\equiv 1$.

Proof. The assertions 1), 2), and 2′) were proved above for the functions $\tilde{F}_k^{(l)}$. Consequently, they are satisfied for any normalization of these functions by means of positive constants.

From (26) applied to $\mathbf{n}$ and $\mathbf{n}_l$, for each $k=1,\ldots,m$, we have

$$\int_{\Delta_k} x^\nu Q_{\mathbf{n},k}(x)\,d|\rho_{\mathbf{n},k}|(x)=0,\qquad \nu=0,\ldots,\big|\mathbf{n}^{k-1}\big|-1,$$

and

$$\int_{\Delta_k} x^\nu Q_{\mathbf{n}_l,k}(x)g_{\mathbf{n},k}(x)\,d|\rho_{\mathbf{n},k}|(x)=0,\qquad \nu=0,\ldots,\big|\mathbf{n}_l^{k-1}\big|-1,$$

where

$$g_{\mathbf{n},k}(x)=\frac{|Q_{\mathbf{n},k-1}(x)Q_{\mathbf{n},k+1}(x)|}{|Q_{\mathbf{n}_l,k-1}(x)Q_{\mathbf{n}_l,k+1}(x)|}\frac{|h_{\mathbf{n}_l,k}(x)|}{|h_{\mathbf{n},k}(x)|},\qquad d\rho_{\mathbf{n},k}(x)=\frac{h_{\mathbf{n},k}(x)\,ds_{r_{k-1}}^{k-1}(x)}{Q_{\mathbf{n},k-1}(x)Q_{\mathbf{n},k+1}(x)}.$$

From (29) and (30)

$$\lim_{\mathbf{n}\in\Lambda'} g_{\mathbf{n},k}(x)=\big|\big(\tilde{F}_{k-1}^{(l)}\tilde{F}_{k+1}^{(l)}\big)(x)\big|^{-1}\tag{32}$$

uniformly on Δ_k.

Fix $k\in\{\tau^{-1}(l)+1,\ldots,m\}$. As mentioned above, for this selection of k we have that $\deg Q_{\mathbf{n}_l,k}=\deg Q_{\mathbf{n},k}=|\mathbf{n}^{k-1}|$. Using Theorems 1 and 2 of [2], and (30), it follows that

$$\lim_{\mathbf{n}\in\Lambda'}\frac{Q_{\mathbf{n}_l,k}(z)}{Q_{\mathbf{n},k}(z)}=\frac{S_k(z)}{S_k(\infty)}=\tilde{S}_k(z)=\tilde{F}_k^{(l)}(z),\qquad \mathcal{K}\subset\overline{\mathbb{C}}\setminus\operatorname{supp}\big(\sigma_k^{k-1}\big),\tag{33}$$

where S_k denotes the Szegő function on $\overline{\mathbb{C}} \setminus \tilde{\Delta}_k$ with respect to the weight $|\tilde{F}_{k-1}^{(l)}(x)\tilde{F}_{k+1}^{(l)}(x)|^{-1}$, $x \in \tilde{\Delta}_k$. The function S_k is uniquely determined by

$$\begin{aligned} &1) \qquad S_k, 1/S_k \in H(\overline{\mathbb{C}} \setminus \tilde{\Delta}_k),\\ &2) \qquad S_k(\infty) > 0,\\ &3) \qquad \left|S_k(x)\right|^2 \frac{1}{|(\tilde{F}_{k-1}^{(l)}\tilde{F}_{k+1}^{(l)})(x)|} = 1, \quad x \in \tilde{\Delta}_k. \end{aligned} \tag{34}$$

Now, fix $k \in \{1, \dots, \tau^{-1}(l)\}$. In this situation $\deg Q_{\mathbf{n}_l,k} = \deg Q_{\mathbf{n},k} + 1 = |\mathbf{n}^{k-1}| + 1$. Let $Q_{\mathbf{n},k}^*(x)$ be the monic polynomial of degree $|\mathbf{n}^{k-1}|$ orthogonal with respect to the varying measure $g_{\mathbf{n},k}\, d|\rho_{\mathbf{n},k}|$. Using the same arguments as above, we have

$$\lim_{\mathbf{n}\in\Lambda'} \frac{Q_{\mathbf{n},k}^*(z)}{Q_{\mathbf{n},k}(z)} = \frac{S_k(z)}{S_k(\infty)} = \tilde{S}_k(z), \qquad \mathcal{K} \subset \overline{\mathbb{C}} \setminus \operatorname{supp}\big(\sigma_k^{k-1}\big). \tag{35}$$

On the other hand, since $\deg Q_{\mathbf{n}_l,k} = \deg Q_{\mathbf{n},k}^* + 1$ and both of these polynomials are orthogonal with respect to the same varying weight, by Theorem 1 of [2] and (30), it follows that

$$\lim_{\mathbf{n}\in\Lambda'} \frac{Q_{\mathbf{n}_l,k}(z)}{Q_{\mathbf{n},k}^*(z)} = \frac{\varphi_k(z)}{\varphi_k'(\infty)} = \tilde{\varphi}_k(z), \qquad \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}\big(\sigma_k^{k-1}\big), \tag{36}$$

where φ_k denotes the conformal representation of $\overline{\mathbb{C}} \setminus \tilde{\Delta}_k$ onto $\{w\colon\ |w| > 1\}$ such that $\varphi_k(\infty) = \infty$ and $\varphi_k'(\infty) > 0$. The function φ_k is uniquely determined by

$$\begin{aligned} &1) \qquad \varphi_k, 1/\varphi_k \in H(\mathbb{C} \setminus \tilde{\Delta}_k),\\ &2) \qquad \varphi_k'(\infty) > 0,\\ &3) \qquad \left|\varphi_k(x)\right| = 1, \quad x \in \tilde{\Delta}_k. \end{aligned} \tag{37}$$

From (35) and (36), we obtain

$$\lim_{\mathbf{n}\in\Lambda'} \frac{Q_{\mathbf{n}_l,k}(z)}{Q_{\mathbf{n},k}(z)} = (\tilde{S}_k\tilde{\varphi}_k)(z) = \tilde{F}_k^{(l)}(z), \qquad \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}\big(\sigma_k^{k-1}\big). \tag{38}$$

Thus,

$$\tilde{F}_k^{(l)} = \begin{cases} \tilde{S}_k\tilde{\varphi}_k, & k = 1, \dots, \tau^{-1}(l),\\ \tilde{S}_k, & k = \tau^{-1}(l) + 1, \dots, m, \end{cases} \tag{39}$$

and from (34) and (39) it follows that

$$\left|\tilde{F}_k^{(l)}(x)\right|^2 \frac{1}{|(\tilde{F}_{k-1}^{(l)}\tilde{F}_{k+1}^{(l)})(x)|} = \frac{1}{\omega_k}, \quad x \in \tilde{\Delta}_k,\ k = 1, \dots, m, \tag{40}$$

where

$$\omega_k = \begin{cases} (S_k\varphi_k')^2(\infty), & k = 1, \ldots, \tau^{-1}(l), \\ S_k^2(\infty), & k = \tau^{-1}(l) + 1, \ldots, m. \end{cases} \tag{41}$$

Now, let us show that there exist positive constants c_k, $k = 1, \ldots, m$, such that the functions $F_k^{(l)} = c_k \tilde{F}_k^{(l)}$ satisfy (31). In fact, according to (40) for any such constants c_k we have that

$$\left|F_k^{(l)}(x)\right|^2 \frac{1}{|(F_{k-1}^{(l)} F_{k+1}^{(l)})(x)|} = \frac{c_k^2}{c_{k-1}c_{k+1}\omega_k}, \quad x \in \tilde{\Delta}_k,\ k = 1, \ldots, m,$$

where $c_0 = c_{m+1} = 1$. The problem reduces to finding appropriate constants c_k such that

$$\frac{c_k^2}{c_{k-1}c_{k+1}\omega_k} = 1, \quad k = 1, \ldots, m. \tag{42}$$

Taking logarithm, we obtain the linear system of equations

$$2\log c_k - \log c_{k-1} - \log c_{k+1} = \log \omega_k, \quad k = 1, \ldots, m \tag{43}$$

($c_0 = c_{m+1} = 1$) on the unknowns $\log c_k$. This system has a unique solution with which we conclude the proof. □

Consider the $(m+1)$-sheeted compact Riemann surface $\mathcal{R}$ introduced in Section 1. Given $l \in \{1, \ldots, m\}$, let $\psi^{(l)}$ be a singled valued function defined on $\mathcal{R}$ onto the extended complex plane satisfying

$$\psi^{(l)}(z) = \frac{C_1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right), \quad z \to \infty^{(0)},$$
$$\psi^{(l)}(z) = C_2 z + \mathcal{O}(1), \quad z \to \infty^{(l)},$$

where C_1 and C_2 are nonzero constants. Since the genus of $\mathcal{R}$ is zero, $\psi^{(l)}$ exists and is uniquely determined up to a multiplicative constant. Consider the branches of $\psi^{(l)}$, corresponding to the different sheets $k = 0, 1, \ldots, m$ of $\mathcal{R}$

$$\psi^{(l)} := \left\{\psi_k^{(l)}\right\}_{k=0}^m.$$

We normalize $\psi^{(l)}$ so that

$$\prod_{k=0}^m \left|\psi_k^{(l)}(\infty)\right| = 1, \quad C_1 \in \mathbb{R} \setminus \{0\}. \tag{44}$$

Certainly, there are two $\psi^{(l)}$ verifying this normalization.

Given an arbitrary function $F(z)$ which has in a neighborhood of infinity a Laurent expansion of the form $F(z) = Cz^k + \mathcal{O}(z^{k-1})$, $C \neq 0$, and $k \in \mathbb{Z}$, we denote

$$\tilde{F} := F/C.$$

C is called the leading coefficient of F. When $C \in \mathbb{R}$, $\mathrm{sg}(F(\infty))$ will represent the sign of C.

Since the product of all the branches $\prod_{k=0}^{m} \psi_k^{(l)}$ is a single valued analytic function in $\overline{\mathbb{C}}$ without singularities, by Liouville's theorem it is constant and because of the normalization introduced above this constant is either 1 or -1. In particular, the function appearing in (4) equals

$$G_0^{(\tau^{-1}(l))}(z) = 1/\tilde{\psi}_0^{(\tau^{-1}(l))}(z) = \prod_{k=1}^{m} \tilde{\psi}_k^{(\tau^{-1}(l))}(z). \tag{45}$$

If $\psi^{(l)}$ is such that $C_1 \in \mathbb{R} \setminus \{0\}$, then

$$\psi^{(l)}(z) = \overline{\psi^{(l)}(\bar{z})}, \quad z \in \mathcal{R}.$$

In fact, let $\phi(z) := \overline{\psi^{(l)}(\bar{z})}$. ϕ and $\psi^{(l)}$ have the same divisor; consequently, there exists a constant C such that $\phi = C\psi^{(l)}$. Comparing the leading coefficients of the Laurent expansion of these functions at $\infty^{(0)}$, we conclude that $C = 1$ since $C_1 \in \mathbb{R} \setminus \{0\}$.

In terms of the branches of $\psi^{(l)}$, the symmetry formula above means that for each $k = 0, 1, \ldots, m$:

$$\psi_k^{(l)} : \overline{\mathbb{R}} \setminus (\tilde{\Delta}_k \cup \tilde{\Delta}_{k+1}) \to \overline{\mathbb{R}}$$

($\tilde{\Delta}_0 = \tilde{\Delta}_{m+1} = \emptyset$); therefore, the coefficients (in particular, the leading one) of the Laurent expansion at ∞ of these branches are real numbers, $\mathrm{sg}(\psi_k^{(l)}(\infty))$ is defined, and

$$\psi_k^{(l)}(x_\pm) = \overline{\psi_k^{(l)}(x_\mp)} = \overline{\psi_{k+1}^{(l)}(x_\pm)}, \quad x \in \tilde{\Delta}_{k+1}. \tag{46}$$

We are ready to state and prove our main result.

Theorem 4.1. *Let $S = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ be a Nikishin system with $\mathrm{supp}(\sigma_k) = \tilde{\Delta}_k \cup e_k$, $k = 1, \ldots, m$, where $\tilde{\Delta}_k$ is a bounded interval of the real line, $|\sigma_k'| > 0$ a.e. on $\tilde{\Delta}_k$, and e_k is a set without accumulation points in $\mathbb{R} \setminus \tilde{\Delta}_k$. Let $\Lambda \subset \mathbb{Z}_+^m(*)$ be a sequence of distinct multi-indices such that $\max_{\mathbf{n}\in\Lambda}(\max_{k=1,\ldots,m} mn_k - |\mathbf{n}|) < \infty$. Let us assume that there exists $l \in \{1, \ldots, m\}$ and a fixed permutation τ of $\{1, \ldots, m\}$ such that for all $\mathbf{n} \in \Lambda$ we have that $\mathbf{n}, \mathbf{n}_l \in \mathbb{Z}_+^m(*, \tau)$. Let $\{Q_{\mathbf{n},k}\}_{k=1}^m$, $\mathbf{n} \in \Lambda$, be the corresponding sequences of polynomials defined in Section 3. Then, for each fixed $k \in \{1, \ldots, m\}$, we have*

$$\lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n}_l,k}(z)}{Q_{\mathbf{n},k}(z)} = \tilde{F}_k^{(l)}(z), \quad z \in \mathcal{K} \subset \mathbb{C} \setminus \mathrm{supp}(\sigma_k^{k-1}), \tag{47}$$

where

$$F_k^{(l)} := \mathrm{sg}\left(\prod_{\nu=k}^{m} \psi_\nu^{(\tau^{-1}(l))}(\infty)\right) \prod_{\nu=k}^{m} \psi_\nu^{(\tau^{-1}(l))}. \tag{48}$$

Proof. Since the families of functions

$$\{Q_{\mathbf{n}_l,k}/Q_{\mathbf{n},k}\}_{\mathbf{n}\in\Lambda}, \quad k=1,\dots,m,$$

are uniformly bounded on each compact subset $\mathcal{K}\subset\mathbb{C}\setminus\operatorname{supp}(\sigma_k^{k-1})$ for all sufficiently large $|\mathbf{n}|$, $\mathbf{n}\in\Lambda$, uniform convergence on compact subsets of the indicated region follows from proving that any convergent subsequence has the same limit. According to Lemma 4.1 the limit functions, appropriately normalized, of a convergent subsequence satisfy the same system of boundary value problems (31). According to Lemma 4.2 in [1] this system has a unique solution.

It remains to show that the functions defined in (48) satisfy (31). When multiplying two consecutive branches, the singularities on the common slit cancel out; therefore, 1) takes place since only the singularities of $\psi_k^{(\tau^{-1}(l))}$ on $\tilde{\Delta}_k$ remain. From the definition of $\psi^{(\tau^{-1}(l))}$ it also follows that for $k=1,\dots,\tau^{-1}(l)$, $F_k^{(l)}$ has at infinity a simple pole, whereas it is regular and different from zero when $k=\tau^{-1}(l)+1,\dots,m$. The factor sign in front of (48) guarantees the positivity claimed in 2) and 2′).

In order to verify 3), notice that $F_k^{(l)}/F_{k-1}^{(l)}=\operatorname{sg}(\psi_{k-1}^{(\tau^{-1}(l))}(\infty))/\psi_{k-1}^{(\tau^{-1}(l))}$. Therefore, if $k=2,\dots,m$,

$$\frac{|F_k^{(l)}(x)|^2}{|F_{k-1}^{(l)}(x)F_{k+1}^{(l)}(x)|}=\frac{|\psi_k^{(\tau^{-1}(l))}(x)|}{|\psi_{k-1}^{(\tau^{-1}(l))}(x)|}=1, \quad x\in\tilde{\Delta}_k,$$

on account of (46). For $k=1$, from the definition and (46)

$$\frac{|F_1^{(l)}(x)|^2}{|F_2^{(l)}(x)|}=\left|\psi_1^{(\tau^{-1}(l))}(x)\right|^2\left|\prod_{\nu=2}^{m}\psi_\nu^{(\tau^{-1}(l))}(x)\right|=\left|\prod_{\nu=0}^{m}\psi_\nu^{(\tau^{-1}(l))}(x)\right|=1, \quad x\in\tilde{\Delta}_1,$$

since $\prod_{\nu=0}^{m}\psi_\nu^{(\tau^{-1}(l))}$ is constantly equal to 1 or -1 on all $\overline{\mathbb{C}}$. □

Theorem 1.1 is a particular case of Theorem 4.1 on account of (45).

Proof of Corollary 1.1. Let

$$\Lambda_\tau=\Lambda\cap\mathbb{Z}_+^m(*,\tau),$$

where τ is a given permutation of $\{1,\dots,m\}$. We are only interested in those Λ_τ with infinitely many terms. There are at most $m!$ such subsequences. For $\mathbf{n}\in\Lambda_\tau$ fixed, denote $\mathbf{n}_{\tau(j)}$, $j\in\{1,\dots,m\}$, the multi-index obtained adding one to all j components $\tau(1),\dots,\tau(j)$ of $\mathbf{n}$. (Notice that this notation differs from that introduced previously for $\mathbf{n}_l$.) In particular, $\mathbf{n}_{\tau(m)}=\mathbf{n}+\mathbf{1}$. It is easy to verify that for all $j\in\{1,\dots,m\}$, $\mathbf{n}_{\tau(j)}\in\Lambda_\tau$. For all $\mathbf{n}\in\Lambda_\tau$ and each $k\in\{1,\dots,m\}$, we have

$$\frac{Q_{\mathbf{n}+\mathbf{1},k}}{Q_{\mathbf{n},k}}=\prod_{j=0}^{m-1}\frac{Q_{\mathbf{n}_{\tau(j+1)},k}}{Q_{\mathbf{n}_{\tau(j)},k}},$$

where $Q_{\mathbf{n}_{\tau(0)},k} = Q_{\mathbf{n},k}$. From (47) it follows that

$$\lim_{\mathbf{n}\in\Lambda_\tau} \frac{Q_{\mathbf{n+1},k}(z)}{Q_{\mathbf{n},k}(z)} = \prod_{l=1}^{m} \tilde{F}_k^{(l)}(z), \qquad \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}\bigl(\sigma_k^{k-1}\bigr).$$

The right side does not depend on l, since all possible values intervene. Therefore, the limit is the same for all τ and thus

$$\lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n+1},k}(z)}{Q_{\mathbf{n},k}(z)} = \prod_{l=1}^{m} \tilde{F}_k^{(l)}(z), \qquad \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}\bigl(\sigma_k^{k-1}\bigr). \tag{49}$$

Formula (5) is (49) for $k = 1$ on account of (45) and (48). □

The following corollary complements Theorem 4.1. The proof is similar to that of Corollary 4.1 in [1].

Corollary 4.1. *Let $S = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ be a Nikishin system with $\operatorname{supp}(\sigma_k) = \tilde{\Delta}_k \cup e_k$, $k = 1, \ldots, m$, where $\tilde{\Delta}_k$ is a bounded interval of the real line, $|\sigma_k'| > 0$ a.e. on $\tilde{\Delta}_k$, and e_k is a set without accumulation points in $\mathbb{R} \setminus \tilde{\Delta}_k$. Let $\Lambda \subset \mathbb{Z}_+^m(*)$ be a sequence of distinct multi-indices such that $\max_{\mathbf{n}\in\Lambda}(\max_{k=1,\ldots,m} mn_k - |\mathbf{n}|) < \infty$. Let us assume that there exists $l \in \{1, \ldots, m\}$ and a fixed permutation τ of $\{1, \ldots, m\}$ such that for all $\mathbf{n} \in \Lambda$ we have that $\mathbf{n}, \mathbf{n}_l \in \mathbb{Z}_+^m(*, \tau)$. Let $\{q_{\mathbf{n},k} = \kappa_{\mathbf{n},k} Q_{\mathbf{n},k}\}_{k=1}^m$, $\mathbf{n} \in \Lambda$, be the system of orthonormal polynomials as defined in* (24) *and $\{K_{\mathbf{n},k}\}_{k=1}^m$, $\mathbf{n} \in \Lambda$, the values given by* (23)*. Then, for each fixed $k = 1, \ldots, m$, we have*

$$\lim_{\mathbf{n}\in\Lambda} \frac{\kappa_{\mathbf{n}_l,k}}{\kappa_{\mathbf{n},k}} = \kappa_k^{(l)}, \tag{50}$$

$$\lim_{\mathbf{n}\in\Lambda} \frac{K_{\mathbf{n}_l,k}}{K_{\mathbf{n},k}} = \kappa_1^{(l)} \cdots \kappa_k^{(l)}, \tag{51}$$

and

$$\lim_{\mathbf{n}\in\Lambda} \frac{q_{\mathbf{n}_l,k}(z)}{q_{\mathbf{n},k}(z)} = \kappa_k^{(l)} \tilde{F}_k^{(l)}(z), \quad z \in \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}\bigl(\sigma_k^{k-1}\bigr), \tag{52}$$

where

$$\kappa_k^{(l)} = \frac{c_k^{(l)}}{\sqrt{c_{k-1}^{(l)} c_{k+1}^{(l)}}}, \qquad c_k^{(l)} = \begin{cases} (F_k^{(l)})'(\infty), & k = 1, \ldots, \tau^{-1}(l), \\ F_k^{(l)}(\infty), & k = \tau^{-1}(l) + 1, \ldots, m, \end{cases} \tag{53}$$

and the $F_k^{(l)}$ are defined by (48).

Proof. By Theorem 4.1, we have limit in (32) along the whole sequence Λ. Reasoning as in the deduction of formulas (33) and (38), but now in connection with orthonormal polynomials (see Theorems 1 and 2 of [2]), it follows that

$$\lim_{\mathbf{n}\in\Lambda} \frac{q_{\mathbf{n}_l,k}(z)}{q_{\mathbf{n},k}(z)} = \begin{cases} (S_k\varphi_k)(z), & k = 1, \ldots, \tau^{-1}(l), \\ S_k(z), & k = \tau^{-1}(l) + 1, \ldots, m, \end{cases} \qquad \mathcal{K} \subset \mathbb{C} \setminus \operatorname{supp}\bigl(\sigma_k^{k-1}\bigr),$$

where S_k is defined in (34). This formula, divided by (33) or (38) according to the value of k gives

$$\lim_{\mathbf{n}\in\Lambda} \frac{\kappa_{\mathbf{n}_l,k}}{\kappa_{\mathbf{n},k}} = \sqrt{\omega_k} = \frac{c_k}{\sqrt{c_{k-1}c_{k+1}}},$$

where ω_k is defined in (41), and the c_k are the normalizing constants found in Lemma 3.1 solving the linear system of equations (43) which ensure that

$$F_k^{(l)} \equiv c_k \tilde{F}_k^{(l)}, \quad k = 1, \ldots, m,$$

with $F_k^{(l)}$ satisfying (31) and thus given by (48). Since $(\tilde{F}_k^{(l)})'(\infty) = 1$, $k = 1, \ldots, \tau^{-1}(l)$, and $(\tilde{F}_k^{(l)})(\infty) = 1$, $k = \tau^{-1}(l) + 1, \ldots, m$, formula (50) immediately follows with $\kappa_k^{(l)}$ as in (53).

From the definition of $\kappa_{\mathbf{n},k}$, we have that

$$K_{\mathbf{n},k} = \kappa_{\mathbf{n},1} \cdots \kappa_{\mathbf{n},k}.$$

Taking the ratio of these constants for the multi-indices $\mathbf{n}$ and $\mathbf{n}_l$ and using (50), we get (51). Formula (52) is an immediate consequence of (50) and (47). □

Remark 4.1. We have imposed two types of restrictions on the class of multi-indices under consideration. The first one refers to being in $\mathbb{Z}_+^m(*)$. This is connected with a long standing question in the theory of multiple orthogonal polynomials; namely, if for any m all multi-indices of a Nikishin system are strongly normal or not. We have proved our results in the largest class of multi-indices known to be strongly normal. Should this conjecture be solved in the positive sense, our methods would allow to eliminate this condition as we have done for the cases $m = 1, 2, 3$.

The second restriction $\max_{\mathbf{n}\in\Lambda}(\max_{k=1,\ldots,m} mn_k - |\mathbf{n}|) < \infty$ is connected with the use of Lemma 3.3. This condition means that all components of the multi-indices are of the same order and that orthogonality is, basically, equally distributed between all measures. The proof of (28) requires the density of certain classes of rational functions with fixed poles (in our case at the zeros of the polynomials $Q_{\mathbf{n},k-1}Q_{\mathbf{n},k+1}$ and numerator of degree twice the order of orthogonality) in the space of continuous functions on a given interval. In general, this is not true if the rational functions are such that the degree of the denominator is much larger in order than that of the numerator (as $|\mathbf{n}| \to \infty$). This is what may occur if we eliminate the restriction above. It can be relaxed to $n_k = |\mathbf{n}|/m + \mathcal{O}(\log|\mathbf{n}|)$, $k = 1, \ldots, m$, without changing the structure of the Riemann surface which describes the solution of the problem, but not much more. In limiting situations (for example, if one of the components of the multi-indices is not allowed to grow at all) some of the sheets may even disappear. The description of the solution in the most general situation is very difficult and technically complicated. On the other hand, in applications, the diagonal case ($n_k = |\mathbf{n}|/m$, $k = 1, \ldots, m$) and nearby indices are the most important.

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