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Journal of Approximation Theory 158 (2009) 214–241
Journal of Approximation Theory
www.elsevier.com/locate/jat

Relative asymptotic of multiple orthogonal polynomials for Nikishin systems

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Received 1 February 2008; received in revised form 25 July 2008; accepted 1 September 2008
Available online 9 November 2008

Communicated by C.K. Chui and H.N. Mhaskar

Dedicated to the memory of G.G. Lorentz

Abstract

We prove the relative asymptotic behavior for the ratio of two sequences of multiple orthogonal polynomials with respect to the Nikishin systems of measures. The first Nikishin system $\mathcal{N}(\sigma_1, \ldots, \sigma_m)$ is such that for each k, σ_k has a constant sign on its compact support $\mathrm{supp}(\sigma_k) \subset \mathbb{R}$ consisting of an interval $\widetilde{\Delta}_k$, on which $|\sigma_k'| > 0$ almost everywhere, and a discrete set without accumulation points in $\mathbb{R} \setminus \widetilde{\Delta}_k$. If $\mathrm{Co}(\mathrm{supp}(\sigma_k)) = \Delta_k$ denotes the smallest interval containing $\mathrm{supp}(\sigma_k)$, we assume that $\Delta_k \cap \Delta_{k+1} = \emptyset$, $k = 1, \ldots, m-1$. The second Nikishin system $\mathcal{N}(r_1\sigma_1, \ldots, r_m\sigma_m)$ is a perturbation of the first by means of rational functions r_k, $k = 1, \ldots, m$, whose zeros and poles lie in $\mathbb{C} \setminus \cup_{k=1}^m \Delta_k$.

Keywords: Multiple orthogonal polynomials; Nikishin systems; Relative asymptotic

1. Introduction

Let σ_1, σ_2 be two finite Borel measures, whose supports $\mathrm{supp}(\sigma_1)$, $\mathrm{supp}(\sigma_2)$ are compact sets contained in non intersecting intervals Δ_1, Δ_2, respectively, of the real line $\mathbb{R}$. Set

$$\mathrm{d}\langle \sigma_1, \sigma_2 \rangle(x) = \int \frac{\mathrm{d}\sigma_2(t)}{x - t}\, \mathrm{d}\sigma_1(x) = \widehat{\sigma}_2(x)\mathrm{d}\sigma_1(x).$$

This expression defines a new measure whose support coincides with that of σ_1. Whenever we find it convenient, we use the differential notation of a measure.

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0021-9045/$ - see front matter © 2009 Published by Elsevier Inc.
doi:10.1016/j.jat.2008.09.002

Let $\Sigma = (\sigma_1, \ldots, \sigma_m)$ be a system of finite Borel measures on the real line with compact support. Δ_k denotes the smallest interval containing the support of σ_k. Assume that $\Delta_k \cap \Delta_{k+1} = \emptyset$, $k = 1, \ldots, m-1$. By definition, $S = (s_1, \ldots, s_m) = \mathcal{N}(\Sigma)$ is called the *Nikishin system* generated by Σ if

$$s_1 = \sigma_1, \qquad s_2 = \langle \sigma_1, \sigma_2 \rangle, \ldots, \qquad s_m = \langle \sigma_1, \langle \sigma_2, \ldots, \sigma_m \rangle \rangle = \langle \sigma_1, \sigma_2, \ldots, \sigma_m \rangle.$$

Such systems were introduced by Nikishin in [1]. Here, we use the notation presented in [2] which is compact and clarifying.

In the sequel, $(\sigma_1, \ldots, \sigma_m)$ will always denote a system of measures such that for each $k = 1, \ldots, m, \sigma_k$ has a constant sign on its compact support (the sign may depend on k). We will write $(s_1, \ldots, s_m) = \mathcal{N}^*(\sigma_1, \ldots, \sigma_m)$, if additionally for each $k = 1, \ldots, m$, $\operatorname{supp}(\sigma_k) \subset \mathbb{R}$ consists of an interval $\widetilde{\Delta}_k$, on which $|\sigma_k'| > 0$ almost everywhere, and a discrete set without accumulation points in $\mathbb{R} \setminus \widetilde{\Delta}_k$. Finally, $(\widetilde{s}_1, \ldots, \widetilde{s}_m) = \mathcal{N}(p_1\sigma_1, \ldots, p_m\sigma_m)$, denotes a Nikishin system where the $p_k, k = 1, \ldots, m$, are monic polynomials with complex coefficients whose zeros lie in $\mathbb{C} \setminus \cup_{k=1}^m \Delta_k$.

Let $(s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ and $Q_{\mathbf{n}}$ (resp. $\widetilde{Q}_{\mathbf{n}}$) be the monic polynomial of smallest degree (not identically equal to zero) such that

$$0 = \int x^\nu Q_{\mathbf{n}}(x) \mathrm{d}s_k(x), \quad \nu = 0, \ldots, n_k - 1, k = 1, \ldots, m, \tag{1}$$

$$0 = \int x^\nu \widetilde{Q}_{\mathbf{n}}(x) \mathrm{d}\widetilde{s}_k(x), \quad \nu = 0, \ldots, n_k - 1, k = 1, \ldots, m, \tag{2}$$

where $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m$. Set $|\mathbf{n}| = n_1 + \cdots + n_m$.

In the general theory of orthogonal polynomials, E.A. Rakhmanov's theorem on ratio asymptotic behavior occupies a significant place (see [3,4], and for a simplified proof [5]) as well as its extension given by Denisov in [6]. In [7] (see also [8]), we studied the ratio asymptotic behavior of sequences of multiple orthogonal polynomials with respect to a Nikishin system of measures with a constant sign extending to this setting the Rakhmanov–Denisov theorem (see Proposition 3.2). In this paper, we find the relative asymptotic behavior of sequences formed by quotients of the form $\widetilde{Q}_{\mathbf{n}}/Q_{\mathbf{n}}$.

The subject of relative asymptotic behavior of sequences of orthogonal polynomials begins with [9] by A. A. Gonchar in which he establishes the relative asymptotic behavior of the denominators of diagonal Padé approximants associated with functions of the form $\int (z - x)^{-1} \mathrm{d}\sigma(x) + r(z)$ and the orthogonal polynomials of the measure σ which is assumed to be supported on an interval Δ of the real line on which $\sigma' > 0$ almost everywhere. Here, r denotes a rational function with complex coefficients whose zeros and poles lie in $\overline{\mathbb{C}} \setminus \Delta$. This turns out to be a key ingredient in his proof of the convergence of the sequence of Padé approximants to such meromorphic Markov type functions. In a series of papers [10–12], Maté–Nevai–Totik extended Szegő's theory of orthogonal polynomials, comparing the asymptotic behavior of two sequences of orthogonal polynomials corresponding to two measures — σ and $g\mathrm{d}\sigma$ — under appropriate assumptions on the measure σ and the weight g (see also [5,13]). A typical example of the application of this type of result in the present context is Corollary 4.2, where we extend the Rakhmanov–Denisov theorem given in [7] to the case of Nikishin systems generated by measures perturbed by rational weights with complex coefficients.

Given the collection of polynomials $(p_1, \ldots, p_m)$, we define

$$\mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m) = \{\mathbf{n} \in \mathbb{Z}_+^m : j < k \Rightarrow n_k + \deg(p_{j+1} \cdots p_k) \le n_j + 1\}.$$

In particular,

$$\mathbb{Z}_+^m(\circledast) = \{\mathbf{n} \in \mathbb{Z}_+^m : j < k \Rightarrow n_k \le n_j + 1\}.$$

A point $z_0 \in \mathbb{C}$ is said to be a 1 attraction point of zeros of a sequence of functions $\{\varphi_\mathbf{n}\}$, $\mathbf{n} \in \Lambda \subset \mathbb{Z}_+^m$, if for each sufficiently small $\varepsilon > 0$ there exists N such that for all $\mathbf{n} \in \Lambda$, $|\mathbf{n}| > N$, the number of zeros (counting multiplicity) of $\varphi_\mathbf{n}$ in $\{z : |z - z_0| < \varepsilon\}$ is 1. A set E is an attractor of the zeros of $\{\varphi_\mathbf{n}\}$, $\mathbf{n} \in \Lambda$, if for each $\varepsilon > 0$ there exists N_0 such that $|\mathbf{n}| > N_0$, $\mathbf{n} \in \Lambda$, implies that all the zeros of $\varphi_\mathbf{n}$ lie in the ε neighborhood of E. Our main result states:

Theorem 1.1. *Let $S = \mathcal{N}^*(\sigma_1, \dots, \sigma_m)$ and $\Lambda \subset \mathbb{Z}_+^m(\circledast; p_1, \dots, p_m)$ be a sequence of distinct multi-indices such that for all $\mathbf{n} \in \Lambda$, $n_1 - n_m \le C$, where C is a constant. Then*

$$\lim_{\mathbf{n} \in \Lambda} \frac{\widetilde{Q}_\mathbf{n}(z)}{Q_\mathbf{n}(z)} = \mathcal{F}(z; p_1, \dots, p_m), \qquad K \subset \overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_1), \tag{3}$$

uniformly on each compact subset K of $\overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_1)$, where $\mathcal{F}$ is analytic and never vanishes in $\overline{\mathbb{C}} \setminus \widetilde{\Delta}_1$. For all sufficiently large $|\mathbf{n}|$, $\mathbf{n} \in \Lambda$, $\deg \widetilde{Q}_\mathbf{n} = |\mathbf{n}|$, $\operatorname{supp}(\sigma_1)$ is an attractor of the zeros of $\{\widetilde{Q}_\mathbf{n}\}$, $\mathbf{n} \in \Lambda$, and each point in $\operatorname{supp}(\sigma_1) \setminus \widetilde{\Delta}_1$ is a 1 attraction point of zeros of $\{\widetilde{Q}_\mathbf{n}\}$, $\mathbf{n} \in \Lambda$. When the coefficients of the polynomials p_k, $k = 1, \dots, m$, are real, the statements remain valid for $\Lambda \subset \mathbb{Z}_+^m(\circledast)$.

An expression for $\mathcal{F}(z; p_1, \dots, p_m)$ is given in (42) at the end of the proof of Theorem 1.1 in Section 4. In the sequel, any limit following the notation used in (3) stands for uniform convergence on each compact subset of the indicated region.

This paper is organized as follows. Sections 2 and 3 contain auxiliary results needed for the proof of Theorem 1.1. Section 4 is dedicated to its proof and deriving several consequences. For example, we show that the same result is valid if we modify the measures in the initial system by rational functions instead of polynomials. These results allow us to extend the Rakhmanov–Denisov theorem on ratio asymptotic behavior to the sequence $\{\widetilde{Q}_\mathbf{n}\}$, $\mathbf{n} \in \Lambda$. In Sections 5 and 6 we study the relative asymptotic behavior of an associated system of second type functions and their zeros.

2. Some lemmas

Obviously, $\mathbb{Z}_+^m(\circledast; p_1, \dots, p_m) \subset \mathbb{Z}_+^m(\circledast)$. If $\mathbf{n} \in \mathbb{Z}_+^m(\circledast)$, it is well known that there exists a unique polynomial $Q_\mathbf{n}$ of degree $\le |\mathbf{n}|$ satisfying the orthogonality relations expressed in (1). Moreover, $Q_\mathbf{n}$ has exactly $|\mathbf{n}|$ simple zeros which lie in the interior of Δ_1 (for example, see [2]).

Let us express the orthogonality relations (2) satisfied by the polynomials $\widetilde{Q}_\mathbf{n}$ in terms of the measures in the initial system.

Lemma 2.1. *For each $k = 1, \dots, m$, we have*

$$\widetilde{s}_k = p_1 l_{k,1} s_1 + p_1 p_2 l_{k,2} s_2 + \cdots + (p_1 \cdots p_k) l_{k,k} s_k, \tag{4}$$

where $l_{k,j}$ is a polynomial of degree $\deg l_{k,j} \le \deg(p_{j+1} \cdots p_k) - 1$, $j < k$, and $l_{k,k} \equiv 1$. In particular, if $\mathbf{n} \in \mathbb{Z}_+^m(\circledast; p_1, \dots, p_m)$, then

$$0 = \int x^\nu \widetilde{Q}_\mathbf{n}(x)(p_1 \cdots p_k)(x) \mathrm{d}s_k(x), \qquad \nu = 0, \dots, n_k - 1, k = 1, \dots, m. \tag{5}$$

Proof. To prove (4), we proceed by induction on m, the number of measures which generate the system. For $m = 1$, (4) is trivial, since $\widetilde{s}_1 = p_1\sigma_1 = p_1 s_1$. Assume that (4) is true for any Nikishin system with $m - 1 \geq 1$ generating measures and let us prove it when the number of generating measures is m.

Fix $k \in \{1, \ldots, m\}$. By definition,

$$\widetilde{s}_k = \langle p_1\sigma_1, \ldots, p_k\sigma_k\rangle = \langle p_1\sigma_1, \langle p_2\sigma_2, \ldots, p_k\sigma_k\rangle\rangle.$$

Consider the Nikishin system $\mathcal{N}(p_2\sigma_2, \ldots, p_k\sigma_k)$ which has at most $m-1$ generating measures. By the induction hypothesis, there exist polynomials $h_2, \ldots, h_k$, $\deg h_j \leq \deg(p_{j+1}\cdots p_k) - 1$, $h_k \equiv 1$, such that

$$\langle p_2\sigma_2, \ldots, p_k\sigma_k\rangle = p_2h_2\sigma_2 + \cdots + (p_2\cdots p_k)h_k\langle\sigma_2, \ldots, \sigma_k\rangle.$$

Inserting this relation above, we have

$$\widetilde{s}_k = \langle p_1\sigma_1, p_2h_2\sigma_2\rangle + \cdots + \langle p_1\sigma_1, (p_2\cdots p_k)h_k\langle\sigma_2, \ldots, \sigma_k\rangle\rangle. \tag{6}$$

Given two measures $\sigma_\alpha, \sigma_\beta$, and a polynomial h, notice that

$$\begin{aligned} \mathrm{d}\langle\sigma_\alpha, h\sigma_\beta\rangle(x) &= \int \frac{(h(t) \mp h(x))\mathrm{d}\sigma_\beta(t)}{x-t}\mathrm{d}\sigma_\alpha(x) \\ &= h^*(x)\mathrm{d}\sigma_\alpha(x) + h(x)\mathrm{d}\langle\sigma_\alpha, \sigma_\beta\rangle(x), \end{aligned}$$

where $\deg h^* = \deg h - 1$. Making use of this property in each term of (6), it follows that

$$\begin{aligned} \widetilde{s}_k = {} & p_1[(p_2h_2)^* + \cdots + (p_2\cdots p_kh_k)^*]\sigma_1 + (p_1p_2)h_2\langle\sigma_1, \sigma_2\rangle + \cdots \\ & + (p_1\cdots p_k)h_k\langle\sigma_1, \ldots, \sigma_k\rangle, \end{aligned}$$

which establishes (4).

Using (4) and the orthogonality relations (2) satisfied by $\widetilde{Q}_{\mathbf{n}}$, it follows that for each $k \in \{1, \ldots, m\}$ and $\nu = 0, \ldots, n_k - 1$,

$$0 = \int x^\nu \widetilde{Q}_{\mathbf{n}}(x)\mathrm{d}\widetilde{s}_k(x) = \sum_{j=1}^{k}\int x^\nu l_{k,j}(x)\widetilde{Q}_{\mathbf{n}}(x)(p_1\cdots p_j)(x)\mathrm{d}s_j(x). \tag{7}$$

In the rest of the proof we assume that $\mathbf{n} \in \mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m)$. When $k = 1$ the last formula reduces to (5). Suppose that (5) holds up to $k - 1$, $1 \leq k - 1 \leq m - 1$, and let us show that it is also satisfied for k.

Let $j \in \{1, \ldots, k-1\}$ and $0 \leq \nu \leq n_k - 1$, then

$$\nu + \deg l_{k,j} \leq n_k - 1 + \deg(p_{j+1}\cdots p_k) - 1 \leq n_j - 1.$$

Therefore, according to the induction hypothesis

$$\int x^\nu l_{k,j}(x)\widetilde{Q}_{\mathbf{n}}(x)(p_1\cdots p_j)(x)\mathrm{d}s_j(x) = 0,$$

and (7) reduces to (5) for the index k. With this we conclude the proof. □

Lemma 2.2. *Let $\mathbf{n} \in \mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m)$. Then, for each $k = 1, \ldots, m$,*

$$0 = \int x^\nu \widetilde{Q}_{\mathbf{n}}(x)(p_1\cdots p_m)(x)\mathrm{d}s_k(x), \quad \nu = 0, \ldots, n_k - \deg(p_{k+1}\cdots p_m) - 1. \tag{8}$$

Proof. In place of x^ν we can put in (5) any polynomial of degree $\le n_k - 1$. So, replacing x^ν by $x^\nu(p_{k+1}\cdots p_m)$ we obtain (8). □

Our next objective is to express the multiple orthogonal polynomials of the perturbed system in terms of multiple orthogonal polynomials of the initial system.

Let $\mathbf{n} \in \mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m)$ and consider the multi-indices

$$\mathbf{n}_j = (n_1 - \deg(p_2\cdots p_m) + j, n_2 - \deg(p_3\cdots p_m), \ldots, n_m), \quad j \ge 0.$$

It is easy to verify that

$$\mathbf{n}_j \in \mathbb{Z}_+^m(\circledast), \quad j \ge 0.$$

Therefore, $\deg Q_{\mathbf{n}_j} = |\mathbf{n}_j|$ and all its $|\mathbf{n}_j|$ simple zeros lie on Δ_1. Moreover, for each $j \ge 0$ and $k = 1, \ldots, m$,

$$0 = \int x^\nu Q_{\mathbf{n}_j}(x) \mathrm{d}s_k(x), \quad \nu = 0, \ldots, n_k - \deg(p_{k+1}\cdots p_m) - 1. \tag{9}$$

Lemma 2.3. *Let $\mathbf{n} \in \mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m)$ and set $R_\mathbf{n} = \widetilde{Q}_\mathbf{n} p_1 \cdots p_m$. There exist unique constants $\lambda_{\mathbf{n},j}$, $j = 0, \ldots, N$, such that*

$$R_\mathbf{n} = \sum_{j=0}^{N} \lambda_{\mathbf{n},j} Q_{\mathbf{n}_j}, \quad N = \deg(p_1 p_2^2 \cdots p_m^m). \tag{10}$$

If j' is such that $\deg R_\mathbf{n} = \deg Q_{\mathbf{n}_{j'}}$ *then $\lambda_{\mathbf{n},j'} = 1$ and $\lambda_{\mathbf{n},j} = 0$, $j'+1 \le j \le N$. In particular, $\lambda_{\mathbf{n},N} = 1$ if and only if* $\deg \widetilde{Q}_\mathbf{n} = |\mathbf{n}|$.

Proof. Since $\deg R_\mathbf{n} \le |\mathbf{n}| + \deg(p_1\cdots p_m)$, and $\{Q_{\mathbf{n}_j}\}$, $j = 0, \ldots, N$, has representatives of all degrees from $|\mathbf{n}| - \deg(p_2 p_3^2 \cdots p_m^{m-1})$ up to $|\mathbf{n}| + \deg(p_1\cdots p_m)$, there exists a unique system of constants $\lambda_{\mathbf{n},j}$, $j = 0 \ldots, N$, such that

$$\deg\left(R_\mathbf{n} - \sum_{j=0}^{N} \lambda_{\mathbf{n},j} Q_{\mathbf{n}_j}\right) \le |\mathbf{n}| - \deg(p_2 p_3^2 \cdots p_m^{m-1}) - 1.$$

From (8) and (9) it follows that

$$R_\mathbf{n} - \sum_{j=0}^{N} \lambda_{\mathbf{n},j} Q_{\mathbf{n}_j} \equiv 0,$$

which is (10). The rest of the statements follow because $R_\mathbf{n}$ is monic. □

Let $\mathbf{n} \in \mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m)$. Define recursively the functions

$$R_{\mathbf{n},0}(z) = R_\mathbf{n}(z), \qquad R_{\mathbf{n},k}(z) = \int \frac{R_{\mathbf{n},k-1}(x)}{z - x} \mathrm{d}\sigma_k(x), \quad k = 1, \ldots, m. \tag{11}$$

In deriving (8), we lost some orthogonality relations. We will recover them in the form of analytic properties of the functions $R_{\mathbf{n},k}$, $k = 0, \ldots, m-1$.

Lemma 2.4. *Fix $\mathbf{n} \in \mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m)$. The following relations take place:*
If z_1 is a zero of $p_1 \cdots p_m$ of multiplicity τ_1, then

$$\Omega_{\mathbf{n}}^{(i)}(z_1) = \left(\frac{R_{\mathbf{n}}}{Q_{\mathbf{n}_0}}\right)^{(i)}(z_1) = 0, \quad i = 0, \ldots, \tau_1 - 1. \tag{12}$$

If z_k is a zero of $p_k \cdots p_m$, $k = 2, \ldots, m$, of multiplicity τ_k, then

$$R_{\mathbf{n},k-1}^{(i)}(z_k) = 0, \quad i = 0, \ldots, \tau_k - 1. \tag{13}$$

Proof. The zeros of $p_1 \cdots p_m$ lie in $\mathbb{C} \setminus \Delta_1$, and those of $Q_{\mathbf{n}_0}$ in Δ_1. Therefore, $\Omega_{\mathbf{n}}$ has a zero at z_1 of multiplicity greater or equal to τ_1 which implies (12).

For simplicity, first we will prove (13) for $k = 2$. By definition

$$R_{\mathbf{n},1}(z) = \int \frac{R_{\mathbf{n}}(x)}{z - x} \mathrm{d}\sigma_1(x).$$

Therefore, for each $i \geq 0$,

$$R_{\mathbf{n},1}^{(i)}(z) = (-1)^i i! \int \frac{R_{\mathbf{n}}(x)}{(z - x)^{i+1}} \mathrm{d}\sigma_1(x), \quad z \in \mathbb{C} \setminus \Delta_1.$$

If z_2 is a zero of $p_2 \cdots p_m$ of multiplicity τ_2, using (5) with $k = 1$ we have that

$$0 = \int \frac{(p_2 \cdots p_m)(x)}{(z_2 - x)^{i+1}} \widetilde{Q}_{\mathbf{n}}(x) p_1(x) \mathrm{d}\sigma_1(x) = \frac{(-1)^i R_{\mathbf{n},1}^{(i)}(z_2)}{i!}, \quad i = 0, \ldots, \tau_2 - 1,$$

which is (13) for $k = 2$. The proof of the general case uses basically the same arguments.

Consider the functions

$$\Phi_{\mathbf{n},k}(z) = \int \frac{R_{\mathbf{n}}(x)}{z - x} \mathrm{d}s_k(x), \quad k = 1, \ldots, m.$$

Notice that $\Phi_{\mathbf{n},1} = R_{\mathbf{n},1}$. For each $i \geq 0$,

$$\Phi_{\mathbf{n},k}^{(i)}(z) = (-1)^i i! \int \frac{R_{\mathbf{n}}(x)}{(z - x)^{i+1}} \mathrm{d}s_k(x), \quad k = 1, \ldots, m.$$

It is easy to verify that for each $k = 2, \ldots, m$,

$$\Phi_{\mathbf{n},k}(z) + (-1)^k R_{\mathbf{n},k}(z) = \int \cdots \int \frac{R_{\mathbf{n}}(x_1)(x_1 - x_k) \mathrm{d}\sigma_1(x_1) \cdots \mathrm{d}\sigma_k(x_k)}{(z - x_1)(x_1 - x_2) \cdots (x_{k-1} - x_k)(z - x_k)}.$$

Taking $x_1 - x_k = x_1 - x_2 + \cdots + x_{k-1} - x_k$, it follows that

$$R_{\mathbf{n},k}(z) = (-1)^{k-1} \Phi_{\mathbf{n},k}(z) + \sum_{l=1}^{k-1} (-1)^{l-1} \widehat{\vartheta}_{l,k}(z) \Phi_{\mathbf{n},l}(z), \quad z \in \mathbb{C} \setminus \left(\bigcup_{l=1}^{m} \Delta_l\right), \tag{14}$$

where $\vartheta_{l,k} = \langle \sigma_k, \sigma_{k-1}, \ldots, \sigma_{l+1} \rangle$. If z_k is a zero of $p_k \cdots p_m$ of multiplicity τ_k ($\leq \tau_{k-1} \leq \cdots \leq \tau_2$), using (5) we obtain that for each $l = 2, \ldots, k$ and $i = 0, \ldots, \tau_k - 1$,

$$0 = \int \frac{(p_l \cdots p_m)(x)}{(z_k - x)^{i+1}} \widetilde{Q}_{\mathbf{n}}(x)(p_1 \cdots p_{l-1})(x) \mathrm{d}s_{l-1}(x) = \frac{(-1)^i \Phi_{\mathbf{n},l-1}^{(i)}(z_k)}{i!}. \tag{15}$$

Now, (13) is a consequence of (14) (with k replaced by $k - 1$), and (15). With this we conclude the proof. $\square$

3. Known asymptotic properties

For each $\mathbf{n} \in \mathbb{Z}_+^m(\circledast)$, define recursively the functions

$$\Psi_{\mathbf{n},0}(z) = Q_{\mathbf{n}}(z), \qquad \Psi_{\mathbf{n},k}(z) = \int \frac{\Psi_{\mathbf{n},k-1}(x)}{z-x} \mathrm{d}\sigma_k(x), \quad k = 1, \ldots, m. \tag{16}$$

In Proposition 1 of [2] it was proved that for each $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m(\circledast)$, $k = 1, \ldots, m$, and $k \leq k + r \leq m$,

$$\int \Psi_{\mathbf{n},k-1}(t)\, t^\nu \mathrm{d}\langle \sigma_k, \ldots, \sigma_{k+r}\rangle(t) = 0, \quad \nu = 0, \ldots, n_{k+r} - 1.$$

From here, the authors deduce that $\Psi_{\mathbf{n},k-1}$, $k = 1, \ldots, m$, has exactly $N_{\mathbf{n},k} = n_k + \cdots + n_m$ zeros in $\mathbb{C} \setminus \Delta_{k-1}$, that they are all simple, and lie in the interior of Δ_k. Let $Q_{\mathbf{n},k}$ be the monic polynomial of degree $N_{\mathbf{n},k}$ whose simple zeros are located at the points where $\Psi_{\mathbf{n},k-1}$ vanishes on Δ_k and let $Q_{\mathbf{n},m+1} \equiv 1$. In Proposition 2 (see also Proposition 3) of [2] the authors show that

$$\int x^\nu \Psi_{\mathbf{n},k-1}(x) \frac{\mathrm{d}\sigma_k(x)}{Q_{\mathbf{n},k+1}(x)} = 0, \quad \nu = 0, \ldots, N_{\mathbf{n},k} - 1, k = 1, \ldots, m. \tag{17}$$

Set

$$H_{\mathbf{n},k}(z) := \frac{Q_{\mathbf{n},k-1}(z)\, \Psi_{\mathbf{n},k-1}(z)}{Q_{\mathbf{n},k}(z)}, \quad k = 1, \ldots, m+1,$$

($H_{\mathbf{n},1}(z) \equiv 1$). It is well known (see (50) in [14]) and easy to verify that

$$H_{\mathbf{n},k+1}(z) = \int \frac{Q^2_{\mathbf{n},k}(x)}{z-x} \frac{H_{\mathbf{n},k}(x)\mathrm{d}\sigma_k(x)}{Q_{\mathbf{n},k-1}(x) Q_{\mathbf{n},k+1}(x)}, \quad k = 1, \ldots, m. \tag{18}$$

From (17), we have that for each multi-index $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m(\circledast)$ there exists an associated system of polynomials

$$\{Q_{\mathbf{n},k}\}_{k=1}^m, \qquad \deg Q_{\mathbf{n},k} = \sum_{\alpha=k}^m n_\alpha =: N_{\mathbf{n},k}, \qquad Q_{\mathbf{n},0} \equiv Q_{\mathbf{n},m+1} \equiv 1.$$

For each $k = 1, \ldots, m$, they satisfy the full system of orthogonality relations

$$\int x^\nu Q_{\mathbf{n},k}(x) \frac{H_{\mathbf{n},k}(x)\mathrm{d}\sigma_k(x)}{Q_{\mathbf{n},k-1}(x) Q_{\mathbf{n},k+1}(x)} = 0, \quad \nu = 0, \ldots, N_{\mathbf{n},k} - 1, \tag{19}$$

with respect to varying measures. Notice that $H_{\mathbf{n},k}$ and $Q_{\mathbf{n},k-1}Q_{\mathbf{n},k+1}$ have a constant sign on Δ_k.

Let $\varepsilon_{\mathbf{n},k}$ be the sign of the measure $H_{\mathbf{n},k}(x)\mathrm{d}\sigma_k(x)/Q_{\mathbf{n},k-1}(x)Q_{\mathbf{n},k+1}(x)$ on $\mathrm{supp}(\sigma_k)$. For each $k = 1, \ldots, m$, set

$$K_{\mathbf{n},k} = \left(\int Q^2_{\mathbf{n},k}(x) \frac{\varepsilon_{\mathbf{n},k} H_{\mathbf{n},k}(x)\mathrm{d}\sigma_k(x)}{Q_{\mathbf{n},k-1}(x) Q_{\mathbf{n},k+1}(x)} \right)^{-1/2}. \tag{20}$$

Take

$$K_{\mathbf{n},0} = 1, \qquad \kappa_{\mathbf{n},k} = \frac{K_{\mathbf{n},k}}{K_{\mathbf{n},k-1}}, \quad k = 1, \ldots, m.$$

Define

$$q_{\mathbf{n},k} = \kappa_{\mathbf{n},k} Q_{\mathbf{n},k}, \qquad h_{\mathbf{n},k} = K^2_{\mathbf{n},k-1} H_{\mathbf{n},k}, \quad k = 1, \dots, m. \tag{21}$$

From (19)

$$\int x^\nu Q_{\mathbf{n},k}(x) \frac{\varepsilon_{\mathbf{n},k} h_{\mathbf{n},k}(x) \mathrm{d}\sigma_k(x)}{Q_{\mathbf{n},k-1}(x) Q_{\mathbf{n},k+1}(x)} = 0, \quad \nu = 0, \dots, N_{\mathbf{n},k} - 1, k = 1, \dots, m,$$

and, with the notation introduced above, it follows that $q_{n,k}$ is orthonormal with respect to the varying measure

$$\frac{\varepsilon_{\mathbf{n},k} h_{\mathbf{n},k}(x) \mathrm{d}\sigma_k(x)}{Q_{\mathbf{n},k-1}(x) Q_{\mathbf{n},k+1}(x)} = d\rho_{\mathbf{n},k}(x).$$

In Lemma 3.3 of [7] (see also Corollary 3 in [13]) we proved

Proposition 3.1. *Let $S = \mathcal{N}^*(\sigma_1, \dots, \sigma_m)$ and $\Lambda \subset \mathbb{Z}^m_+(\circledast)$ be a sequence of multi-indices such that for all $\mathbf{n} \in \Lambda$, $n_1 - n_m \le C$, where C is a constant. Then, for each fixed $k = 1, \dots, m$, we have*

$$\lim_{\mathbf{n}\in\Lambda} \varepsilon_{\mathbf{n},k} h_{\mathbf{n},k+1}(z) = \frac{1}{\sqrt{(z-b_k)(z-a_k)}}, \qquad K \subset \overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_k), \tag{22}$$

where $[a_k, b_k] = \widetilde{\Delta}_k$. The square root is taken so that $\sqrt{(z-b_k)(z-a_k)} > 0$ for $z = x > b_k$. $\operatorname{supp}(\sigma_k)$ *is an attractor of the zeros of $\{Q_{\mathbf{n},k}\}$, $\mathbf{n} \in \Lambda$, and each point of* $\operatorname{supp}(\sigma_k) \setminus \widetilde{\Delta}_k$ *is a 1 attraction point of zeros of $\{Q_{\mathbf{n},k}\}$, $\mathbf{n} \in \Lambda$.*

In the proof of our main result, we use the asymptotic behavior of the polynomials $Q_{\mathbf{n},k}$, $k = 1, \dots, m$, and the functions $\Psi_{\mathbf{n},k}$, $k = 1, \dots, m$, when $\mathbf{n}$ runs through a sequence of multi-indices $\Lambda \subset \mathbf{Z}^m_+(\circledast)$. In order to describe these asymptotic formulas we need to introduce some notions.

Consider the $(m+1)$-sheeted Riemann surface

$$\mathcal{R} = \overline{\bigcup_{k=0}^{m} \mathcal{R}_k},$$

formed by the consecutively "glued" sheets

$$\mathcal{R}_0 := \overline{\mathbb{C}} \setminus \widetilde{\Delta}_1, \qquad \mathcal{R}_k := \overline{\mathbb{C}} \setminus \{\widetilde{\Delta}_k \cup \widetilde{\Delta}_{k+1}\}, \quad k = 1, \dots, m-1, \qquad \mathcal{R}_m = \overline{\mathbb{C}} \setminus \widetilde{\Delta}_m,$$

where the upper and lower banks of the slits of two neighboring sheets are identified. Fix $l \in \{1, \dots, m\}$. Let $\psi^{(l)}$, $l = 1, \dots, m$, be a single valued rational function on $\mathcal{R}$ whose divisor consists of a simple zero at the point $\infty^{(0)} \in \mathcal{R}_0$ and a simple pole at the point $\infty^{(l)} \in \mathcal{R}_l$. Therefore,

$$\psi^{(l)}(z) = C_1/z + \mathcal{O}(1/z^2), \quad z \to \infty^{(0)}, \qquad \psi^{(l)}(z) = C_2 z + \mathcal{O}(1), \quad z \to \infty^{(l)},$$

where C_1 and C_2 are constants different from zero. Since the genus of $\mathcal{R}$ equals zero, such a single valued function on $\mathcal{R}$ exists and it is uniquely determined except for a multiplicative constant. We denote the branches of the algebraic function $\psi^{(l)}$, corresponding to the different sheets $k = 0, \dots, m$ of $\mathcal{R}$ by

$$\psi^{(l)} := \{\psi^{(l)}_k\}_{k=0}^m.$$

We normalize $\psi^{(l)}$ so that

$$\prod_{k=0}^{m} |\psi_k^{(l)}(\infty)| = 1, \qquad C_1 \in \mathbb{R}\backslash\{0\}. \tag{23}$$

Certainly, there are two $\psi^{(l)}$ verifying this normalization.

Since the product of all the branches $\prod_{k=0}^{m} \psi_k^{(l)}$ is a single valued analytic function in $\overline{\mathbb{C}}$ without singularities, by Liouville's Theorem, it is constant, and because of the normalization introduced above, this constant is either 1 or -1. Since $\psi^{(l)}$ is such that $C_1 \in \mathbb{R} \setminus \{0\}$, then

$$\psi^{(l)}(z) = \overline{\psi^{(l)}(\overline{z})}, \quad z \in \mathcal{R}.$$

In fact, let $\phi(z) := \overline{\psi^{(l)}(\overline{z})}$. ϕ and $\psi^{(l)}$ have the same divisor; consequently, there exists a constant C such that $\phi = C\psi^{(l)}$. Comparing the leading coefficients of the Laurent expansion of these functions at $\infty^{(0)}$, we conclude that $C = 1$.

Given an arbitrary function $F(z)$ which has, in a neighborhood of infinity, a Laurent expansion of the form $F(z) = Cz^k + \mathcal{O}(z^{k-1})$, $C \neq 0$, and $k \in \mathbb{Z}$, we denote

$$\widetilde{F} := F/C.$$

(For simplicity in writing, we write $\widetilde{F}_k^{(l)}$ instead of the more appropriate $\widetilde{F_k^{(l)}}$.) C is called the leading coefficient of F. When $C \in \mathbb{R} \setminus \{0\}$, $\operatorname{sg}(F(\infty))$ represents the sign of C.

In terms of the branches of $\psi^{(l)}$, the symmetry formula above means that for each $k = 0, 1, \ldots, m$

$$\psi_k^{(l)} : \overline{\mathbb{R}} \setminus (\widetilde{\Delta}_k \cup \widetilde{\Delta}_{k+1}) \longrightarrow \overline{\mathbb{R}} \tag{24}$$

($\widetilde{\Delta}_0 = \widetilde{\Delta}_{m+1} = \emptyset$); therefore, the coefficients (in particular, the leading one) of the Laurent expansion at ∞ of these branches are real numbers and $\operatorname{sg}(\psi_k^{(l)}(\infty))$ is defined. It also expresses that

$$\psi_k^{(l)}(x_\pm) = \overline{\psi_k^{(l)}(x_\mp)} = \overline{\psi_{k+1}^{(l)}(x_\pm)}, \quad x \in \widetilde{\Delta}_{k+1}.$$

For any fixed multi-index $\mathbf{n} = (n_1, \ldots, n_m)$, set

$$\mathbf{n}^l := (n_1, \ldots, n_{l-1}, n_l + 1, n_{l+1}, \ldots, n_m).$$

In [7] (see also [8]) the authors prove

Proposition 3.2. *Let $S = \mathcal{N}^*(\sigma_1, \ldots, \sigma_m)$ and $\Lambda \subset \mathbb{Z}_+^m(\circledast)$ be a sequence of multi-indices such that for all $\mathbf{n} \in \Lambda$ and some fixed $l \in \{1, \ldots, m\}$, we have that $\mathbf{n}^l \in \mathbb{Z}_+^m(\circledast)$ and $n_1 - n_m \leq C$, where C is a constant. Let $\{q_{\mathbf{n},k} = \kappa_{\mathbf{n},k} Q_{\mathbf{n},k}\}_{k=1}^m$, $\mathbf{n} \in \Lambda$, be the system of orthonormal polynomials defined in (21) and $\{K_{\mathbf{n},k}\}_{k=1}^m$, $\mathbf{n} \in \Lambda$, the values given by (20). Then, for each fixed $k = 1, \ldots, m$, we have*

$$\lim_{\mathbf{n} \in \Lambda} \frac{\kappa_{\mathbf{n}^l,k}}{\kappa_{\mathbf{n},k}} = \kappa_k^{(l)}, \tag{25}$$

$$\lim_{\mathbf{n} \in \Lambda} \frac{K_{\mathbf{n}^l,k}}{K_{\mathbf{n},k}} = \kappa_1^{(l)} \cdots \kappa_k^{(l)}, \tag{26}$$

and

$$\lim_{\mathbf{n}\in\Lambda} \frac{q_{\mathbf{n}^l,k}(z)}{q_{\mathbf{n},k}(z)} = \kappa_k^{(l)} \widetilde{F}_k^{(l)}(z), \qquad K \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_k), \tag{27}$$

where

$$\kappa_k^{(l)} = \frac{c_k^{(l)}}{\sqrt{c_{k-1}^{(l)} c_{k+1}^{(l)}}}, \quad c_k^{(l)} = \begin{cases} (F_k^{(l)})'(\infty), & k = 1, \ldots, l, \\ F_k^{(l)}(\infty), & k = l+1, \ldots, m, \end{cases} \tag{28}$$

$(c_0^{(l)} = c_{m+1}^{(l)} = 1)$ *and*

$$F_k^{(l)} := \delta_{k,l} \prod_{\nu=k}^{m} \psi_\nu^{(l)}, \tag{29}$$

with $\delta_{k,l} = sg\left(\prod_{\nu=k}^m \psi_\nu^{(l)}(\infty)\right)$.

4. Proof of Theorem 1.1

When $l = 1$, it is possible to find an algebraic function $\psi^{(1)}$ satisfying

$$\prod_{k=0}^{m} \psi_k^{(1)}(\infty) = 1, \qquad C_1 \in \mathbb{R} \setminus \{0\}. \tag{30}$$

Let $(a, b)_k$ denote the interval (a, b) on the sheet $\mathcal{R}_k$. We distinguish two cases. Suppose that $\widetilde{\Delta}_1 = [a_1, b_1]$ is to the left of $\widetilde{\Delta}_2 = [a_2, b_2]$. Take $\psi^{(1)}$ verifying (23) with $C_1 = \lim_{z\to\infty} z\psi_0^{(1)}(z) > 0$. Because of (24), the restriction of $\psi^{(1)}$ to $(-\infty, a_1]_0 \cup (-\infty, a_1]_1$ establishes a bicontinuous bijection onto the interval $(-\infty, 0)$ of the real line. It follows that $\psi_1^{(1)}(x) \to -\infty, x \to -\infty, x \in \mathbb{R}$, which means that $C_2 > 0$, and $\psi_k^{(1)}(\infty) > 0, k = 2, \ldots, m$. Therefore, $\prod_{k=0}^m \psi_k^{(1)}(\infty) > 0$. If $\widetilde{\Delta}_1$ is to the right of $\widetilde{\Delta}_2$, take $\psi^{(1)}$ satisfying (23) with $C_1 < 0$. Now, the restriction of $\psi^{(1)}$ to $[b_1, +\infty)_0 \cup [b_1, +\infty)_1$ establishes a bicontinuous bijection onto $(-\infty, 0)$. It follows that $\psi_1^{(1)}(x) \to -\infty, x \to +\infty, x \in \mathbb{R}$, which means that $C_2 < 0$, and $\psi_k^{(1)}(\infty) > 0, k = 2, \ldots, m$. Again, $\prod_{k=0}^m \psi_k^{(1)}(\infty) > 0$.

Throughout the rest of the paper, when $\widetilde{\Delta}_1$ is to the left of $\widetilde{\Delta}_2$, we will select $\psi^{(1)}$ so that $\operatorname{sg}(\psi_k^{(1)}(\infty)) = 1$, for all $k = 0, \ldots, m$. If $\widetilde{\Delta}_1$ is to the right of $\widetilde{\Delta}_2$, we will take $\psi^{(1)}$ so that $\operatorname{sg}(\psi_0^{(1)}(\infty)) = \operatorname{sg}(\psi_1^{(1)}(\infty)) = -1$ and $\operatorname{sg}(\psi_k^{(1)}(\infty)) = 1$, for all $k = 2, \ldots, m$.

In general, for any $l \in \{1, \ldots, m\}$ and $\psi^{(l)}$ verifying (23), we know that

$$\prod_{\nu=0}^{m} \psi_\nu^{(l)}(\infty) \in \{-1, 1\}.$$

Let $\Lambda \subset \mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m)$ be an infinite sequence of distinct multi-indices such that $n_1 - n_m \le C, \mathbf{n} \in \Lambda$. According to (25)–(29), for each fixed $j \ge 0$,

$$\lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n}_{j+1}}(z)}{Q_{\mathbf{n}_j}(z)} = \widetilde{F}_1^{(1)}(z) = \frac{\operatorname{sg}(\psi_0^{(1)}(\infty))}{c_1^{(1)}\psi_0^{(1)}(z)} =: \varphi_0(z), \qquad K \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_1). \tag{31}$$

(Notice that (30) implies that $\prod_{\nu=0}^m \psi_\nu^{(1)}(z) \equiv 1$.)

Using (10),

$$\Omega_{\mathbf{n}} = \frac{R_{\mathbf{n}}}{Q_{\mathbf{n}_0}} = \sum_{j=0}^{N} \lambda_{\mathbf{n},j} \frac{Q_{\mathbf{n}_j}}{Q_{\mathbf{n}_0}}, \qquad N = \deg(p_1 p_2^2 \cdots p_m^m).$$

Set

$$\lambda_{\mathbf{n}}^* = \left(\sum_{j=0}^{N} |\lambda_{\mathbf{n},j}| \right)^{-1}.$$

At least one of the numbers in the sum is 1 so $\lambda_{\mathbf{n}}^*$ is finite. Define

$$\lambda_{\mathbf{n}}^* \Omega_{\mathbf{n}} = \sum_{j=0}^{N} \lambda_{\mathbf{n},j}^* \frac{Q_{\mathbf{n}_j}}{Q_{\mathbf{n}_0}}, \qquad \sum_{j=0}^{N} |\lambda_{\mathbf{n},j}^*| = 1. \tag{32}$$

Because of (31) and (32), the family $\{\lambda_{\mathbf{n}}^* \Omega_{\mathbf{n}}\}$, $\mathbf{n} \in \Lambda$, is normal in $\mathbb{C} \setminus \operatorname{supp}(\sigma_1)$, and any convergent subsequence $\{\lambda_{\mathbf{n}}^* \Omega_{\mathbf{n}}\}$, $\mathbf{n} \in \Lambda' \subset \Lambda$, converges to

$$\lim_{\mathbf{n} \in \Lambda'} \lambda_{\mathbf{n}}^* \Omega_{\mathbf{n}}(z) = p_{\Lambda'}(\varphi_0(z)) = \sum_{j=0}^{N} \lambda_j \varphi_0^j(z), \qquad K \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_1).$$

That is, $p_{\Lambda'}(w)$ is a polynomial of degree $\leq N$, not identically equal to zero since $\sum_{j=0}^{N} |\lambda_j| = 1$. We will show that $p_{\Lambda'}$ does not depend on the subsequence taken. This implies the existence of limit along all Λ. To this aim, we will uniquely determine N zeros of $p_{\Lambda'}$.

Let z_1 be one of the zeros of $p_1 \cdots p_m$ and τ_1 its multiplicity. Using (12) and the Weierstrass theorem, it follows that

$$(p_{\Lambda'} \circ \varphi_0)^{(i)}(z_1) = 0, \quad i = 0, \ldots, \tau_1 - 1.$$

Since φ_0 is one to one in $\mathbb{C} \setminus \widetilde{\Delta}_1$, we conclude that $p_{\Lambda'}(w)$ is divisible by

$$(w - \varphi_0(z_1))^{\tau_1}.$$

We will detect the rest of the zeros of $p_{\Lambda'}(w)$ in virtue of (13). Consider the sequence $\{\lambda_{\mathbf{n}}^* R_{\mathbf{n},k-1}\}$, $\mathbf{n} \in \Lambda'$. From (10), (11) and (16)

$$\lambda_{\mathbf{n}}^* R_{\mathbf{n},k-1}(z) = \sum_{j=0}^{N} \lambda_{\mathbf{n},j}^* \Psi_{\mathbf{n}_j,k-1}(z).$$

Multiplying this equation by $\varepsilon_{\mathbf{n}_0,k-1} K_{\mathbf{n}_0,k-1}^2 Q_{\mathbf{n}_0,k-1}/Q_{\mathbf{n}_0,k}$ and using the definition of $h_{\mathbf{n},k}$, we obtain

$$\frac{\lambda_{\mathbf{n}}^* \varepsilon_{\mathbf{n}_0,k-1} K_{\mathbf{n}_0,k-1}^2 (Q_{\mathbf{n}_0,k-1} R_{\mathbf{n},k-1})(z)}{Q_{\mathbf{n}_0,k}(z)} = \sum_{j=0}^{N} \lambda_{\mathbf{n},j}^* \frac{K_{\mathbf{n}_0,k-1}^2}{K_{\mathbf{n}_j,k-1}^2} \frac{Q_{\mathbf{n}_0,k-1}(z)}{Q_{\mathbf{n}_j,k-1}(z)} \frac{Q_{\mathbf{n}_j,k}(z)}{Q_{\mathbf{n}_0,k}(z)} \frac{\varepsilon_{\mathbf{n}_0,k-1}}{\varepsilon_{\mathbf{n}_j,k-1}} \varepsilon_{\mathbf{n}_j,k-1} h_{\mathbf{n}_j,k}(z).$$

From (25)–(27), for each $j \geq 0$ and $k = 2, \ldots, m$,

$$\lim_{\mathbf{n}\in\Lambda'} \frac{K^2_{\mathbf{n}_j,k-1}}{K^2_{\mathbf{n}_{j+1},k-1}} \frac{Q_{\mathbf{n}_j,k-1}(z)}{Q_{\mathbf{n}_{j+1},k-1}(z)} \frac{Q_{\mathbf{n}_{j+1},k}(z)}{Q_{\mathbf{n}_j,k}(z)} = \frac{\widetilde{F}^{(1)}_k(z)}{(\kappa^{(1)}_1 \cdots \kappa^{(1)}_{k-1})^2 \widetilde{F}^{(1)}_{k-1}(z)},$$

uniformly on compact subsets of $\mathbb{C} \setminus (\mathrm{supp}(\sigma_{k-1}) \cup \mathrm{supp}(\sigma_k))$. On account of (28) and the expression of the functions $F^{(1)}_k$,

$$\frac{\widetilde{F}^{(1)}_k(z)}{(\kappa^{(1)}_1 \cdots \kappa^{(1)}_{k-1})^2 \widetilde{F}^{(1)}_{k-1}(z)} = \frac{\mathrm{sg}(\psi^{(1)}_{k-1}(\infty))}{c^{(1)}_1 \psi^{(1)}_{k-1}(z)} =: \varphi_{k-1}(z). \tag{33}$$

Let us consider the ratios $\varepsilon_{\mathbf{n}_{j+1},k}/\varepsilon_{\mathbf{n}_j,k}$, $k = 1, \ldots, m-1$, $j \geq 0$. Recall that $\varepsilon_{\mathbf{n},k}$ is, by definition, the sign of the measure $H_{\mathbf{n},k}(x)\mathrm{d}\sigma_k(x)/(Q_{\mathbf{n},k-1}Q_{\mathbf{n},k+1})(x)$ on Δ_k. Notice that for each fixed $k = 2, \ldots, m$ the polynomials $Q_{\mathbf{n}_j,k}$ have the same degree for all $j \geq 0$; therefore, they all have the same sign on any interval disjoint from Δ_k. On the other hand, the polynomials $Q_{\mathbf{n}_j,1}$ have degrees that increase one by one with j. Hence, if Δ_1 is to the left of Δ_2, all the polynomials $Q_{\mathbf{n}_j,1}$ have the same sign on Δ_2 whereas, if Δ_1 is to the right of Δ_2, the sign of these polynomials alternates on Δ_2 as j increases one by one. Taking these facts into consideration, it is easy to see that for all $j \geq 0$, the measures $H_{\mathbf{n}_j,1}(x)\mathrm{d}\sigma_1(x)/Q_{\mathbf{n}_j,2}(x) = \mathrm{d}\sigma_1(x)/Q_{\mathbf{n}_j,2}(x)$, have the same sign; therefore, for all $j \geq 0$, $\varepsilon_{\mathbf{n}_{j+1},1}/\varepsilon_{\mathbf{n}_j,1} = 1$ and the functions $H_{\mathbf{n}_j,2}$ have the same sign on Δ_2 (see (18)). Hence, the measures $H_{\mathbf{n}_j,2}(x)\mathrm{d}\sigma_2(x)/(Q_{\mathbf{n}_j,1}Q_{\mathbf{n}_j,3})(x)$ have the same sign if Δ_1 is to the left of Δ_2 and an alternate sign as j increases when Δ_1 is to the right of Δ_2. Thus, for all $j \geq 0$, $\varepsilon_{\mathbf{n}_{j+1},2}/\varepsilon_{\mathbf{n}_j,2} = 1$ when Δ_1 is to the left of Δ_2 and $\varepsilon_{\mathbf{n}_{j+1},2}/\varepsilon_{\mathbf{n}_j,2} = -1$ when Δ_1 is to the right of Δ_2. By the same token (see (18)), for all $j \geq 0$ the functions $H_{\mathbf{n}_j,3}$ have the same sign on Δ_3 when Δ_1 is to the left of Δ_2 and an alternate sign when Δ_1 is to the right of Δ_2. From now on the situation repeats and for each fixed $k = 2, \ldots, m-1$, and all $j \geq 0$, $\varepsilon_{\mathbf{n}_{j+1},k}/\varepsilon_{\mathbf{n}_j,k} = 1$ when Δ_1 is to the left of Δ_2 while $\varepsilon_{\mathbf{n}_{j+1},k}/\varepsilon_{\mathbf{n}_j,k} = -1$ when Δ_1 is to the right of Δ_2.

Let $\delta = 1$ when Δ_1 is to the left of Δ_2 and $\delta = -1$ if Δ_1 is to the right of Δ_2. Using (25)–(28), it follows that

$$\lim_{\mathbf{n}\in\Lambda'} \lambda^*_{\mathbf{n}} \varepsilon_{\mathbf{n}_0,k-1} K^2_{\mathbf{n}_0,k-1} \frac{Q_{\mathbf{n}_0,k-1}(z) R_{\mathbf{n},k-1}(z)}{Q_{\mathbf{n}_0,k}(z)}$$
$$= \begin{cases} \dfrac{1}{\sqrt{(z-b_1)(z-a_1)}} \displaystyle\sum_{j=0}^{N} \lambda_j \varphi_1^j(z), & k = 2, \\ \dfrac{1}{\sqrt{(z-b_{k-1})(z-a_{k-1})}} \displaystyle\sum_{j=0}^{N} \lambda_j (\delta\varphi_{k-1})^j(z), & k = 3, \ldots, m, \end{cases}$$
$$= \begin{cases} \dfrac{1}{\sqrt{(z-b_1)(z-a_1)}} p_{\Lambda'}(\varphi_1(z)), & k = 2, \\ \dfrac{1}{\sqrt{(z-b_{k-1})(z-a_{k-1})}} p_{\Lambda'}(\delta\varphi_{k-1}(z)), & k = 3, \ldots, m, \end{cases} \tag{34}$$

uniformly on each compact subset K of $\mathbb{C} \setminus (\mathrm{supp}(\sigma_{k-1}) \cup \mathrm{supp}(\sigma_k))$.

Let z_k be one of the zeros of $p_k \cdots p_m$, $k = 2, \ldots, m$, and τ_k its multiplicity. Using (34) and (13), and the Weierstrass theorem, it follows that

$$(p_{\Lambda'} \circ \varphi_1)^{(i)}(z_2) = 0, \quad i = 0, \ldots, \tau_2 - 1,$$

and

$$(p_{\Lambda'} \circ (\delta\varphi_{k-1}))^{(i)}(z_k) = 0, i = 0, \ldots, \tau_k - 1, \quad k = 3, \ldots, m.$$

Since φ_{k-1} is one to one in $\mathbb{C} \setminus (\widetilde{\Delta}_{k-1} \cup \widetilde{\Delta}_k)$, we conclude that $p_{\Lambda'}(w)$ is divisible by

$$(w - \varphi_1(z_2))^{\tau_2},$$

and

$$(w - \delta\varphi_{k-1}(z_k))^{\tau_k}, \quad k = 3, \ldots, m.$$

Therefore, the following sets are formed by zeros of $p_{\Lambda'}$:

$$\mathcal{Z}_0 := \{\varphi_0(z_1) : z_1 \text{ is a zero of } p_1 \cdots p_m\},$$
$$\mathcal{Z}_1 := \{\varphi_1(z_2) : z_2 \text{ is a zero of } p_2 \cdots p_m\},$$
$$\mathcal{Z}_k := \{\delta\varphi_k(z_{k+1}) : z_{k+1} \text{ is a zero of } p_{k+1} \cdots p_m\}, \quad 2 \leq k \leq m-1.$$

Assume, first, that $\delta = 1$. Recall that in this case we selected $\psi^{(1)}$ so that $\mathrm{sg}(\psi_k^{(1)}(\infty)) = 1$ for all $0 \leq k \leq m$. Therefore the functions $\varphi_0, \varphi_1, \delta\varphi_k, 2 \leq k \leq m-1$, are the first m branches of $1/c_1^{(1)}\psi^{(1)}$. If $\delta = -1$, since $\psi^{(1)}$ was chosen so that $\mathrm{sg}(\psi_0^{(1)}(\infty)) = \mathrm{sg}(\psi_1^{(1)}(\infty)) = -1$ and $\mathrm{sg}(\psi_k^{(1)}(\infty)) = 1, 2 \leq k \leq m$, the functions $\varphi_0, \varphi_1, \delta\varphi_k, 2 \leq k \leq m-1$, are now the first m branches of $-1/c_1^{(1)}\psi^{(1)}$. In any case, since $\psi^{(1)} : \mathcal{R} \longrightarrow \overline{\mathbb{C}}$ is bijective, it follows that the zero sets $\mathcal{Z}_k, 0 \leq k \leq m-1$ are pairwise disjoint. Therefore, we have detected $N = \deg(p_1 p_2^2 \cdots p_m^m)$ zeros (counting multiplicities) of the polynomial $p_{\Lambda'}$ and their location does not depend on the subsequence $\Lambda' \subset \Lambda$.

Let

$$(p_k \cdots p_m)(z) = \prod_{\nu=1}^{l_k} (z - z_{k,\nu})^{\tau_{k,\nu}},$$

where $\{z_{k,1}, \ldots, z_{k,l_k}\}$ are the distinct zeros of $p_k \cdots p_m$. Then

$$p_{\Lambda'}(w) = c \prod_{k=1}^{2} \prod_{\nu=1}^{l_k} (w - \varphi_{k-1}(z_{k,\nu}))^{\tau_{k,\nu}} \prod_{k=3}^{m} \prod_{\nu=1}^{l_k} (w - \delta\varphi_{k-1}(z_{k,\nu}))^{\tau_{k,\nu}},$$

where c is uniquely defined by the conditions that it is a positive constant such that the sum of the moduli of the coefficients of $p_{\Lambda'}$ equals one; moreover,

$$0 < c = \lim_{\mathrm{n} \in \Lambda} \lambda_{\mathbf{n}}^* < \infty.$$

Consequently, uniformly on each compact subset $K \subset \mathbb{C} \setminus \mathrm{supp}(\sigma_1)$,

$$\lim_{\mathbf{n} \in \Lambda} \frac{R_{\mathbf{n}}(z)}{Q_{\mathbf{n}_0}(z)} = \prod_{k=1}^{2} \prod_{\nu=1}^{l_k} (\varphi_0(z) - \varphi_{k-1}(z_{k,\nu}))^{\tau_{k,\nu}} \prod_{k=3}^{m} \prod_{\nu=1}^{l_k} (\varphi_0(z) - \delta\varphi_{k-1}(z_{k,\nu}))^{\tau_{k,\nu}}. \tag{35}$$

From (25) and (27), it follows that

$$\lim_{\mathbf{n} \in \Lambda} \frac{Q_{\mathbf{n}}(z)}{Q_{\mathbf{n}_0}(z)} = (\widetilde{F}_1^{(1)}(z))^{\deg(p_2 \cdots p_m)} \cdots (\widetilde{F}_1^{(m-1)}(z))^{\deg(p_m)}. \tag{36}$$

Combining (35) and (36), we get

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{Q}_{\mathbf{n}}(z)}{Q_{\mathbf{n}}(z)} = \mathcal{F}(z; p_1, \ldots, p_m), \qquad K \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_1),$$

where ($\varphi_0(z) = \widetilde{F}_1^{(1)}(z)$)

$$\mathcal{F}(z; p_1, \ldots, p_m) = \prod_{\nu=1}^{l_1} \left(\frac{\varphi_0(z) - \varphi_0(z_{1,\nu})}{z - z_{1,\nu}} \right)^{\tau_{1,\nu}} \prod_{\nu=1}^{l_2} \left(1 - \frac{\varphi_1(z_{2,\nu})}{\varphi_0(z)} \right)^{\tau_{2,\nu}} \times \prod_{k=3}^{m} \prod_{\nu=1}^{l_k} \left(\frac{\varphi_0(z) - \delta\varphi_{k-1}(z_{k,\nu})}{\widetilde{F}_1^{(k-1)}(z)} \right)^{\tau_{k,\nu}}.$$

Let us simplify the expression above. From the definition of the functions φ_k, and taking into account that $\delta = \operatorname{sg}(\psi_0^{(1)}(\infty))$, it follows that

$$1 - \frac{\varphi_1(z_{2,\nu})}{\varphi_0(z)} = 1 - \frac{\psi_0^{(1)}(z)}{\psi_1^{(1)}(z_{2,\nu})}.$$

It is easy to see that for $l \geq 2$ the following equation holds:

$$\frac{1}{\psi^{(1)}(z)} - \frac{1}{\psi^{(1)}(\infty^{(l-1)})} = \frac{C_0^{(l-1)}}{C_0^{(1)} \psi^{(l-1)}(z)}, \tag{37}$$

where

$$\psi^{(1)}(z) = C_0^{(1)}/z + \mathcal{O}(1/z^2), \quad z \to \infty^{(0)},$$
$$\psi^{(l-1)}(z) = C_0^{(l-1)}/z + \mathcal{O}(1/z^2), \quad z \to \infty^{(0)}.$$

For $k \geq 3$ (recall that $\prod_{\nu=0}^{m} \psi_\nu^{(l)}(\infty) \in \{-1, 1\}$ when $l \geq 2$), we have that

$$\widetilde{F}_1^{(k-1)}(z) = \frac{\operatorname{sg}(\psi_0^{(k-1)})(\infty)}{c_1^{(k-1)} \psi_0^{(k-1)}(z)}.$$

Thus

$$\frac{\varphi_0(z) - \delta\varphi_{k-1}(z_{k,\nu})}{\widetilde{F}_1^{(k-1)}(z)} = \frac{c_1^{(k-1)} \psi_0^{(k-1)}(z)}{c_1^{(1)} \operatorname{sg}(\psi_0^{(k-1)}(\infty))} \left(\frac{\operatorname{sg}(\psi_0^{(1)}(\infty))}{\psi_0^{(1)}(z)} - \frac{\delta}{\psi_{k-1}^{(1)}(z_{k,\nu})} \right). \tag{38}$$

From (37), it follows that

$$\psi_0^{(k-1)}(z) \left(\frac{1}{\psi_0^{(1)}(z)} - \frac{1}{\psi_{k-1}^{(1)}(\infty)} \right) = \frac{C_0^{(k-1)}}{C_0^{(1)}}.$$

Therefore,

$$\psi_0^{(k-1)}(z) \left(\frac{1}{\psi_0^{(1)}(z)} - \frac{\delta}{\psi_{k-1}^{(1)}(z_{k,\nu})} \right) = \frac{C_0^{(k-1)}}{C_0^{(1)}} + \left(\frac{\psi_0^{(k-1)}(z)}{\psi_{k-1}^{(1)}(\infty)} - \frac{\delta\psi_0^{(k-1)}(z)}{\psi_{k-1}^{(1)}(z_{k,\nu})} \right). \tag{39}$$

It is straightforward to check that

$$\frac{c_1^{(k-1)}}{c_1^{(1)}}\frac{C_0^{(k-1)}}{C_0^{(1)}} = \frac{\mathrm{sg}(\psi_0^{(k-1)}(\infty))}{\mathrm{sg}(\psi_0^{(1)}(\infty))}. \tag{40}$$

Evaluating (37) at $z_{k,\nu}$ we obtain

$$\frac{1}{\psi_{k-1}^{(1)}(z_{k,\nu})} - \frac{1}{\psi_{k-1}^{(1)}(\infty)} = \frac{C_0^{(k-1)}}{C_0^{(1)}\psi_{k-1}^{(k-1)}(z_{k,\nu})}. \tag{41}$$

Assume that Δ_1 is to the left of Δ_2, then $\delta = \mathrm{sg}(\psi_0^{(1)}(\infty)) = 1$. From (38)–(41), we find that

$$\frac{\varphi_0(z) - \delta\varphi_{k-1}(z_{k,\nu})}{\widetilde{F}_1^{(k-1)}(z)} = 1 - \frac{\psi_0^{(k-1)}(z)}{\psi_{k-1}^{(k-1)}(z_{k,\nu})}.$$

If Δ_1 is to the right of Δ_2, then $\delta = \mathrm{sg}(\psi_0^{(1)}(\infty)) = -1$. Applying (38)–(41), we obtain again

$$\frac{\varphi_0(z) - \delta\varphi_{k-1}(z_{k,\nu})}{\widetilde{F}_1^{(k-1)}(z)} = 1 - \frac{\psi_0^{(k-1)}(z)}{\psi_{k-1}^{(k-1)}(z_{k,\nu})}.$$

Therefore,

$$\mathcal{F}(z; p_1, \ldots, p_m) = \prod_{\nu=1}^{l_1}\left(\frac{\varphi_0(z) - \varphi_0(z_{1,\nu})}{z - z_{1,\nu}}\right)^{\tau_{1,\nu}} \prod_{k=2}^{m}\prod_{\nu=1}^{l_k}\left(1 - \frac{\psi_0^{(k-1)}(z)}{\psi_{k-1}^{(k-1)}(z_{k,\nu})}\right)^{\tau_{k,\nu}}. \tag{42}$$

(We did not substitute φ_0 in terms of $\psi_0^{(1)}$ (see (31)) in the first group of products for simplicity in the final expression.)

We have proved (3) on compact subsets of $\mathbb{C} \setminus \mathrm{supp}(\sigma_1)$. Using the maximum principle, it follows that the same is true on compact subsets of $\overline{\mathbb{C}} \setminus \mathrm{supp}(\sigma_1)$. Notice that $\mathcal{F}$ is analytic and has no zero in $\overline{\mathbb{C}} \setminus \widetilde{\Delta}_1$. For all $\mathbf{n} \in \Lambda$, $\deg Q_{\mathbf{n}} = |\mathbf{n}|$, $\mathrm{supp}(\sigma_1)$ is an attractor of the zeros of $\{Q_{\mathbf{n}}\}, \mathbf{n} \in \Lambda$, and each point in $\mathrm{supp}(\sigma_1) \setminus \widetilde{\Delta}_1$ is a 1 attraction point of zeros of $\{Q_{\mathbf{n}}\}, \mathbf{n} \in \Lambda$; therefore, the statements concerning $\deg \widetilde{Q}_{\mathbf{n}}$ and the asymptotic behavior of the zeros of these polynomials follow from (3), on account of the argument principle and the corresponding behavior of the zeros of the polynomials $Q_{\mathbf{n}}$ described in Proposition 3.1.

In order to prove the last statement, let us assume that the polynomials $p_k, k = 1, \ldots, m$, have real coefficients and $\Lambda \subset \mathbf{Z}_+^m(\circledast)$. Notice that, in this case, the polynomials $\widetilde{Q}_{\mathbf{n}}$ are the multiple orthogonal polynomials with respect to the Nikishin system $\mathcal{N}(p_1\sigma_1, \ldots, p_m\sigma_m)$ generated by real measures with a constant sign. Thus, Proposition 3.2 can be applied to them. Given Λ we construct the auxiliary sequence $\Lambda(\diamond)$ as follows. To each $\mathbf{n} = (n_1, \ldots, n_m) \in \Lambda$ we associate $\mathbf{n}_\diamond = (n_1, n_2 - \deg(p_2), \ldots, n_m - \deg(p_2 \cdots p_m))$ (we disregard those multi-indices in Λ for which a component of $\mathbf{n}_\diamond$ would turn out to be negative which, according to the assumptions on Λ there can be, at most, a finite number of such $\mathbf{n}$). It is easy to see that $\Lambda(\diamond) \subset \mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m)$.

Choose consecutive multi-indices running from $\mathbf{n}_\diamond$ to $\mathbf{n}$ so that each one of them belongs to $\mathbb{Z}_+^m(\circledast)$. We can write $Q_{\mathbf{n}}/Q_{\mathbf{n}_\diamond}$ as the product of quotients of the corresponding monic multiple orthogonal polynomials. The same can be done with $\widetilde{Q}_{\mathbf{n}}/\widetilde{Q}_{\mathbf{n}_\diamond}$. According to (25) and (27), there exists an analytic function $G(z)$ in $\mathbb{C} \setminus \widetilde{\Delta}_1$, which is never zero, such that

$$\lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n}}(z)}{Q_{\mathbf{n}_\diamond}(z)} = \lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{Q}_{\mathbf{n}}(z)}{\widetilde{Q}_{\mathbf{n}_\diamond}(z)} = G(z), \qquad K \subset \mathbb{C} \setminus \mathrm{supp}(\sigma_1).$$

Since

$$\frac{\widetilde{Q}_{\mathbf{n}}(z)}{Q_{\mathbf{n}}(z)} = \frac{\widetilde{Q}_{\mathbf{n}}(z)}{\widetilde{Q}_{\mathbf{n}_\diamond}(z)} \frac{\widetilde{Q}_{\mathbf{n}_\diamond}(z)}{Q_{\mathbf{n}_\diamond}(z)} \frac{Q_{\mathbf{n}_\diamond}(z)}{Q_{\mathbf{n}}(z)},$$

using Theorem 1.1 on the ratio in the middle and the previous limits on the other two ratios, the last statement readily follows. □

We can easily extend the main result to the more general case when the perturbation on the initial system is carried out by rational functions.

Corollary 4.1. *Let $S = \mathcal{N}^*(\sigma_1, \ldots, \sigma_m)$. Consider the perturbed Nikishin system $\mathcal{N}(\frac{p_1}{q_1}\sigma_1, \ldots, \frac{p_m}{q_m}\sigma_m)$, where p_k, q_k denote relatively prime polynomials whose zeros lie in $\mathbb{C} \setminus \cup_{k=1}^m \Delta_k$. Let $\Lambda \subset \mathbb{Z}_+^m(\circledast; p_1q_1, \ldots, p_mq_m)$ be a sequence of distinct multi-indices such that for all $\mathbf{n} \in \Lambda$, $n_1 - n_m \leq C$, where C is a constant. Let $\widetilde{Q}_{\mathbf{n}}$ be the monic multiple orthogonal polynomial of smallest degree relative to the Nikishin system $\mathcal{N}(\frac{p_1}{q_1}\sigma_1, \ldots, \frac{p_m}{q_m}\sigma_m)$ and $\mathbf{n}$. Then*

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{Q}_{\mathbf{n}}(z)}{Q_{\mathbf{n}}(z)} = \frac{\mathcal{F}(z; p_1, \ldots, p_m)}{\mathcal{F}(z; q_1, \ldots, q_m)}, \qquad K \subset \overline{\mathbb{C}} \setminus \mathrm{supp}(\sigma_1). \tag{43}$$

For all sufficiently large $|\mathbf{n}|$, $\mathbf{n} \in \Lambda$, $\deg \widetilde{Q}_{\mathbf{n}} = |\mathbf{n}|$, $\mathrm{supp}(\sigma_1)$ is an attractor of the zeros of $\{\widetilde{Q}_{\mathbf{n}}\}$, $\mathbf{n} \in \Lambda$, and each point in $\mathrm{supp}(\sigma_1) \setminus \widetilde{\Delta}_1$ is a 1 attraction point of zeros of $\{\widetilde{Q}_{\mathbf{n}}\}$, $\mathbf{n} \in \Lambda$. When the polynomials $p_k, q_k, k = 1, \ldots, m$, have real coefficients, the statements remain valid for $\Lambda \subset \mathbb{Z}_+^m(\circledast)$.

Proof. Notice that $\mathcal{N}(\frac{p_1}{q_1}\sigma_1, \ldots, \frac{p_m}{q_m}\sigma_m) = \mathcal{N}(\frac{p_1\overline{q}_1}{|q_1|^2}\sigma_1, \ldots, \frac{p_m\overline{q}_m}{|q_m|^2}\sigma_m)$, where $\overline{q}_k$ denotes the polynomial obtained conjugating the coefficients of q_k. Let $Q^*_{\mathbf{n}}$ be the $\mathbf{n}$th monic multiple orthogonal polynomial with respect to the Nikishin system $\mathcal{N}(\frac{\sigma_1}{|q_1|^2}, \ldots, \frac{\sigma_m}{|q_m|^2})$ generated by measures with constant sign.

Using Theorem 1.1,

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{Q}_{\mathbf{n}}(z)}{Q^*_{\mathbf{n}}(z)} = \mathcal{F}(z; p_1\overline{q}_1, \ldots, p_m\overline{q}_m), \qquad K \subset \overline{\mathbb{C}} \setminus \mathrm{supp}(\sigma_1)$$

and, considering the last remark of the same theorem, we also have

$$\lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n}}(z)}{Q^*_{\mathbf{n}}(z)} = \mathcal{F}(z; q_1\overline{q}_1, \ldots, q_m\overline{q}_m), \qquad K \subset \overline{\mathbb{C}} \setminus \mathrm{supp}(\sigma_1).$$

On the other hand,

$$\frac{\mathcal{F}(z; p_1\overline{q}_1, \ldots, p_m\overline{q}_m)}{\mathcal{F}(z; q_1\overline{q}_1, \ldots, q_m\overline{q}_m)} = \frac{\mathcal{F}(z; p_1, \ldots, p_m)}{\mathcal{F}(z; q_1, \ldots, q_m)}$$

because, in the products defining the functions on the left hand side, all the factors connected with the zeros of the $\overline{q}_k$ cancel out. Consequently, (43) takes place. The rest of the statements of the corollary are proved, following arguments similar to those employed in the proof of Theorem 1.1. □

The previous results allow us to derive ratio asymptotic behavior for the multiple orthogonal polynomials of our perturbed Nikishin systems.

Corollary 4.2. *Let $S = \mathcal{N}^*(\sigma_1, \ldots, \sigma_m)$ Consider the perturbed Nikishin system $\mathcal{N}(\frac{p_1}{q_1}\sigma_1, \ldots, \frac{p_m}{q_m}\sigma_m)$, where p_k, q_k denote relatively prime polynomials whose zeros lie in $\mathbb{C} \setminus \cup_{k=1}^m \Delta_k$. Let $\Lambda \subset \mathbb{Z}_+^m(\circledast; p_1q_1, \ldots, p_mq_m)$ be a sequence of distinct multi-indices such that for all $\mathbf{n} \in \Lambda$ and some fixed $l \in \{1, \ldots, m\}$, we have that $\mathbf{n}^l \in \mathbb{Z}_+^m(\circledast; p_1q_1, \ldots, p_mq_m)$ and $n_1 - n_m \leq C$, where C is a constant. Let $\widetilde{Q}_\mathbf{n}$ be the monic multiple orthogonal polynomial of smallest degree with respect to the Nikishin system $\mathcal{N}(\frac{p_1}{q_1}\sigma_1, \ldots, \frac{p_m}{q_m}\sigma_m)$ and $\mathbf{n}$. Then*

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{Q}_{\mathbf{n}^l}(z)}{\widetilde{Q}_\mathbf{n}(z)} = \lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n}^l}(z)}{Q_\mathbf{n}(z)} = \widetilde{F}_1^{(l)}(z), \qquad K \subset \mathbb{C} \setminus \operatorname{supp}(\sigma_1).$$

Proof. Since

$$\frac{\widetilde{Q}_{\mathbf{n}^l}(z)}{\widetilde{Q}_\mathbf{n}(z)} = \frac{\widetilde{Q}_{\mathbf{n}^l}(z)}{Q_{\mathbf{n}^l}(z)} \frac{Q_{\mathbf{n}^l}(z)}{Q_\mathbf{n}(z)} \frac{Q_\mathbf{n}(z)}{\widetilde{Q}_\mathbf{n}(z)},$$

the result follows immediately applying Proposition 3.2 and Corollary 4.1. $\square$

5. Relative asymptotic behavior of second type functions

Let $\widetilde{Q}_\mathbf{n}$ be the monic polynomial of smallest degree satisfying (2). Set

$$\begin{aligned} \widetilde{\Psi}_{n,0}(z) &:= \widetilde{Q}_\mathbf{n}(z), \\ \widetilde{\Psi}_{n,k}(z) &:= \int \frac{\widetilde{\Psi}_{n,k-1}(x)}{z-x} p_k(x)\, \mathrm{d}\sigma_k(x), \quad 1 \leq k \leq m. \end{aligned} \tag{44}$$

Lemma 5.1. *If $n_j \geq \deg(p_{j+1} \cdots p_m)$, $j = 1, \ldots, m-1$, then $R_{\mathbf{n},k}(z) = (p_{k+1} \cdots p_m)(z)\widetilde{\Psi}_{\mathbf{n},k}(z)$, $z \in \mathbb{C} \setminus \operatorname{supp}(\sigma_k)$, $k = 0, 1, \ldots, m$, $(R_{\mathbf{n},m} = \widetilde{\Psi}_{\mathbf{n},m})$.*

Proof. We proceed by induction on k. The case $k = 0$ is trivial since by definition, $R_{\mathbf{n},0}(z) = (p_1 \cdots p_m)(z)\widetilde{Q}_\mathbf{n}(z)$. Assume that the result holds for $k-1$, and let us prove it for k. We have

$$\begin{aligned} R_{\mathbf{n},k}(z) &= \int \frac{R_{\mathbf{n},k-1}(x)}{z-x} \mathrm{d}\sigma_k(x) = \int \frac{\widetilde{\Psi}_{\mathbf{n},k-1}(x)(p_k \cdots p_m)(x)}{z-x} \mathrm{d}\sigma_k(x) \\ &= (p_{k+1} \cdots p_m)(z)\widetilde{\Psi}_{\mathbf{n},k}(z) + \int \widetilde{\Psi}_{\mathbf{n},k-1}(x) l(x) p_k(x) \mathrm{d}\sigma_k(x), \end{aligned}$$

where $l(x)$ is a polynomial of degree $\deg(p_{k+1} \cdots p_m) - 1$. Now, for $k \leq k+r \leq m$, the functions $\widetilde{\Psi}_{\mathbf{n},k}$ satisfy the orthogonality relations (see in [2] that the proof presented there is also valid for complex measures)

$$\int \widetilde{\Psi}_{\mathbf{n},k-1}(t) t^\nu \mathrm{d}\langle p_k\sigma_k, \ldots, p_{k+r}\sigma_{k+r}\rangle(t) = 0, \quad \nu = 0, \ldots, n_{k+r} - 1.$$

In particular, $\int \widetilde{\Psi}_{\mathbf{n},k-1}(t) t^\nu p_k(t) \mathrm{d}\sigma_k(t) = 0$ if $\nu \leq n_k - 1$. Thus, since we are assuming that $n_k \geq \deg(p_{k+1} \cdots p_m)$, we get that

$$\int \widetilde{\Psi}_{\mathbf{n},k-1}(x) l(x) p_k(x) \mathrm{d}\sigma_k(x) = 0$$

and the result follows. $\square$

Remark 5.1. The condition $n_k \geq \deg(p_{k+1} \cdots p_m)$, $k = 1, \ldots, m-1$, is automatically satisfied by the components of multi-indices $\mathbf{n}$ with norm sufficiently large that belong to a sequence $\Lambda \subset \mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m)$ such that for all $\mathbf{n} \in \Lambda$, $n_1 - n_m \leq C$, where C is a constant. In fact, it is satisfied for all $\mathbf{n} \in \mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m)$ such that $n_m \geq 1$.

Now, we need to introduce some notations. Let

$$\delta_k := \begin{cases} 1, & \text{if } \Delta_k \text{ is to the left of } \Delta_{k+1}, \\ -1, & \text{if } \Delta_k \text{ is to the right of } \Delta_{k+1}. \end{cases}$$

For $k \geq 2$, set

$$\Delta_{k,l} := \begin{cases} -\delta_k \delta_{k-1}, & \text{if } l \geq k+1, \\ \delta_{k-1}, & \text{if } l \in \{k-1, k\}, \\ 1, & \text{if } l \leq k-2. \end{cases}$$

If $k = 1$,

$$\Delta_{1,l} := \begin{cases} 1, & \text{if } l = 1, \\ -\delta_1, & \text{if } l \geq 2. \end{cases}$$

Lemma 5.2. *For any* $\mathbf{n}, \mathbf{n}^l \in \mathbb{Z}_+^m(\circledast)$

$$\frac{\varepsilon_{\mathbf{n}^l,k}}{\varepsilon_{\mathbf{n},k}} = \prod_{i=1}^{k} \Delta_{i,l}. \tag{45}$$

Proof. By definition, $\varepsilon_{\mathbf{n},k}$ is the sign of the measure $\frac{H_{\mathbf{n},k}(x)\mathrm{d}\sigma_k(x)}{Q_{\mathbf{n},k-1}(x)Q_{\mathbf{n},k+1}(x)}$ on $\operatorname{supp}(\sigma_k)$. We will denote by $\operatorname{sign}(f, \Delta)$ the sign of a function f on the interval Δ. Thus

$$\frac{\varepsilon_{\mathbf{n}^l,k}}{\varepsilon_{\mathbf{n},k}} = \operatorname{sign}\left(\frac{H_{\mathbf{n}^l,k} Q_{\mathbf{n},k-1} Q_{\mathbf{n},k+1}}{H_{\mathbf{n},k} Q_{\mathbf{n}^l,k-1} Q_{\mathbf{n}^l,k+1}}, \Delta_k\right). \tag{46}$$

If $l \geq k-1$, since $\deg(Q_{\mathbf{n}^l,k-1}) = 1 + \deg(Q_{\mathbf{n},k-1})$, we have that

$$\operatorname{sign}\left(Q_{\mathbf{n},k-1}/Q_{\mathbf{n}^l,k-1}, \Delta_k\right) = \delta_{k-1}, \tag{47}$$

and if $l \leq k-2$, since $\deg(Q_{\mathbf{n}^l,k-1}) = \deg(Q_{\mathbf{n}^l,k-1})$, we obtain

$$\operatorname{sign}\left(Q_{\mathbf{n},k-1}/Q_{\mathbf{n}^l,k-1}, \Delta_k\right) = 1. \tag{48}$$

By similar arguments, we know that for $l \geq k+1$,

$$\operatorname{sign}\left(Q_{\mathbf{n},k+1}/Q_{\mathbf{n}^l,k+1}, \Delta_k\right) = -\delta_k, \tag{49}$$

and if $l \leq k$,

$$\operatorname{sign}\left(Q_{\mathbf{n},k+1}/Q_{\mathbf{n}^l,k+1}, \Delta_k\right) = 1. \tag{50}$$

Finally, from (18) it follows that

$$\frac{H_{\mathbf{n}^l,k}(x)}{H_{\mathbf{n},k}(x)} = \frac{\int_{\Delta_{k-1}} \frac{Q^2_{\mathbf{n}^l,k-1}(t)}{x-t} \frac{H_{\mathbf{n}^l,k-1}(t)\mathrm{d}\sigma_{k-1}(t)}{Q_{\mathbf{n}^l,k-2}(t)Q_{\mathbf{n}^l,k}(t)}}{\int_{\Delta_{k-1}} \frac{Q^2_{\mathbf{n},k-1}(t)}{x-t} \frac{H_{\mathbf{n},k-1}(t)\mathrm{d}\sigma_{k-1}(t)}{Q_{\mathbf{n},k-2}(t)Q_{\mathbf{n},k}(t)}}.$$

Therefore,

$$\operatorname{sign}\left(H_{\mathbf{n}^l,k}/H_{\mathbf{n},k},\, \Delta_k\right) = \frac{\varepsilon_{\mathbf{n}^l,k-1}}{\varepsilon_{\mathbf{n},k-1}}. \tag{51}$$

From (46)–(51) we conclude that

$$\frac{\varepsilon_{\mathbf{n}^l,k}}{\varepsilon_{\mathbf{n},k}} = \Delta_{k,l}\frac{\varepsilon_{\mathbf{n}^l,k-1}}{\varepsilon_{\mathbf{n},k-1}}.$$

Since $H_{\mathbf{n}^l,1} \equiv H_{\mathbf{n},1} \equiv Q_{\mathbf{n}^l,0} \equiv Q_{\mathbf{n},0} \equiv 1$, we have that $\varepsilon_{\mathbf{n}^l,1}$ is the sign of the measure $\frac{\mathrm{d}\sigma_1(x)}{Q_{\mathbf{n}^l,2}(x)}$ on Δ_1, and $\varepsilon_{\mathbf{n},1}$ is the sign of the measure $\frac{\mathrm{d}\sigma_1(x)}{Q_{\mathbf{n},2}(x)}$ on Δ_1. Therefore, we have (45). $\square$

Definition 5.1. We define the following functions

$$\varphi_{k-1}^{(j)}(z) := \frac{\operatorname{sg}(\psi_{k-1}^{(j)}(\infty))}{c_1^{(j)}\psi_{k-1}^{(j)}(z)}, \quad 1 \le j \le m-1. \tag{52}$$

Notice that $\varphi_{k-1}^{(1)} = \varphi_{k-1}$, where φ_{k-1} was previously defined in (33).

Theorem 5.1. *Let $S = \mathcal{N}^*(\sigma_1, \ldots, \sigma_m)$ and $\Lambda \subset \mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m)$ be a sequence of distinct multi-indices such that for all $\mathbf{n} \in \Lambda, n_1 - n_m \le C$, where C is a constant. Then, for each $k \in \{0, 1, \ldots, m\}$,*

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{\Psi}_{\mathbf{n},k}(z)}{\Psi_{\mathbf{n},k}(z)} = G_k(z; p_1, \ldots, p_m), \qquad K \subset \overline{\mathbb{C}} \setminus (\operatorname{supp}(\sigma_k) \cup \operatorname{supp}(\sigma_{k+1})), \tag{53}$$

where G_k is analytic and never vanishes in the indicated region. For each $k = \{0, \ldots, m-1\}$ and all sufficiently large $|\mathbf{n}|, \mathbf{n} \in \Lambda$, $\widetilde{\Psi}_{\mathbf{n},k}$ has exactly $N_{\mathbf{n},k+1} = n_{k+1} + \cdots + n_m$ zeros in $\overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_k)$, $\operatorname{supp}(\sigma_{k+1})$ is an attractor of the zeros of $\{\widetilde{\Psi}_{\mathbf{n},k}\}, \mathbf{n} \in \Lambda$, in this region, and each point in $\operatorname{supp}(\sigma_{k+1}) \setminus \widetilde{\Delta}_{k+1}$ is a 1 attraction point of zeros of $\{\widetilde{\Psi}_{\mathbf{n},k}\}, \mathbf{n} \in \Lambda$. When the coefficients of the polynomials $p_k, k = 1, \ldots, m$, are real, all the statements above remain valid for $\Lambda \subset \mathbb{Z}_+^m(\circledast)$. An expression for G_k is given in (56)–(57).

Proof. For $k = 0$, (53) is (3) since $\widetilde{\Psi}_{\mathbf{n},0} = \widetilde{Q}_{\mathbf{n}}$ and $\Psi_{\mathbf{n},0} = Q_{\mathbf{n}}$; therefore,

$$G_0(z; p_1, \ldots, p_m) = \mathcal{F}(z; p_1, \ldots, p_m).$$

By (34), we know that

$$\lim_{\mathbf{n}\in\Lambda} \frac{\lambda_{\mathbf{n}}^* \varepsilon_{\mathbf{n}_0,k-1} K_{\mathbf{n}_0,k-1}^2 (Q_{\mathbf{n}_0,k-1} R_{\mathbf{n},k-1})(z)}{Q_{\mathbf{n}_0,k}(z)} = \begin{cases} \dfrac{1}{\sqrt{(z-b_1)(z-a_1)}} p_\Lambda(\varphi_1(z)), & k = 2, \\ \dfrac{1}{\sqrt{(z-b_{k-1})(z-a_{k-1})}} p_\Lambda(\delta\varphi_{k-1}(z)), & k = 3, \ldots, m, \end{cases}$$

uniformly on compact subsets of $\mathbb{C} \setminus (\operatorname{supp}(\sigma_{k-1}) \cup \operatorname{supp}(\sigma_k))$. Also, see (22),

$$\lim_{\mathbf{n}\in\Lambda} \varepsilon_{\mathbf{n}_0,k-1} h_{\mathbf{n}_0,k}(z) = \frac{1}{\sqrt{(z-b_{k-1})(z-a_{k-1})}}, \qquad K \subset \overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_{k-1}).$$

Thus, since $\lim_{\mathbf{n}\in\Lambda}\lambda_{\mathbf{n}}^* = c$, we conclude that

$$\lim_{\mathbf{n}\in\Lambda}\frac{R_{\mathbf{n},k-1}(z)}{\Psi_{\mathbf{n}_0,k-1}(z)} = \lim_{\mathbf{n}\in\Lambda} K^2_{\mathbf{n}_0,k-1}\frac{(Q_{\mathbf{n}_0,k-1}R_{\mathbf{n},k-1})(z)}{(h_{\mathbf{n}_0,k}Q_{\mathbf{n}_0,k})(z)}$$
$$= \begin{cases} p_\Lambda(\varphi_1(z))/c, & k=2,\\ p_\Lambda(\delta\varphi_{k-1}(z))/c, & k=3,\ldots,m, \end{cases} \tag{54}$$

uniformly on compact subsets of $\mathbb{C}\setminus(\operatorname{supp}(\sigma_{k-1})\cup\operatorname{supp}(\sigma_k))$.

Recall that $\mathbf{n}_j = (n_1 - \deg(p_2\cdots p_m) + j, n_2 - \deg(p_3\cdots p_m), \ldots, n_m)$. It is easy to see that

$$\frac{\Psi_{\mathbf{n}_0,k-1}}{\Psi_{\mathbf{n}_j,k-1}} = \frac{Q_{\mathbf{n}_0,k}}{Q_{\mathbf{n}_j,k}}\frac{Q_{\mathbf{n}_j,k-1}}{Q_{\mathbf{n}_0,k-1}}\frac{\varepsilon_{\mathbf{n}_0,k-1}h_{\mathbf{n}_0,k}}{\varepsilon_{\mathbf{n}_j,k-1}h_{\mathbf{n}_j,k}}\frac{\varepsilon_{\mathbf{n}_j,k-1}}{\varepsilon_{\mathbf{n}_0,k-1}}\frac{K^2_{\mathbf{n}_j,k-1}}{K^2_{\mathbf{n}_0,k-1}}.$$

From this expression, applying Proposition 3.2 and (45), we obtain that the following limit holds uniformly on compact subsets of $\mathbb{C}\setminus(\operatorname{supp}(\sigma_{k-1})\cup\operatorname{supp}(\sigma_k))$

$$\lim_{\mathbf{n}\in\Lambda}\frac{\Psi_{\mathbf{n}_0,k-1}(z)}{\Psi_{\mathbf{n}_j,k-1}(z)} = (\Delta_{k-1,1}\cdots\Delta_{1,1})^j\left(\frac{\widetilde{F}^{(1)}_{k-1}(z)}{\widetilde{F}^{(1)}_{k}(z)}\right)^j(\kappa_1^{(1)}\cdots\kappa_{k-1}^{(1)})^{2j}.$$

Now, from (28) and (29), we have

$$\frac{\widetilde{F}^{(1)}_{k-1}(z)}{\widetilde{F}^{(1)}_{k}(z)} = \frac{c_k^{(1)}}{c_{k-1}^{(1)}}\operatorname{sg}(\psi^{(1)}_{k-1}(\infty))\psi^{(1)}_{k-1}(z),$$

and from (28)

$$(\kappa_1^{(1)}\cdots\kappa_{k-1}^{(1)})^2 = c_1^{(1)}\frac{c_{k-1}^{(1)}}{c_k^{(1)}}.$$

Thus,

$$\lim_{\mathbf{n}\in\Lambda}\frac{\Psi_{\mathbf{n}_0,k-1}(z)}{\Psi_{\mathbf{n}_j,k-1}(z)} = (\Delta_{k-1,1}\cdots\Delta_{1,1})^j(c_1^{(1)}\operatorname{sg}(\psi^{(1)}_{k-1}(\infty))\psi^{(1)}_{k-1}(z))^j.$$

Set

$$\Xi_k := (\Delta_{k-1,1}\cdots\Delta_{1,1})^{\deg(p_2\cdots p_m)}\cdots(\Delta_{k-1,m-1}\cdots\Delta_{1,m-1})^{\deg(p_m)}. \tag{55}$$

Using the same arguments employed above, on an appropriate consecutive collection of multi-indices, one proves that

$$\lim_{\mathbf{n}\in\Lambda}\frac{\Psi_{\mathbf{n}_0,k-1}(z)}{\Psi_{\mathbf{n},k-1}(z)} = \Xi_k\prod_{j=1}^{m-1}\frac{1}{(\varphi^{(j)}_{k-1}(z))^{\deg(p_{j+1}\cdots p_m)}},$$

uniformly on compact subsets of $\mathbb{C}\setminus(\operatorname{supp}(\sigma_{k-1})\cup\operatorname{supp}(\sigma_k))$. Therefore, writing

$$\frac{R_{\mathbf{n},k-1}(z)}{\Psi_{\mathbf{n},k-1}(z)} = \frac{R_{\mathbf{n},k-1}(z)}{\Psi_{\mathbf{n}_0,k-1}(z)}\frac{\Psi_{\mathbf{n}_0,k-1}(z)}{\Psi_{\mathbf{n},k-1}(z)},$$

using the expression of p_Λ, applying (54), and Lemma 5.1, for $k=2$ we get

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{\Psi}_{\mathbf{n},1}(z)}{\Psi_{\mathbf{n},1}(z)} = \Xi_2 \prod_{\nu=1}^{l_1} (\varphi_1(z) - \varphi_0(z_{1,\nu}))^{\tau_{1,\nu}} \prod_{\nu=1}^{l_2} \left(\frac{1}{\varphi_1(z)} \frac{\varphi_1(z) - \varphi_1(z_{2,\nu})}{z - z_{2,\nu}} \right)^{\tau_{2,\nu}}$$
$$\times \prod_{j=3}^{m} \prod_{\nu=1}^{l_j} \left(\frac{\varphi_1(z) - \delta\varphi_{j-1}(z_{j,\nu})}{\varphi_1^{(j-1)}(z)} \right)^{\tau_{j,\nu}} \tag{56}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus (\mathrm{supp}(\sigma_1) \cup \mathrm{supp}(\sigma_2))$, and for $k \geq 3$ we obtain

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{\Psi}_{\mathbf{n},k-1}(z)}{\Psi_{\mathbf{n},k-1}(z)} = \Xi_k \prod_{\nu=1}^{l_1} (\delta\varphi_{k-1}(z) - \varphi_0(z_{1,\nu}))^{\tau_{1,\nu}} \prod_{\nu=1}^{l_2} \left(\frac{\delta\varphi_{k-1}(z) - \varphi_1(z_{2,\nu})}{\varphi_{k-1}(z)} \right)^{\tau_{2,\nu}}$$
$$\times \prod_{\nu=1}^{l_k} \left(\frac{\delta\varphi_{k-1}(z) - \delta\varphi_{k-1}(z_{k,\nu})}{\varphi_{k-1}^{(k-1)}(z)(z - z_{k,\nu})} \right)^{\tau_{k,\nu}} \prod_{j=3, j\neq k}^{m} \prod_{\nu=1}^{l_j} \left(\frac{\delta\varphi_{k-1}(z) - \delta\varphi_{j-1}(z_{j,\nu})}{\varphi_{k-1}^{(j-1)}(z)} \right)^{\tau_{j,\nu}} \tag{57}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus (\mathrm{supp}(\sigma_{k-1}) \cup \mathrm{supp}(\sigma_k))$. Therefore, (53) is proved.

From the expression of the limit functions one sees that G_k does not vanish in $\overline{\mathbb{C}} \setminus (\mathrm{supp}(\sigma_k) \cup \mathrm{supp}(\sigma_{k+1}))$. The statements concerning the number of zeros of $\widetilde{\Psi}_{\mathbf{n},k}$ for $k \in \{0, \ldots, m-1\}$ and their limit behavior follows at once from (53), on account of the argument principle and the corresponding behavior of the zeros of the polynomials $Q_{\mathbf{n},k+1}$ described in Proposition 3.1. Recall that the zeros of $Q_{\mathbf{n},k+1}$ are those of $\Psi_{\mathbf{n},k}$ in $\overline{\mathbb{C}} \setminus \mathrm{supp}(\sigma_k)$.

Now, let us assume that the coefficients of the polynomials p_k are real and $\Lambda \subset \mathbb{Z}_+^m(\circledast)$. Since

$$\frac{\Psi_{\mathbf{n}^l,k-1}}{\Psi_{\mathbf{n},k-1}} = \frac{Q_{\mathbf{n}^l,k}}{Q_{\mathbf{n},k}} \frac{Q_{\mathbf{n},k-1}}{Q_{\mathbf{n}^l,k-1}} \frac{\varepsilon_{\mathbf{n}^l,k-1} h_{\mathbf{n}^l,k}}{\varepsilon_{\mathbf{n},k-1} h_{\mathbf{n},k}} \frac{\varepsilon_{\mathbf{n},k-1}}{\varepsilon_{\mathbf{n}^l,k-1}} \frac{K^2_{\mathbf{n},k-1}}{K^2_{\mathbf{n}^l,k-1}},$$

applying (26), (27), (22) and (45), we conclude that

$$\lim_{\mathbf{n}\in\Lambda} \frac{\Psi_{\mathbf{n}^l,k-1}(z)}{\Psi_{\mathbf{n},k-1}(z)}, \qquad K \subset \mathbb{C} \setminus (\mathrm{supp}(\sigma_{k-1}) \cup \mathrm{supp}(\sigma_k)),$$

holds and the limit does not vanish in the indicated region.

Since each measure $p_k \sigma_k$ is real with a constant sign, we can define the polynomials $\widetilde{Q}_{\mathbf{n},k}$, $1 \leq k \leq m$, as the monic polynomials of degree $N_{\mathbf{n},k}$ whose simple zeros are located at the points where $\widetilde{\Psi}_{n,k-1}$ vanishes on Δ_k. Let $\widetilde{Q}_{\mathbf{n},0} \equiv \widetilde{Q}_{\mathbf{n},m+1} \equiv 1$. We also introduce the associated notions

$$\widetilde{H}_{\mathbf{n},k} := \frac{\widetilde{Q}_{\mathbf{n},k-1} \widetilde{\Psi}_{\mathbf{n},k-1}}{\widetilde{Q}_{\mathbf{n},k}}, \quad k = 1, \ldots, m+1, \tag{58}$$

$\widetilde{\varepsilon}_{\mathbf{n},k}$ as the sign of $\widetilde{H}_{\mathbf{n},k}(x) p_k(x) \mathrm{d}\sigma_k(x) / \widetilde{Q}_{\mathbf{n},k-1}(x) \widetilde{Q}_{\mathbf{n},k+1}(x)$ on $\mathrm{supp}(\sigma_k)$, and

$$\widetilde{K}_{\mathbf{n},k} := \left(\int \widetilde{Q}^2_{\mathbf{n},k}(x) \frac{\widetilde{\varepsilon}_{\mathbf{n},k} \widetilde{H}_{\mathbf{n},k}(x) p_k(x) \mathrm{d}\sigma_k(x)}{\widetilde{Q}_{\mathbf{n},k-1}(x) \widetilde{Q}_{\mathbf{n},k+1}(x)} \right)^{-1/2}. \tag{59}$$

The formulas (26), (27), (22), and (45) are independent of the orthogonality measures, hence

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{\Psi}_{\mathbf{n}^l,k-1}(z)}{\widetilde{\Psi}_{\mathbf{n},k-1}(z)} = \lim_{\mathbf{n}\in\Lambda} \frac{\Psi_{\mathbf{n}^l,k-1}(z)}{\Psi_{\mathbf{n},k-1}(z)}.$$

Applying the same argument used in the last two paragraphs, of the proof of Theorem 1.1, we conclude that (53) is valid for $\Lambda \subset \mathbb{Z}_+^m(\circledast)$.

The rest of the statements regarding the zeros of $\widetilde{\Psi}_{\mathbf{n},k}$ and their limit behavior follows, as in the case of polynomials with complex coefficients. □

Corollary 5.1. *Let $S = \mathcal{N}^*(\sigma_1, \ldots, \sigma_m)$. Consider the perturbed Nikishin system $\mathcal{N}(\frac{p_1}{q_1}\sigma_1, \ldots, \frac{p_m}{q_m}\sigma_m)$, where p_k, q_k denote relatively prime polynomials whose zeros lie in $\mathbb{C} \setminus \cup_{k=1}^m \Delta_k$. Let $\Lambda \subset \mathbb{Z}_+^m(\circledast; p_1q_1, \ldots, p_mq_m)$ be a sequence of distinct multi-indices such that for all $\mathbf{n} \in \Lambda$, $n_1 - n_m \leq C$, where C is a constant. Let $\widetilde{Q}_{\mathbf{n}}$ be the monic multiple orthogonal polynomial of smallest degree relative to the Nikishin system $\mathcal{N}(\frac{p_1}{q_1}\sigma_1, \ldots, \frac{p_m}{q_m}\sigma_m)$ and $\mathbf{n}$, whereas $\widetilde{\Psi}_{\mathbf{n},k}, 0 \leq k \leq m$, denote the second type functions defined in (44), with p_k replaced by p_k/q_k. Then, for each $k \in \{0, \ldots, m\}$,*

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{\Psi}_{n,k}(z)}{\Psi_{n,k}(z)} = \frac{G_k(z; p_1, \ldots, p_m)}{G_k(z; q_1, \ldots, q_m)}, \qquad K \subset \overline{\mathbb{C}} \setminus (\mathrm{supp}(\sigma_k) \cup \mathrm{supp}(\sigma_{k+1})). \tag{60}$$

For each $k = \{0, \ldots, m-1\}$ and all sufficiently large $|\mathbf{n}|$, $\mathbf{n} \in \Lambda$, $\widetilde{\Psi}_{\mathbf{n},k}$ has exactly $N_{\mathbf{n},k+1}$ zeros in $\overline{\mathbb{C}} \setminus \mathrm{supp}(\sigma_k)$, $\mathrm{supp}(\sigma_{k+1})$ is an attractor of the zeros of $\{\widetilde{\Psi}_{\mathbf{n},k}\}$, $\mathbf{n} \in \Lambda$, in this region, and each point in $\mathrm{supp}(\sigma_{k+1}) \setminus \widetilde{\Delta}_{k+1}$ is a 1 attraction point of zeros of $\{\widetilde{\Psi}_{\mathbf{n},k}\}$, $\mathbf{n} \in \Lambda$. When the polynomials $p_k, q_k, k = 1, \ldots, m$, have real coefficients, all the statements remain valid when $\Lambda \subset \mathbb{Z}_+^m(\circledast)$.

Proof. We consider the auxiliary Nikishin system

$$S_1 := \mathcal{N}\left(\frac{\sigma_1}{|q_1|^2}, \ldots, \frac{\sigma_m}{|q_m|^2}\right),$$

and define the related second type functions

$$\Psi^*_{n,0}(z) := Q^*_{\mathbf{n}}(z),$$
$$\Psi^*_{n,k}(z) := \int \frac{\Psi^*_{n,k-1}(x)}{z-x}\frac{\mathrm{d}\sigma_k(x)}{|q_k(x)|^2}, \quad 1 \leq k \leq m,$$

where $Q^*_{\mathbf{n}}$ denotes the multiple orthogonal polynomial associated to S_1 and $\mathbf{n}$.

Notice that if we perturb the generator of system S_1 multiplying the k-th measure by the real polynomial $|q_k|^2$ we get the generator of the original Nikishin system S. Thus, applying Theorem 5.1, we obtain that for all $k \in \{0, \ldots, m\}$

$$\lim_{\mathbf{n}\in\Lambda} \frac{\Psi_{n,k}(z)}{\Psi^*_{n,k}(z)} = G_k(z; |q_1|^2, \ldots, |q_m|^2), \qquad K \subset \overline{\mathbb{C}} \setminus (\mathrm{supp}(\sigma_k) \cup \mathrm{supp}(\sigma_{k+1})).$$

The perturbed system $S_2 := \mathcal{N}(\frac{p_1}{q_1}\sigma_1, \ldots, \frac{p_m}{q_m}\sigma_m)$ can be written as

$$S_2 = \mathcal{N}\left(p_1\overline{q}_1\frac{\sigma_1}{|q_1|^2}, \ldots, p_m\overline{q}_m\frac{\sigma_m}{|q_m|^2}\right).$$

Therefore, employing the same argument

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{\Psi}_{n,k}(z)}{\Psi^*_{n,k}(z)} = G_k(z; p_1\overline{q}_1, \ldots, p_m\overline{q}_m), \qquad K \subset \overline{\mathbb{C}} \setminus (\mathrm{supp}(\sigma_k) \cup \mathrm{supp}(\sigma_{k+1})).$$

We conclude that

$$\lim_{\mathbf{n}\in\Lambda}\frac{\widetilde{\Psi}_{n,k}(z)}{\Psi_{n,k}(z)}=\frac{G_k(z;p_1\overline{q}_1,\ldots,p_m\overline{q}_m)}{G_k(z;q_1\overline{q}_1,\ldots,q_m\overline{q}_m)}=\frac{G_k(z;p_1,\ldots,p_m)}{G_k(z;q_1,\ldots,q_m)},$$

uniformly on compact subsets of $\overline{\mathbb{C}}\setminus(\mathrm{supp}(\sigma_k)\cup\mathrm{supp}(\sigma_{k+1}))$. The statements concerning the zeros can be proved as in the case of polynomial perturbation.

When the polynomials $p_k, q_k, k = 1,\ldots,m$, have real coefficients, it follows from Theorem 5.1 that (60) remains valid for $\Lambda\subset\mathbb{Z}_+^m(\circledast)$. The statements concerning the zeros are derived immediately. □

6. Relative asymptotic behavior for the polynomials $Q_{\mathbf{n},k}$

In this section, we will restrict our attention to the case when the polynomials $p_k, q_k, k = 1,\ldots,m$, have real coefficients (and of course their zeros lie in $\mathbb{C}\setminus\cup_{k=1}^m\Delta_k$). Accordingly, we use the objects $\widetilde{Q}_{\mathbf{n},k}$, $\widetilde{H}_{\mathbf{n},k}$, $\widetilde{K}_{\mathbf{n},k}$, and $\widetilde{\varepsilon}_{\mathbf{n},k}$, introduced at the end of the proof of Theorem 5.1 (see (58) and (59)). Here, we study the relative asymptotic behavior of the ratios $\widetilde{Q}_{\mathbf{n},k}/Q_{\mathbf{n},k}$.

Lemma 6.1. *For any* $\mathbf{n}\in\mathbb{Z}_+^m(\circledast)$

$$\frac{\varepsilon_{\mathbf{n},k}}{\widetilde{\varepsilon}_{\mathbf{n},k}}=\prod_{i=1}^{k}\mathrm{sign}\,(p_i,\mathrm{supp}(\sigma_i))\,. \tag{61}$$

Proof. By definition $\varepsilon_{\mathbf{n},k}$ is the sign of $H_{\mathbf{n},k}(x)\mathrm{d}\sigma_k(x)/Q_{\mathbf{n},k-1}(x)Q_{\mathbf{n},k+1}(x)$ on $\mathrm{supp}(\sigma_k)$ and $\widetilde{\varepsilon}_{\mathbf{n},k}$ is the sign of $\widetilde{H}_{\mathbf{n},k}(x)p_k(x)\mathrm{d}\sigma_k(x)/\widetilde{Q}_{\mathbf{n},k-1}(x)\widetilde{Q}_{\mathbf{n},k+1}(x)$ on $\mathrm{supp}(\sigma_k)$. If $k = 1$ these measures reduce, respectively, to $\mathrm{d}\sigma_1(x)/Q_{\mathbf{n},2}(x)$ and $p_1(x)\mathrm{d}\sigma_1(x)/\widetilde{Q}_{\mathbf{n},2}(x)$. Since $Q_{\mathbf{n},2}$ and $\widetilde{Q}_{\mathbf{n},2}$ are monic polynomials of the same degree and their zeros are located in Δ_2, which is disjoint with $\mathrm{supp}(\sigma_1)$, it follows that $Q_{\mathbf{n},2}$ and $\widetilde{Q}_{\mathbf{n},2}$ have the same sign on $\mathrm{supp}(\sigma_1)$. Therefore,

$$\frac{\varepsilon_{\mathbf{n},1}}{\widetilde{\varepsilon}_{\mathbf{n},1}}=\mathrm{sign}\,(p_1,\mathrm{supp}(\sigma_1))\,.$$

To conclude the proof we show that

$$\frac{\varepsilon_{\mathbf{n},k}}{\widetilde{\varepsilon}_{\mathbf{n},k}}=\mathrm{sign}\,(p_k,\mathrm{supp}(\sigma_k))\,\frac{\varepsilon_{\mathbf{n},k-1}}{\widetilde{\varepsilon}_{\mathbf{n},k-1}}.$$

Notice that $Q_{\mathbf{n},k-1}$ and $\widetilde{Q}_{\mathbf{n},k-1}$ have the same sign on $\mathrm{supp}(\sigma_k)$ by an argument similar to the one explained above. The same holds for $Q_{\mathbf{n},k+1}$ and $\widetilde{Q}_{\mathbf{n},k+1}$. Therefore

$$\frac{\varepsilon_{\mathbf{n},k}}{\widetilde{\varepsilon}_{\mathbf{n},k}}=\frac{\mathrm{sign}\left(H_{\mathbf{n},k},\mathrm{supp}(\sigma_k)\right)}{\mathrm{sign}\left(p_k\widetilde{H}_{\mathbf{n},k},\mathrm{supp}(\sigma_k)\right)}.$$

By (18), we know that

$$H_{\mathbf{n},k}(x)=\int_{\Delta_{k-1}}\frac{Q^2_{\mathbf{n},k-1}(t)}{x-t}\frac{H_{\mathbf{n},k-1}(t)\mathrm{d}\sigma_{k-1}(t)}{Q_{\mathbf{n},k-2}(t)Q_{\mathbf{n},k}(t)},$$

and

$$\widetilde{H}_{\mathbf{n},k}(x)=\int_{\Delta_{k-1}}\frac{\widetilde{Q}^2_{\mathbf{n},k-1}(t)}{x-t}\frac{\widetilde{H}_{\mathbf{n},k-1}(t)p_{k-1}(t)\mathrm{d}\sigma_{k-1}(t)}{\widetilde{Q}_{\mathbf{n},k-2}(t)\widetilde{Q}_{\mathbf{n},k}(t)}.$$

Consequently,

$$\frac{\operatorname{sign}\left(H_{\mathbf{n},k}, \operatorname{supp}(\sigma_k)\right)}{\operatorname{sign}\left(\widetilde{H}_{\mathbf{n},k}, \operatorname{supp}(\sigma_k)\right)} = \frac{\varepsilon_{\mathbf{n},k-1}}{\widetilde{\varepsilon}_{\mathbf{n},k-1}},$$

and the claim follows. □

We are ready to state and prove

Theorem 6.1. *Let $S = \mathcal{N}^*(\sigma_1, \ldots, \sigma_m)$ and $\Lambda \subset \mathbb{Z}_+^m(\circledast)$ be a sequence of distinct multi-indices such that for all $\mathbf{n} \in \Lambda, n_1 - n_m \le C$, where C is a constant. Assume that the polynomials $p_k, k = 1, \ldots, m$, have real coefficients. For each $k \in \{1, \ldots, m\}$,*

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{Q}_{\mathbf{n},k}(z)}{Q_{\mathbf{n},k}(z)} = \mathcal{F}_k(z; p_1, \ldots, p_m), \qquad K \subset \overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_k), \tag{62}$$

where $\mathcal{F}_k(z; p_1, \ldots, p_m)$ is analytic and never vanishes in $\overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_k)$ and

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{K}^2_{\mathbf{n},k}}{K^2_{\mathbf{n},k}} = \frac{\prod_{i=1}^{k} \operatorname{sign}(p_i, \operatorname{supp}(\sigma_i))}{G_k(\infty; p_1, \ldots, p_m)}. \tag{63}$$

For $k \in \{1, \ldots, m-1\}$ and $z \in \overline{\mathbb{C}} \setminus (\operatorname{supp}(\sigma_k) \cup \operatorname{supp}(\sigma_{k+1}))$

$$\mathcal{F}_{k+1}(z; p_1, \ldots, p_m) = \prod_{i=0}^{k} \frac{G_i(z; p_1, \ldots, p_m)}{G_i(\infty; p_1, \ldots, p_m)}, \tag{64}$$

where $G_i(z; p_1, \ldots, p_m)$ is the function given in (53).

Proof. If $\Lambda \subset \mathbb{Z}_+^m(\circledast; p_1, \ldots, p_m)$, from (34) and Lemma 5.1, we have that

$$\lim_{\mathbf{n}\in\Lambda} \lambda^*_{\mathbf{n}} \varepsilon_{\mathbf{n}_0,k-1} K^2_{\mathbf{n}_0,k-1} \frac{Q_{\mathbf{n}_0,k-1}(z)(p_k \cdots p_m)(z)\widetilde{\Psi}_{\mathbf{n},k-1}(z)}{Q_{\mathbf{n}_0,k}(z)}$$
$$= \begin{cases} \dfrac{1}{\sqrt{(z-b_1)(z-a_1)}} p_\Lambda(\varphi_1(z)), & k = 2, \\ \dfrac{1}{\sqrt{(z-b_{k-1})(z-a_{k-1})}} p_\Lambda(\delta\varphi_{k-1}(z)), & k = 3, \ldots, m. \end{cases} \tag{65}$$

By Proposition 3.1, we know that

$$\lim_{\mathbf{n}\in\Lambda} \widetilde{\varepsilon}_{\mathbf{n},k} \widetilde{K}^2_{\mathbf{n},k} \widetilde{H}_{\mathbf{n},k+1}(z) = \frac{1}{\sqrt{(z-b_k)(z-a_k)}}, \qquad K \subset \overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_k), \tag{66}$$

where $[a_k, b_k] = \widetilde{\Delta}_k$. Formula (58) implies

$$\frac{\lambda^*_{\mathbf{n}} \varepsilon_{\mathbf{n}_0,k-1} K^2_{\mathbf{n}_0,k-1} Q_{\mathbf{n}_0,k-1}(z)(p_k \cdots p_m)(z)\widetilde{\Psi}_{\mathbf{n},k-1}(z)}{\widetilde{\varepsilon}_{\mathbf{n},k-1} \widetilde{K}^2_{\mathbf{n},k-1} \widetilde{H}_{\mathbf{n},k}(z) Q_{\mathbf{n}_0,k}(z)}$$
$$= \lambda^*_{\mathbf{n}} \frac{\varepsilon_{\mathbf{n}_0,k-1}}{\widetilde{\varepsilon}_{\mathbf{n},k-1}} \frac{K^2_{\mathbf{n}_0,k-1}}{\widetilde{K}^2_{\mathbf{n},k-1}} \frac{Q_{\mathbf{n}_0,k-1}(z)}{\widetilde{Q}_{\mathbf{n},k-1}(z)} \frac{\widetilde{Q}_{\mathbf{n},k}(z)}{Q_{\mathbf{n}_0,k}(z)} (p_k \cdots p_m)(z). \tag{67}$$

Using (65), (66), and (67), we obtain

$$\lim_{\mathbf{n}\in\Lambda} \lambda^*_{\mathbf{n}} \frac{\varepsilon_{\mathbf{n}_0,k-1}}{\widetilde{\varepsilon}_{\mathbf{n},k-1}} \frac{K^2_{\mathbf{n}_0,k-1}}{\widetilde{K}^2_{\mathbf{n},k-1}} \frac{Q_{\mathbf{n}_0,k-1}(z)}{\widetilde{Q}_{\mathbf{n},k-1}(z)} \frac{\widetilde{Q}_{\mathbf{n},k}(z)}{Q_{\mathbf{n}_0,k}(z)} (p_k \cdots p_m)(z) = \begin{cases} p_\Lambda(\varphi_1(z)), & k=2, \\ p_\Lambda(\delta\varphi_{k-1}(z)), & k=3,\dots,m. \end{cases} \tag{68}$$

Using the results on ratio asymptotic for the constants $K_{\mathbf{n},k}, \widetilde{K}_{\mathbf{n},k}$ and the polynomials $Q_{\mathbf{n},k}, \widetilde{Q}_{\mathbf{n},k}$, it follows that (68) is also valid for $\Lambda \subset \mathbb{Z}^m_+(\circledast)$.

Since

$$\frac{\varepsilon_{\mathbf{n}_0,k-1}}{\widetilde{\varepsilon}_{\mathbf{n},k-1}} = \frac{\varepsilon_{\mathbf{n}_0,k-1}}{\varepsilon_{\mathbf{n},k-1}} \frac{\varepsilon_{\mathbf{n},k-1}}{\widetilde{\varepsilon}_{\mathbf{n},k-1}},$$

applying Lemma 6.1, (45), and (55), we obtain

$$\frac{\varepsilon_{\mathbf{n}_0,k-1}}{\widetilde{\varepsilon}_{\mathbf{n},k-1}} = \Xi_k \prod_{i=1}^{k-1} \operatorname{sign}\left(p_i, \operatorname{supp}(\sigma_i)\right). \tag{69}$$

We have

$$\frac{K^2_{\mathbf{n}_0,k-1}}{\widetilde{K}^2_{\mathbf{n},k-1}} = \frac{K^2_{\mathbf{n}_0,k-1}}{K^2_{\mathbf{n},k-1}} \frac{K^2_{\mathbf{n},k-1}}{\widetilde{K}^2_{\mathbf{n},k-1}}, \tag{70}$$

and by (26)

$$\lim_{\mathbf{n}\in\Lambda} \frac{K^2_{\mathbf{n}_0,k-1}}{K^2_{\mathbf{n},k-1}} = \prod_{i=1}^{m-1} (\kappa_1^{(i)} \cdots \kappa_{k-1}^{(i)})^{-2\deg(p_{i+1}\cdots p_m)}. \tag{71}$$

Write

$$\frac{Q_{\mathbf{n}_0,k-1}(z)}{\widetilde{Q}_{\mathbf{n},k-1}(z)} = \frac{Q_{\mathbf{n}_0,k-1}(z)}{Q_{\mathbf{n},k-1}(z)} \frac{Q_{\mathbf{n},k-1}(z)}{\widetilde{Q}_{\mathbf{n},k-1}(z)}, \tag{72}$$

and

$$\frac{\widetilde{Q}_{\mathbf{n},k}(z)}{Q_{\mathbf{n}_0,k}(z)} = \frac{\widetilde{Q}_{\mathbf{n},k}(z)}{Q_{\mathbf{n},k}(z)} \frac{Q_{\mathbf{n},k}(z)}{Q_{\mathbf{n}_0,k}(z)}. \tag{73}$$

Notice that

$$\lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n}_0,k-1}(z)}{Q_{\mathbf{n},k-1}(z)} = \prod_{i=1}^{m-1} (\widetilde{F}^{(i)}_{k-1}(z))^{-\deg(p_{i+1}\cdots p_m)}. \tag{74}$$

$$\lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n},k}(z)}{Q_{\mathbf{n}_0,k}(z)} = \prod_{i=1}^{m-1} (\widetilde{F}^{(i)}_{k}(z))^{\deg(p_{i+1}\cdots p_m)}. \tag{75}$$

From (29) and (28)it follows that

$$\frac{\widetilde{F}^{(i)}_k(z)}{\widetilde{F}^{(i)}_{k-1}(z)} = \frac{c^{(i)}_{k-1}}{c^{(i)}_k} \frac{\operatorname{sg}(\psi^{(i)}_{k-1}(\infty))}{\psi^{(i)}_{k-1}(z)},$$

$$(\kappa_1^{(i)} \cdots \kappa_{k-1}^{(i)})^2 = \frac{c_1^{(i)} c_{k-1}^{(i)}}{c_k^{(i)}}.$$

Therefore, using (52), we get

$$\frac{\widetilde{F}_k^{(i)}(z)}{\widetilde{F}_{k-1}^{(i)}(z)(\kappa_1^{(i)}\cdots\kappa_{k-1}^{(i)})^2} = \varphi_{k-1}^{(i)}(z). \tag{76}$$

Taking into consideration (69)–(76), we conclude that

$$\begin{aligned}
&\lim_{\mathbf{n}\in\Lambda} \lambda_{\mathbf{n}}^* \frac{\varepsilon_{\mathbf{n}_0,k-1}}{\widetilde{\varepsilon}_{\mathbf{n},k-1}} \frac{K^2_{\mathbf{n}_0,k-1}}{\widetilde{K}^2_{\mathbf{n},k-1}} \frac{Q_{\mathbf{n}_0,k-1}(z)}{\widetilde{Q}_{\mathbf{n},k-1}(z)} \frac{\widetilde{Q}_{\mathbf{n},k}(z)}{Q_{\mathbf{n}_0,k}(z)} (p_k\cdots p_m)(z) \\
&\quad = c\,\Xi_k \prod_{i=1}^{k-1} \operatorname{sign}(p_i, \operatorname{supp}(\sigma_i)) \prod_{i=1}^{m-1} (\varphi_{k-1}^{(i)}(z))^{\deg(p_{i+1}\cdots p_m)} \lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n},k-1}(z)}{\widetilde{Q}_{\mathbf{n},k-1}(z)} \\
&\qquad \times (p_k\cdots p_m)(z) \lim_{\mathbf{n}\in\Lambda} \frac{K^2_{\mathbf{n},k-1}}{\widetilde{K}^2_{\mathbf{n},k-1}} \frac{\widetilde{Q}_{\mathbf{n},k}(z)}{Q_{\mathbf{n},k}(z)},
\end{aligned} \tag{77}$$

provided that the limits on the right hand side exist.

In Theorem 1.1 we proved (62) for $k = 1$. Assume that $k = 2$. Eq. (68) and (77) yield

$$\lim_{\mathbf{n}\in\Lambda} \frac{K^2_{\mathbf{n},1}}{\widetilde{K}^2_{\mathbf{n},1}} \frac{\widetilde{Q}_{\mathbf{n},2}(z)}{Q_{\mathbf{n},2}(z)} = \frac{p_\Lambda(\varphi_1(z))\mathcal{F}(z;p_1,\ldots,p_m)}{c\,\Xi_2 \operatorname{sign}(p_1,\operatorname{supp}(\sigma_1))\,(p_2\cdots p_m)(z) \prod_{i=1}^{m-1} (\varphi_1^{(i)}(z))^{\deg(p_{i+1}\cdots p_m)}},$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_2)$. Using (56), we have

$$\frac{p_\Lambda(\varphi_1(z))}{c\,\Xi_2\,(p_2\cdots p_m)(z) \prod_{i=1}^{m-1} (\varphi_1^{(i)}(z))^{\deg(p_{i+1}\cdots p_m)}} = G_1(z;p_1,\ldots,p_m).$$

Consequently,

$$\lim_{\mathbf{n}\in\Lambda} \frac{K^2_{\mathbf{n},1}}{\widetilde{K}^2_{\mathbf{n},1}} \frac{\widetilde{Q}_{\mathbf{n},2}(z)}{Q_{\mathbf{n},2}(z)} = \frac{\mathcal{F}(z;p_1,\ldots,p_m)\,G_1(z;p_1,\ldots,p_m)}{\operatorname{sign}(p_1,\operatorname{supp}(\sigma_1))}.$$

Evaluating at infinity, we obtain ($\mathcal{F}(\infty;p_1,\ldots,p_m) = 1$)

$$\lim_{\mathbf{n}\in\Lambda} \frac{K^2_{\mathbf{n},1}}{\widetilde{K}^2_{\mathbf{n},1}} = \frac{G_1(\infty;p_1,\ldots,p_m)}{\operatorname{sign}(p_1,\operatorname{supp}(\sigma_1))}.$$

Therefore, (63) and (64) are satisfied for $k = 1$, since $G_0 = \mathcal{F}$.

Define the functions

$$\mathcal{F}_k(z;p_1,\ldots,p_m) := \lim_{n\in\Lambda} \frac{\widetilde{Q}_{\mathbf{n},k}(z)}{Q_{\mathbf{n},k}(z)}$$

provided the limit exists. From (57) it follows that for any $k \geq 3$,

$$\frac{p_\Lambda(\delta\varphi_{k-1}(z))}{c\,\Xi_k\,(p_k\cdots p_m)(z) \prod_{i=1}^{m-1} (\varphi_{k-1}^{(i)}(z))^{\deg(p_{i+1}\cdots p_m)}} = G_{k-1}(z;p_1,\ldots,p_m).$$

As a consequence, using (77), we obtain that for any $k \geq 3$,

$$\lim_{\mathbf{n}\in\Lambda} \frac{K^2_{\mathbf{n},k-1}}{\widetilde{K}^2_{\mathbf{n},k-1}} \frac{\widetilde{Q}_{\mathbf{n},k}(z)}{Q_{\mathbf{n},k}(z)} = \frac{\mathcal{F}_{k-1}(z;\, p_1,\ldots,p_m)\, G_{k-1}(z;\, p_1,\ldots,p_m)}{\prod_{i=1}^{k-1} \operatorname{sign}(p_i, \operatorname{supp}(\sigma_i))}.$$

Therefore, using an induction process, one proves (62)–(64). □

Corollary 6.1. *Let $S = \mathcal{N}^*(\sigma_1,\ldots,\sigma_m)$. Consider the perturbed Nikishin system $\mathcal{N}(\frac{p_1}{q_1}\sigma_1,\ldots,\frac{p_m}{q_m}\sigma_m)$, where p_k, q_k denote relatively prime polynomials with real coefficients whose zeros lie in $\mathbb{C}\setminus\cup_{k=1}^m \Delta_k$. Let $\Lambda \subset \mathbb{Z}^m_+(\circledast)$ be a sequence of distinct multi-indices such that for all $\mathbf{n}\in\Lambda$, $n_1 - n_m \leq C$, where C is a constant. Let $\widetilde{Q}_{\mathbf{n},k}$, $1 \leq k \leq m$, be the monic polynomials of degree $N_{\mathbf{n},k}$ whose simple zeros are located at the points where $\widetilde{\Psi}_{n,k-1}$ vanishes on Δ_k, where $\widetilde{\Psi}_{\mathbf{n},k}$, $0 \leq k \leq m$, denote the second type functions defined in (44), with p_k replaced by p_k/q_k. Let $\widetilde{K}_{\mathbf{n},k}$, $1 \leq k \leq m$ be the constants defined in (59), with p_k replaced by p_k/q_k. Then, for each $k \in \{1,\ldots,m\}$,*

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{Q}_{\mathbf{n},k}(z)}{Q_{\mathbf{n},k}(z)} = \frac{\mathcal{F}_k(z;\, p_1,\ldots,p_m)}{\mathcal{F}_k(z;\, q_1,\ldots,q_m)}, \qquad K \subset \overline{\mathbb{C}}\setminus \operatorname{supp}(\sigma_k), \tag{78}$$

and

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{K}^2_{\mathbf{n},k}}{K^2_{\mathbf{n},k}} = \prod_{i=1}^{k} \operatorname{sign}(p_i/q_i, \operatorname{supp}(\sigma_i))\, \frac{G_k(\infty;\, q_1,\ldots,q_m)}{G_k(\infty;\, p_1,\ldots,p_m)}. \tag{79}$$

Proof. By $Q^*_{\mathbf{n},k}$ denote polynomials associated with the auxiliary Nikishin system $\mathcal{N}(\sigma_1/q_1,\ldots,\sigma_m/q_m)$, corresponding to the indices $\mathbf{n}$, k. On account of Theorem 6.1, we have that

$$\lim_{\mathbf{n}\in\Lambda} \frac{\widetilde{Q}_{\mathbf{n},k}(z)}{Q^*_{\mathbf{n},k}(z)} = \mathcal{F}_k(z;\, p_1,\ldots,p_m), \qquad K \subset \overline{\mathbb{C}}\setminus \operatorname{supp}(\sigma_k).$$

and

$$\lim_{\mathbf{n}\in\Lambda} \frac{Q_{\mathbf{n},k}(z)}{Q^*_{\mathbf{n},k}(z)} = \mathcal{F}_k(z;\, q_1,\ldots,q_m), \qquad K \subset \overline{\mathbb{C}}\setminus \operatorname{supp}(\sigma_k).$$

Therefore, (78) is obtained. Using the same idea, (79) follows from (63). □

Remark 6.1. Theorem 5.1 and Corollary 5.1 allow us to define polynomials $\widetilde{Q}_{\mathbf{n},k}$, $k = 1,\ldots,m$, in the case when p_k, q_k have complex coefficients as those monic polynomials which carry the zeros of $\widetilde{\Psi}_{\mathbf{n},k-1}$ lying in $\mathbb{C}\setminus\Delta_{k-1}$. For such polynomials $\widetilde{Q}_{\mathbf{n},k}$, results analogous to those expressed in Theorem 6.1 and Corollary 6.1 can be proved.

Acknowledgments

Both authors acknowledge support from grants MTM 2006-13000-C03-02 of Min. de Ciencia y Tecnología and CCG 06–UC3M/ESP–0690 of Universidad Carlos III de Madrid-Comunidad de Madrid.

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