# Asymptotics of greedy energy sequences on the unit circle and the sphere

A. López-García

University of Central Florida

Joint work with R.E. McCleary

CAOPA Zoom Seminar, September 7, 2020

# Energy of a point configuration

Let $\omega = (x_1, \ldots, x_N)$ be a tuple of $N \geq 2$ points in $\mathbb{R}^p$. With $|x_i - x_j|$ denoting the Euclidean distance between $x_i$ and $x_j$:

The **logarithmic energy** of $\omega$ is

$$E_0(\omega) := \sum_{i \neq j} \log \frac{1}{|x_i - x_j|} = 2 \sum_{i<j} \log \frac{1}{|x_i - x_j|}.$$

# Energy of a point configuration

Let $\omega = (x_1, \ldots, x_N)$ be a tuple of $N \geq 2$ points in $\mathbb{R}^p$. With $|x_i - x_j|$ denoting the Euclidean distance between $x_i$ and $x_j$:

The **logarithmic energy** of $\omega$ is

$$E_0(\omega) := \sum_{i \neq j} \log \frac{1}{|x_i - x_j|} = 2 \sum_{i < j} \log \frac{1}{|x_i - x_j|}.$$

For a parameter $s > 0$, the **Riesz $s$-energy** of $\omega$ is

$$E_s(\omega) := \sum_{i \neq j} \frac{1}{|x_i - x_j|^s} = 2 \sum_{i < j} \frac{1}{|x_i - x_j|^s}.$$

# Energy of a point configuration

Let $\omega = (x_1, \dots, x_N)$ be a tuple of $N \geq 2$ points in $\mathbb{R}^p$. With $|x_i - x_j|$ denoting the Euclidean distance between $x_i$ and $x_j$:

The **logarithmic energy** of $\omega$ is

$$E_0(\omega) := \sum_{i \neq j} \log \frac{1}{|x_i - x_j|} = 2 \sum_{i<j} \log \frac{1}{|x_i - x_j|}.$$

For a parameter $s > 0$, the **Riesz $s$-energy** of $\omega$ is

$$E_s(\omega) := \sum_{i \neq j} \frac{1}{|x_i - x_j|^s} = 2 \sum_{i<j} \frac{1}{|x_i - x_j|^s}.$$

For a parameter $\lambda > 0$, the **$\lambda$-energy** of $\omega$ is

$$H_\lambda(\omega) := \sum_{i \neq j} |x_i - x_j|^\lambda = 2 \sum_{i<j} |x_i - x_j|^\lambda.$$

# Optimal energy configurations

Let $K \subset \mathbb{R}^p$ be a compact set, not finite. Throughout the talk, $K$ satisfies these conditions.

The energy functionals $E_0$ and $E_s$, $s > 0$, are lower semicontinuous functions of $(x_1, \ldots, x_N)$, so they attain their minimum value on $K^N = K \times \cdots \times K$. That is, for each $N \geq 2$, there exists $\omega_{N,s} \in K^N$, in general not unique, such that

$$E_s(\omega_{N,s}) = \min_{\omega \in K^N} E_s(\omega).$$

# Optimal energy configurations

Let $K \subset \mathbb{R}^p$ be a compact set, not finite. Throughout the talk, $K$ satisfies these conditions.

The energy functionals $E_0$ and $E_s$, $s > 0$, are lower semicontinuous functions of $(x_1, \ldots, x_N)$, so they attain their minimum value on $K^N = K \times \cdots \times K$. That is, for each $N \geq 2$, there exists $\omega_{N,s} \in K^N$, in general not unique, such that

$$E_s(\omega_{N,s}) = \min_{\omega \in K^N} E_s(\omega).$$

In the logarithmic case ($s = 0$), and for $K \subset \mathbb{C}$, the configurations $\omega_{N,0}$ are called **Fekete sets** on $K$. In general, the configurations $\omega_{N,s}$ are called ***N*-point minimal *s*-energy configurations** on $K$.

# Optimal energy configurations

Let $K \subset \mathbb{R}^p$ be a compact set, not finite. Throughout the talk, $K$ satisfies these conditions.

The energy functionals $E_0$ and $E_s$, $s > 0$, are lower semicontinuous functions of $(x_1, \ldots, x_N)$, so they attain their minimum value on $K^N = K \times \cdots \times K$. That is, for each $N \geq 2$, there exists $\omega_{N,s} \in K^N$, in general not unique, such that

$$E_s(\omega_{N,s}) = \min_{\omega \in K^N} E_s(\omega).$$

In the logarithmic case ($s = 0$), and for $K \subset \mathbb{C}$, the configurations $\omega_{N,0}$ are called **Fekete sets** on $K$. In general, the configurations $\omega_{N,s}$ are called ***N*-point minimal *s*-energy configurations** on $K$.

Clearly, for each $\lambda > 0$ and $N \geq 2$, there exists $\omega_{N,\lambda} \in K^N$, in general not unique, such that

$$H_\lambda(\omega_{N,\lambda}) = \max_{\omega \in K^N} H_\lambda(\omega).$$

$\omega_{N,\lambda}$ is an ***N*-point maximal $\lambda$-energy configuration** on $K$.

# Logarithmic energy in the plane

Let $\mathcal{P}(K)$ be the space of all Borel probability measures on $K$. Assume $K \subset \mathbb{C}$. For $\mu \in \mathcal{P}(K)$,

$$I_0(\mu) := \iint \log \frac{1}{|x-y|}\, d\mu(x)\, d\mu(y),$$

$$U^\mu(z) := \int \log \frac{1}{|z-t|}\, d\mu(t).$$

Let

$$W_0(K) := \inf_{\mu \in \mathcal{P}(K)} I_0(\mu) \qquad \textbf{(Robin constant of } K\textbf{)}$$

$$C_0(K) := e^{-W_0(K)} \qquad \textbf{(Logarithmic capacity of } K\textbf{)}$$

# Logarithmic energy in the plane

Let $\mathcal{P}(K)$ be the space of all Borel probability measures on $K$. Assume $K \subset \mathbb{C}$. For $\mu \in \mathcal{P}(K)$,

$$I_0(\mu) := \iint \log \frac{1}{|x-y|} \, d\mu(x) \, d\mu(y),$$
$$U^\mu(z) := \int \log \frac{1}{|z-t|} \, d\mu(t).$$

Let

$$W_0(K) := \inf_{\mu \in \mathcal{P}(K)} I_0(\mu) \qquad \textbf{(Robin constant of } K\textbf{)}$$
$$C_0(K) := e^{-W_0(K)} \qquad \textbf{(Logarithmic capacity of } K\textbf{)}$$

**Theorem (Fekete-Szegő)**

*If $(\omega_{N,0})_{N\geq 2}$ is a sequence of Fekete sets on $K$, then $\left(\frac{E_0(\omega_{N,0})}{N(N-1)}\right)_{N\geq 2}$ is monotonically increasing and its limit is $W_0(K)$.*

If $C_0(K) > 0$, then $\frac{1}{N} \sum_{x \in \omega_{N,0}} \delta_x \xrightarrow{*} \mu_K$, where $\mu_K$ is the equilibrium measure for $K$.

# Edrei-Leja sequences

In a 1939 work, A. Edrei introduced the following inductive construction of a sequence $(a_n)_{n=0}^{\infty}$ on a compact set $K \subset \mathbb{C}$:

1) Pick $a_0 \in K$ arbitrarily. Let $a_0$ be the first selected point of the sequence.
2) For each $n \geq 1$, assuming that $a_0, \ldots, a_{n-1}$ have been selected, pick the next point of the sequence $a_n \in K$ so that

$$\prod_{i=0}^{n-1} |a_n - a_i| = \max_{z \in K} \prod_{i=0}^{n-1} |z - a_i|. \tag{1}$$

The sequence $(a_n)_{n=0}^{\infty}$ is an **Edrei-Leja sequence** on $K$.

# Edrei-Leja sequences

In a 1939 work, A. Edrei introduced the following inductive construction of a sequence $(a_n)_{n=0}^{\infty}$ on a compact set $K \subset \mathbb{C}$:

1) Pick $a_0 \in K$ arbitrarily. Let $a_0$ be the first selected point of the sequence.
2) For each $n \geq 1$, assuming that $a_0, \ldots, a_{n-1}$ have been selected, pick the next point of the sequence $a_n \in K$ so that

$$\prod_{i=0}^{n-1} |a_n - a_i| = \max_{z \in K} \prod_{i=0}^{n-1} |z - a_i|. \tag{1}$$

The sequence $(a_n)_{n=0}^{\infty}$ is an **Edrei-Leja sequence** on $K$.

(1) is equivalent to

$$\sum_{i=0}^{n-1} \log \frac{1}{|a_n - a_i|} = \inf_{z \in K} \sum_{i=0}^{n-1} \log \frac{1}{|z - a_i|},$$

or

$$E_0((a_0, \ldots, a_{n-1}, a_n)) = \inf_{z \in K} E_0((a_0, \ldots, a_{n-1}, z)).$$

In his work, Edrei showed that for the configurations $\alpha_N := (a_0, \ldots, a_{N-1})$, we have

$$\lim_{N\to\infty} \frac{E_0(\alpha_N)}{N^2} = W_0(K). \tag{2}$$

In his work, Edrei showed that for the configurations $\alpha_N := (a_0, \ldots, a_{N-1})$, we have

$$\lim_{N\to\infty} \frac{E_0(\alpha_N)}{N^2} = W_0(K). \tag{2}$$

The proof is short: For every $N \geq 2$, $E_0(\alpha_N) \geq E_0(\omega_{N,0})$, so by Fekete-Szegő,

$$\liminf_{N\to\infty} \frac{E_0(\alpha_N)}{N^2} \geq \lim_{N\to\infty} \frac{E_0(\omega_{N,0})}{N^2} = W_0(K).$$

In his work, Edrei showed that for the configurations $\alpha_N := (a_0, \ldots, a_{N-1})$, we have

$$\lim_{N\to\infty} \frac{E_0(\alpha_N)}{N^2} = W_0(K). \tag{2}$$

The proof is short: For every $N \geq 2$, $E_0(\alpha_N) \geq E_0(\omega_{N,0})$, so by Fekete-Szegő,

$$\liminf_{N\to\infty} \frac{E_0(\alpha_N)}{N^2} \geq \lim_{N\to\infty} \frac{E_0(\omega_{N,0})}{N^2} = W_0(K).$$

If $W_0(K) = +\infty$, we are done. Suppose $W_0(K) < +\infty$ (or $C_0(K) > 0$).

$$E_0(\alpha_N) = 2\sum_{i=1}^{N-1}\sum_{j<i} \log\frac{1}{|a_i - a_j|} \leq 2\sum_{i=1}^{N-1}\sum_{j<i} \log\frac{1}{|z - a_j|}, \qquad \forall z \in K.$$

In his work, Edrei showed that for the configurations $\alpha_N := (a_0, \ldots, a_{N-1})$, we have

$$\lim_{N\to\infty} \frac{E_0(\alpha_N)}{N^2} = W_0(K). \tag{2}$$

The proof is short: For every $N \geq 2$, $E_0(\alpha_N) \geq E_0(\omega_{N,0})$, so by Fekete-Szegő,

$$\liminf_{N\to\infty} \frac{E_0(\alpha_N)}{N^2} \geq \lim_{N\to\infty} \frac{E_0(\omega_{N,0})}{N^2} = W_0(K).$$

If $W_0(K) = +\infty$, we are done. Suppose $W_0(K) < +\infty$ (or $C_0(K) > 0$).

$$E_0(\alpha_N) = 2\sum_{i=1}^{N-1}\sum_{j<i} \log\frac{1}{|a_i - a_j|} \leq 2\sum_{i=1}^{N-1}\sum_{j<i} \log\frac{1}{|z - a_j|}, \qquad \forall z \in K.$$

Integrating this inequality with respect to $d\mu_K(z)$,

$$E_0(\alpha_N) \leq 2\sum_{i=1}^{N-1}\sum_{j<i} U^{\mu_K}(a_j) \leq 2\sum_{i=1}^{N-1}\sum_{j<i} W_0(K) = N(N-1)W_0(K).$$

and the result follows. Q.E.D.

In his work, Edrei showed that for the configurations $\alpha_N := (a_0, \ldots, a_{N-1})$, we have

$$\lim_{N\to\infty} \frac{E_0(\alpha_N)}{N^2} = W_0(K). \tag{2}$$

The proof is short: For every $N \geq 2$, $E_0(\alpha_N) \geq E_0(\omega_{N,0})$, so by Fekete-Szegő,

$$\liminf_{N\to\infty} \frac{E_0(\alpha_N)}{N^2} \geq \lim_{N\to\infty} \frac{E_0(\omega_{N,0})}{N^2} = W_0(K).$$

If $W_0(K) = +\infty$, we are done. Suppose $W_0(K) < +\infty$ (or $C_0(K) > 0$).

$$E_0(\alpha_N) = 2\sum_{i=1}^{N-1}\sum_{j<i} \log\frac{1}{|a_i - a_j|} \leq 2\sum_{i=1}^{N-1}\sum_{j<i} \log\frac{1}{|z - a_j|}, \qquad \forall z \in K.$$

Integrating this inequality with respect to $d\mu_K(z)$,

$$E_0(\alpha_N) \leq 2\sum_{i=1}^{N-1}\sum_{j<i} U^{\mu_K}(a_j) \leq 2\sum_{i=1}^{N-1}\sum_{j<i} W_0(K) = N(N-1)W_0(K).$$

and the result follows. Q.E.D.

If $C_0(K) > 0$, then (2) implies $\frac{1}{N}\sum_{i=0}^{N-1} \delta_{a_i} \xrightarrow{*} \mu_K$.

# On the unit circle $S^1$

On the unit circle $S^1$, $N$-point Fekete sets are the configurations formed by $N$ equally spaced points, such as the set of all $N$th roots of unity. Also, $E_0(\omega_{N,0}) = -N \log N$, for all $N \geq 2$.

# On the unit circle $S^1$

On the unit circle $S^1$, $N$-point Fekete sets are the configurations formed by $N$ equally spaced points, such as the set of all $N$th roots of unity. Also, $E_0(\omega_{N,0}) = -N \log N$, for all $N \geq 2$.

For an Edrei-Leja sequence $(a_n)_{n=0}^{\infty}$ on $S^1$, what is the behavior of $E_0(\alpha_N)$, $\alpha_N = (a_0, \ldots, a_{N-1})$, as $N \to \infty$ ?

# On the unit circle $S^1$

On the unit circle $S^1$, $N$-point Fekete sets are the configurations formed by $N$ equally spaced points, such as the set of all $N$th roots of unity. Also, $E_0(\omega_{N,0}) = -N \log N$, for all $N \geq 2$.

For an Edrei-Leja sequence $(a_n)_{n=0}^{\infty}$ on $S^1$, what is the behavior of $E_0(\alpha_N)$, $\alpha_N = (a_0, \ldots, a_{N-1})$, as $N \to \infty$ ?

Edrei's result shows that

$$\lim_{N\to\infty} \frac{E_0(\alpha_N)}{N^2} = 0 = W_0(S^1).$$

Is $E_0(\alpha_N) \sim -N \log N$ ?

# On the unit circle $S^1$

In a work on interpolatory properties of Edrei-Leja sequences on the unit circle, J-P. Calvi and P. Van Manh proved the following identity:

**Theorem (Calvi, Van Manh, 2011)**

*Let $(a_n)_{n=0}^{\infty}$ be an Edrei-Leja sequence on $S^1$. For all $n \geq 1$,*

$$\prod_{i=0}^{n-1} |a_n - a_i| = 2^{\tau(n)}$$

*where $\tau(n)$ is the number of 1's in the binary representation of $n$.*

# On the unit circle $S^1$

In a work on interpolatory properties of Edrei-Leja sequences on the unit circle, J-P. Calvi and P. Van Manh proved the following identity:

**Theorem (Calvi, Van Manh, 2011)**

*Let $(a_n)_{n=0}^{\infty}$ be an Edrei-Leja sequence on $S^1$. For all $n \geq 1$,*

$$\prod_{i=0}^{n-1} |a_n - a_i| = 2^{\tau(n)}$$

*where $\tau(n)$ is the number of 1's in the binary representation of n.*

Bialas-Ciez and Calvi (2012) also showed how to describe geometrically the configuration $\alpha_N = (a_0, \ldots, a_{N-1})$ in terms of the binary representation of $N$.

# On the unit circle $S^1$

In a work on interpolatory properties of Edrei-Leja sequences on the unit circle, J-P. Calvi and P. Van Manh proved the following identity:

**Theorem (Calvi, Van Manh, 2011)**

*Let $(a_n)_{n=0}^{\infty}$ be an Edrei-Leja sequence on $S^1$. For all $n \geq 1$,*

$$\prod_{i=0}^{n-1} |a_n - a_i| = 2^{\tau(n)}$$

*where $\tau(n)$ is the number of 1's in the binary representation of $n$.*

Bialas-Ciez and Calvi (2012) also showed how to describe geometrically the configuration $\alpha_N = (a_0, \ldots, a_{N-1})$ in terms of the binary representation of $N$.

In particular, for every $k \geq 1$, the first $2^k$ points of an Edrei-Leja sequence are equally spaced on $S^1$.

# On the unit circle $S^1$

Using the identity of Calvi and Van Manh, it was shown by López and Wagner (2015) that

$$\lim_{N\to\infty} \frac{E_0(\alpha_N)}{N \log N} = -1.$$

# On the unit circle $S^1$

Using the identity of Calvi and Van Manh, it was shown by López and Wagner (2015) that

$$\lim_{N\to\infty} \frac{E_0(\alpha_N)}{N\log N} = -1.$$

In view of this, it's natural to study the sequence $(E_0(\alpha_N) + N\log N)_N$ (second-order asymptotics).

**Theorem (López, Wagner, 2015)**

*For every $N \geq 2$,*

$$0 \leq \frac{E_0(\alpha_N) + N\log N}{N} < \log(4/3).$$

# On the unit circle $S^1$

Using the identity of Calvi and Van Manh, it was shown by López and Wagner (2015) that

$$\lim_{N\to\infty} \frac{E_0(\alpha_N)}{N \log N} = -1.$$

In view of this, it's natural to study the sequence $(E_0(\alpha_N) + N \log N)_N$ (second-order asymptotics).

## Theorem (López, Wagner, 2015)

*For every $N \geq 2$,*

$$0 \leq \frac{E_0(\alpha_N) + N \log N}{N} < \log(4/3).$$

*The lower bound is attained iff $N = 2^k$, $k \geq 1$, and*

$$\limsup_{N\to\infty} \frac{E_0(\alpha_N) + N \log N}{N} = \log(4/3).$$

# On the unit circle $S^1$

Using the identity of Calvi and Van Manh, it was shown by López and Wagner (2015) that

$$\lim_{N\to\infty} \frac{E_0(\alpha_N)}{N\log N} = -1.$$

In view of this, it's natural to study the sequence $(E_0(\alpha_N) + N\log N)_N$ (second-order asymptotics).

**Theorem (López, Wagner, 2015)**

*For every $N \geq 2$,*

$$0 \leq \frac{E_0(\alpha_N) + N\log N}{N} < \log(4/3).$$

*The lower bound is attained iff $N = 2^k$, $k \geq 1$, and*

$$\limsup_{N\to\infty} \frac{E_0(\alpha_N) + N\log N}{N} = \log(4/3).$$

So the sequence $\frac{E_0(\alpha_N)}{-N\log N}$ converges, but the sequence $\frac{E_0(\alpha_N)+N\log N}{N}$ diverges!

# The sequence $\frac{E_0(\alpha_N)+N\log N}{N}$

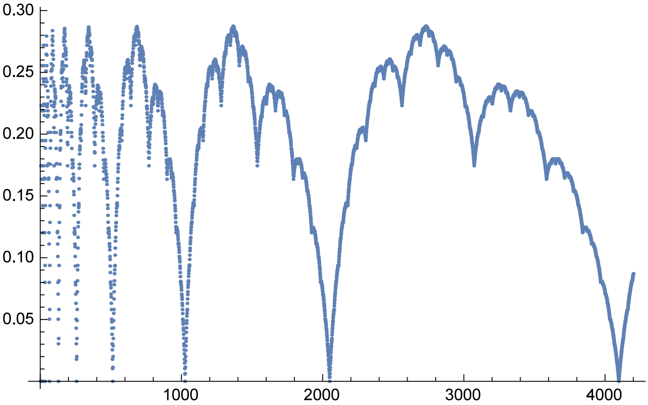


Figure: The first 4200 points of the sequence $\frac{E_0(\alpha_N)+N\log N}{N}$. The limsup is $\log(4/3) \approx 0.2876$.

We have a **doubling periodicity** property: For all $N \geq 2$,

$$\frac{E_0(\alpha_N) + N\log N}{N} = \frac{E_0(\alpha_{2N}) + 2N\log(2N)}{2N}.$$

What can be said about the behavior of $\prod_{i=0}^{n-1} |a_n - a_i| = 2^{\tau(n)}$? How should this sequence be normalized?

What can be said about the behavior of $\prod_{i=0}^{n-1} |a_n - a_i| = 2^{\tau(n)}$? How should this sequence be normalized?

General result: Suppose $K \subset \mathbb{C}$ is compact, $C_0(K) > 0$, $(a_n)_{n=0}^{\infty}$ is an Edrei-Leja sequence on $K$, and let

$$P_N(z) := \prod_{n=0}^{N-1} (z - a_n),$$
$$\|P_N\|_K := \sup_{z \in K} |P_N(z)|.$$

Leja and Górski proved

$$\lim_{N\to\infty} \|P_N\|_K^{1/N} = C_0(K). \tag{3}$$

What can be said about the behavior of $\prod_{i=0}^{n-1} |a_n - a_i| = 2^{\tau(n)}$? How should this sequence be normalized?

General result: Suppose $K \subset \mathbb{C}$ is compact, $C_0(K) > 0$, $(a_n)_{n=0}^{\infty}$ is an Edrei-Leja sequence on $K$, and let

$$P_N(z) := \prod_{n=0}^{N-1} (z - a_n),$$

$$\|P_N\|_K := \sup_{z \in K} |P_N(z)|.$$

Leja and Górski proved

$$\lim_{N\to\infty} \|P_N\|_K^{1/N} = C_0(K). \quad (3)$$

In the case of the unit circle $K = S^1$, $C_0(S^1) = 1$, so (3) implies

$$\lim_{N\to\infty} \frac{\log \|P_N\|_{S^1}}{N} = \lim_{N\to\infty} \frac{\log(2)\tau(N)}{N} = 0.$$

So $\log \|P_N\|_{S^1} = o(N)$.

## Theorem (López, McCleary)

*On the unit circle, for all* $N \geq 1$,

$$0 < \frac{\log \|P_N\|_{S^1}}{\log(N+1)} \leq 1.$$

*The upper bound is attained iff* $N = 2^k - 1$, $k \geq 1$. *Also,*

$$\liminf_{N\to\infty} \frac{\log \|P_N\|_{S^1}}{\log(N+1)} = 0.$$

## Theorem (López, McCleary)

*On the unit circle, for all $N \geq 1$,*

$$0 < \frac{\log \|P_N\|_{S^1}}{\log(N+1)} \leq 1.$$

*The upper bound is attained iff $N = 2^k - 1$, $k \geq 1$. Also,*

$$\liminf_{N\to\infty} \frac{\log \|P_N\|_{S^1}}{\log(N+1)} = 0.$$

## Proof.

By Calvi-Van Manh, $\|P_N\|_{S^1} = 2^{\tau(N)}$. The $\tau$ function has the property

$$N \geq 2^{\tau(N)} - 1, \qquad N \geq 1,$$

with equality iff $N = 2^k - 1$ for some $k \geq 1$. We have

$$\frac{\log \|P_N\|_{S^1}}{\log(N+1)} = \frac{\log(2^{\tau(N)})}{\log(N+1)} \leq \frac{\log(N+1)}{\log(N+1)} = 1.$$

Also, $\log \|P_N\|_{S^1} > 0$ since $\|P_N\|_{S^1} \geq 2$.

## Theorem (López, McCleary)

*On the unit circle, for all $N \geq 1$,*

$$0 < \frac{\log \|P_N\|_{S^1}}{\log(N+1)} \leq 1.$$

*The upper bound is attained iff $N = 2^k - 1$, $k \geq 1$. Also,*

$$\liminf_{N \to \infty} \frac{\log \|P_N\|_{S^1}}{\log(N+1)} = 0.$$

## Proof.

By Calvi-Van Manh, $\|P_N\|_{S^1} = 2^{\tau(N)}$. The $\tau$ function has the property

$$N \geq 2^{\tau(N)} - 1, \qquad N \geq 1,$$

with equality iff $N = 2^k - 1$ for some $k \geq 1$. We have

$$\frac{\log \|P_N\|_{S^1}}{\log(N+1)} = \frac{\log(2^{\tau(N)})}{\log(N+1)} \leq \frac{\log(N+1)}{\log(N+1)} = 1.$$

Also, $\log \|P_N\|_{S^1} > 0$ since $\|P_N\|_{S^1} \geq 2$. Taking $N = 2^k$, we have $\|P_{2^k}\|_{S^1} = 2$, so $\log \|P_{2^k}\|_{S^1} / \log(2^k + 1) \longrightarrow 0$ as $k \to \infty$. □

# The sequence $\frac{\log \|P_N\|_{S^1}}{\log(N+1)}$

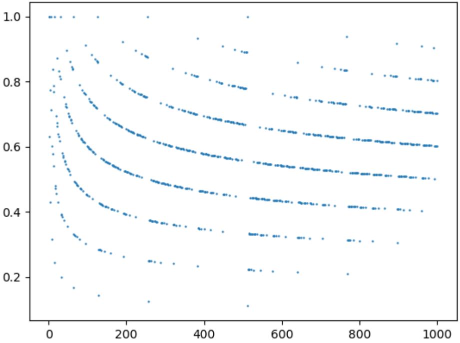


Figure: The first 1000 points of the sequence $\frac{\log \|P_N\|_{S^1}}{\log(N+1)}$.

We have $\tau(N) = \tau(2N)$ for all $N$, so for each $m \geq 1$, the subsequence

$$\left( \frac{\log \|P_{2^k(2^m-1)}\|_{S^1}}{\log(2^k(2^m-1)+1)} \right)_{k=0}^{\infty}$$

decreases from 1 to 0 (the numerator is the constant).

# $\lambda$-energy on the unit sphere $S^d$

Let $\lambda > 0$.

For configurations $\omega = (x_1, \ldots, x_N)$ on the unit sphere $S^d \subset \mathbb{R}^{d+1}$ we consider the $\lambda$-energy

$$H_\lambda(\omega) := \sum_{i \neq j} |x_i - x_j|^\lambda = 2 \sum_{i<j} |x_i - x_j|^\lambda.$$

# $\lambda$-energy on the unit sphere $S^d$

Let $\lambda > 0$.

For configurations $\omega = (x_1, \ldots, x_N)$ on the unit sphere $S^d \subset \mathbb{R}^{d+1}$ we consider the $\lambda$-energy

$$H_\lambda(\omega) := \sum_{i \neq j} |x_i - x_j|^\lambda = 2 \sum_{i<j} |x_i - x_j|^\lambda.$$

For $\mu \in \mathcal{P}(S^d)$, let

$$I_\lambda(\mu) := \iint |x - y|^\lambda d\mu(x)\, d\mu(y).$$

We say that $\sigma$ is a **maximal distribution** if

$$I_\lambda(\sigma) = \sup_{\mu \in \mathcal{P}(S^d)} I_\lambda(\mu).$$

Let $\sigma_d$ denote the normalized uniform (Lebesgue) measure on $S^d$.

For the continuous energy problem we have:

### Theorem (G. Björck, 1956)

*For $0 < \lambda < 2$, the measure $\sigma_d$ is the unique maximal distribution in $\mathcal{P}(S^d)$. For $\lambda = 2$, a distribution $\sigma \in \mathcal{P}(S^d)$ is maximal if and only if its center of mass is at the origin. For $\lambda > 2$, a distribution $\sigma \in \mathcal{P}(S^d)$ is maximal if and only if it is of the form $\sigma = \frac{1}{2}(\delta_a + \delta_{-a})$ for some $a \in S^d$.*

For the continuous energy problem we have:

### Theorem (G. Björck, 1956)

*For $0 < \lambda < 2$, the measure $\sigma_d$ is the unique maximal distribution in $\mathcal{P}(S^d)$. For $\lambda = 2$, a distribution $\sigma \in \mathcal{P}(S^d)$ is maximal if and only if its center of mass is at the origin. For $\lambda > 2$, a distribution $\sigma \in \mathcal{P}(S^d)$ is maximal if and only if it is of the form $\sigma = \frac{1}{2}(\delta_a + \delta_{-a})$ for some $a \in S^d$.*

So the most interesting range for the $\lambda$-energy problem on $S^d$ is $0 < \lambda < 2$, independently of $d$.

For the continuous energy problem we have:

**Theorem (G. Björck, 1956)**

*For $0 < \lambda < 2$, the measure $\sigma_d$ is the unique maximal distribution in $\mathcal{P}(S^d)$. For $\lambda = 2$, a distribution $\sigma \in \mathcal{P}(S^d)$ is maximal if and only if its center of mass is at the origin. For $\lambda > 2$, a distribution $\sigma \in \mathcal{P}(S^d)$ is maximal if and only if it is of the form $\sigma = \frac{1}{2}(\delta_a + \delta_{-a})$ for some $a \in S^d$.*

So the most interesting range for the $\lambda$-energy problem on $S^d$ is $0 < \lambda < 2$, independently of $d$.

On $S^1$, for $0 < \lambda < 2$, the $N$-point configurations $\omega_{N,\lambda}$ that satisfy

$$H_\lambda(\omega_{N,\lambda}) = \max_{\omega \in (S^1)^N} H_\lambda(\omega)$$

are the configurations formed by $N$ equally spaced points.

# Greedy $\lambda$-energy sequences

Given $\lambda > 0$, a sequence $(a_n)_{n=0}^{\infty} \subset S^d$ is a **greedy $\lambda$-energy sequence** on $S^d$ if for every $n \geq 1$,

$$\sum_{k=0}^{n-1} |a_n - a_k|^{\lambda} = \max_{x \in S^d} \sum_{k=0}^{n-1} |x - a_k|^{\lambda}.$$

# Greedy $\lambda$-energy sequences

Given $\lambda > 0$, a sequence $(a_n)_{n=0}^{\infty} \subset S^d$ is a **greedy $\lambda$-energy sequence** on $S^d$ if for every $n \geq 1$,

$$\sum_{k=0}^{n-1} |a_n - a_k|^{\lambda} = \max_{x \in S^d} \sum_{k=0}^{n-1} |x - a_k|^{\lambda}.$$

Notation:

$$\alpha_{N,\lambda} := (a_0, \ldots, a_{N-1})$$

$$\sigma_{N,\lambda} := \frac{1}{N} \sum_{k=0}^{N-1} \delta_{a_k}$$

$$U_{N,\lambda}(x) := \sum_{k=0}^{N-1} |x - a_k|^{\lambda}$$

# Distribution

The following properties are valid in any dimension $d \geq 1$.

1) **Symmetry property:** Let $\lambda > 0$ be arbitrary. For every $k \geq 0$,

$$a_{2k+1} = -a_{2k}.$$

More precisely, after $a_0, \ldots, a_{2k}$ have been selected, there is a unique possible choice of $a_{2k+1}$, which is $-a_{2k}$.

# Distribution

The following properties are valid in any dimension $d \geq 1$.

1) **Symmetry property:** Let $\lambda > 0$ be arbitrary. For every $k \geq 0$,

$$a_{2k+1} = -a_{2k}.$$

More precisely, after $a_0, \ldots, a_{2k}$ have been selected, there is a unique possible choice of $a_{2k+1}$, which is $-a_{2k}$.

2) **Uniform distribution:** For $0 < \lambda < 2$, we have $\sigma_{N,\lambda} \xrightarrow{*} \sigma_d$, as $N \to \infty$.

# Distribution

The following properties are valid in any dimension $d \geq 1$.

1) **Symmetry property:** Let $\lambda > 0$ be arbitrary. For every $k \geq 0$,

$$a_{2k+1} = -a_{2k}.$$

More precisely, after $a_0, \ldots, a_{2k}$ have been selected, there is a unique possible choice of $a_{2k+1}$, which is $-a_{2k}$.

2) **Uniform distribution:** For $0 < \lambda < 2$, we have $\sigma_{N,\lambda} \xrightarrow{*} \sigma_d$, as $N \to \infty$.

3) If $\lambda > 2$, the greedy $\lambda$-energy sequence $(a_n)_{n=0}^{\infty}$ concentrates on the opposite points $a_0, -a_0$:

$$\{a_{2k}, a_{2k+1}\} = \{a_0, -a_0\}, \qquad \text{for all } k \geq 0.$$

# Distribution

The following properties are valid in any dimension $d \geq 1$.

1) **Symmetry property:** Let $\lambda > 0$ be arbitrary. For every $k \geq 0$,

$$a_{2k+1} = -a_{2k}.$$

More precisely, after $a_0, \ldots, a_{2k}$ have been selected, there is a unique possible choice of $a_{2k+1}$, which is $-a_{2k}$.

2) **Uniform distribution:** For $0 < \lambda < 2$, we have $\sigma_{N,\lambda} \xrightarrow{*} \sigma_d$, as $N \to \infty$.

3) If $\lambda > 2$, the greedy $\lambda$-energy sequence $(a_n)_{n=0}^{\infty}$ concentrates on the opposite points $a_0, -a_0$:

$$\{a_{2k}, a_{2k+1}\} = \{a_0, -a_0\}, \qquad \text{for all } k \geq 0.$$

4) If $\lambda = 2$, the sequence $(\sigma_{N,2})$ may be divergent, but any convergent subsequence converges to a measure $\sigma$ with center of mass at the origin.

# First order asymptotics

From now on we assume $0 < \lambda < 2$.

# First order asymptotics

From now on we assume $0 < \lambda < 2$.

Using classical arguments, one can prove: For any greedy $\lambda$-energy sequence $(a_n)_{n=0}^{\infty} \subset S^d$,

$$\lim_{N\to\infty} \frac{H_\lambda(\alpha_{N,\lambda})}{N^2} = I_\lambda(\sigma_d), \qquad \alpha_{N,\lambda} = (a_0, \ldots, a_{N-1}),$$

$$\lim_{n\to\infty} \frac{U_{N,\lambda}(a_N)}{N} = I_\lambda(\sigma_d), \qquad U_{N,\lambda}(x) = \sum_{k=0}^{N-1} |x - a_k|^\lambda.$$

# First order asymptotics

From now on we assume $0 < \lambda < 2$.

Using classical arguments, one can prove: For any greedy $\lambda$-energy sequence $(a_n)_{n=0}^{\infty} \subset S^d$,

$$\lim_{N\to\infty} \frac{H_\lambda(\alpha_{N,\lambda})}{N^2} = I_\lambda(\sigma_d), \qquad \alpha_{N,\lambda} = (a_0, \ldots, a_{N-1}),$$

$$\lim_{n\to\infty} \frac{U_{N,\lambda}(a_N)}{N} = I_\lambda(\sigma_d), \qquad U_{N,\lambda}(x) = \sum_{k=0}^{N-1} |x - a_k|^\lambda.$$

These limits suggest the analysis of the sequences $(H_\lambda(\alpha_{N,\lambda}) - I_\lambda(\sigma_d)N^2)$ and $(U_{N,\lambda}(a_N) - I_\lambda(\sigma_d)N)$. We have analyzed these sequences in the case $d = 1$ (the unit circle).

# Binary representation of energy and potential in the case $d = 1$

Let's define

$$\mathcal{L}_\lambda(N) := \sum_{0 \le k \ne \ell \le N-1} |e^{\frac{2\pi i k}{N}} - e^{\frac{2\pi i \ell}{N}}|^\lambda \qquad (\lambda\text{-energy of the } N\text{-th roots of unity})$$

$$\mathcal{U}_\lambda(N) := \sum_{k=0}^{N-1} |e^{\frac{2\pi i k}{N}} - e^{\frac{\pi i}{N}}|^\lambda \qquad (\text{potential of the } N\text{-th roots of unity at } e^{\frac{\pi i}{N}})$$

# Binary representation of energy and potential in the case $d = 1$

Let's define

$$\mathcal{L}_\lambda(N) := \sum_{0 \leq k \neq \ell \leq N-1} |e^{\frac{2\pi i k}{N}} - e^{\frac{2\pi i \ell}{N}}|^\lambda \qquad (\lambda\text{-energy of the } N\text{-th roots of unity})$$

$$\mathcal{U}_\lambda(N) := \sum_{k=0}^{N-1} |e^{\frac{2\pi i k}{N}} - e^{\frac{\pi i}{N}}|^\lambda \qquad (\text{potential of the } N\text{-th roots of unity at } e^{\frac{\pi i}{N}})$$

## Lemma (López, McCleary)

*Suppose that $N \geq 2$ has the binary representation*

$$N = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_p}, \qquad n_1 > n_2 > \cdots > n_p \geq 0.$$

*Then,*

$$U_{N,\lambda}(a_N) = \sum_{k=1}^{p} \mathcal{U}_\lambda(2^{n_k}),$$

# Binary representation of energy and potential in the case $d = 1$

Let's define

$$\mathcal{L}_\lambda(N) := \sum_{0\le k\ne \ell \le N-1} |e^{\frac{2\pi i k}{N}} - e^{\frac{2\pi i \ell}{N}}|^\lambda \qquad (\lambda\text{-energy of the } N\text{-th roots of unity})$$

$$\mathcal{U}_\lambda(N) := \sum_{k=0}^{N-1} |e^{\frac{2\pi i k}{N}} - e^{\frac{\pi i}{N}}|^\lambda \qquad (\text{potential of the } N\text{-th roots of unity at } e^{\frac{\pi i}{N}})$$

## Lemma (López, McCleary)

*Suppose that $N \ge 2$ has the binary representation*

$$N = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_p}, \qquad n_1 > n_2 > \cdots > n_p \ge 0.$$

*Then,*

$$U_{N,\lambda}(a_N) = \sum_{k=1}^{p} \mathcal{U}_\lambda(2^{n_k}),$$

$$H_\lambda(\alpha_{N,\lambda}) = \sum_{k=1}^{p-1} \left( \sum_{j=k+1}^{p} 2^{n_j - n_k} \right) \mathcal{L}_\lambda(2^{n_k+1}) + \sum_{k=1}^{p} \left( 1 - \sum_{j=k+1}^{p} 2^{n_j - n_k + 1} \right) \mathcal{L}_\lambda(2^{n_k}).$$

# The sequence $(U_{N,\lambda}(a_N) - I_\lambda(\sigma_1)N)$

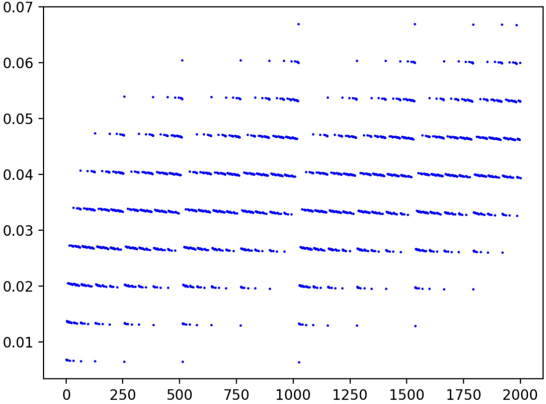


Figure: The first 2000 points of the sequence $(U_{N,\lambda}(a_N) - I_\lambda(\sigma_1)N)$ for $\lambda = 0.01$

# The sequence $(U_{N,\lambda}(a_N) - I_\lambda(\sigma_1)N)$

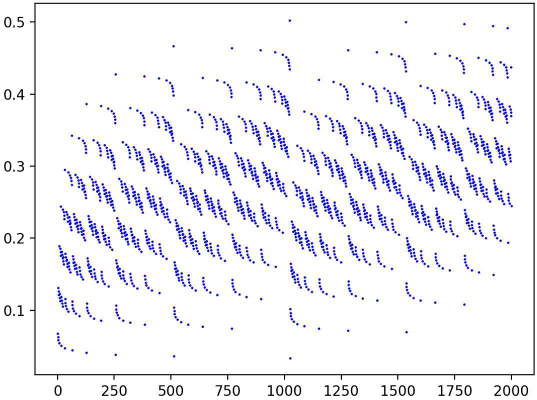


Figure: The first 2000 points of the sequence $(U_{N,\lambda}(a_N) - I_\lambda(\sigma_1)N)$ for $\lambda = 0.1$

# The sequence $(U_{N,\lambda}(a_N) - I_\lambda(\sigma_1)N)$

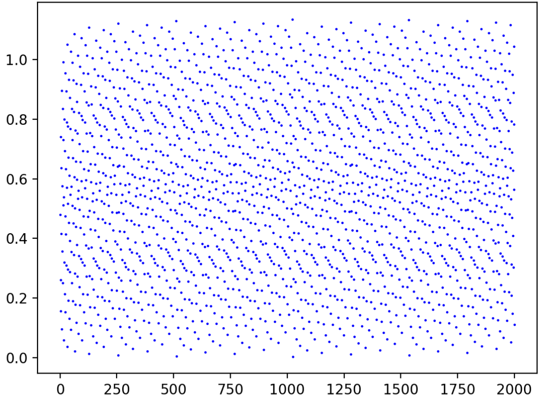


Figure: The first 2000 points of the sequence $(U_{N,\lambda}(a_N) - I_\lambda(\sigma_1)N)$ for $\lambda = 0.7$

# The sequence $(U_{N,\lambda}(a_N) - I_\lambda(\sigma_1)N)$

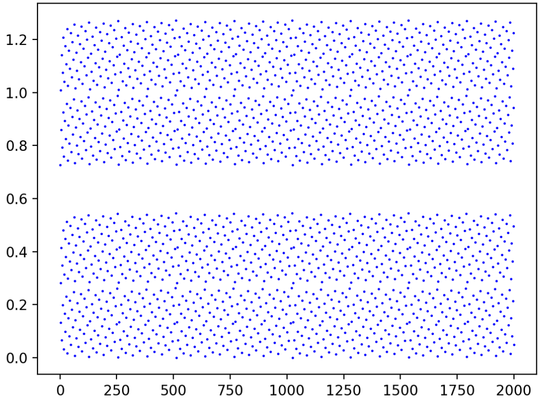


Figure: The first 2000 points of the sequence $(U_{N,\lambda}(a_N) - I_\lambda(\sigma_1)N)$ for $\lambda = 1$

# The sequence $(U_{N,\lambda}(a_N) - I_\lambda(\sigma_1)N)$

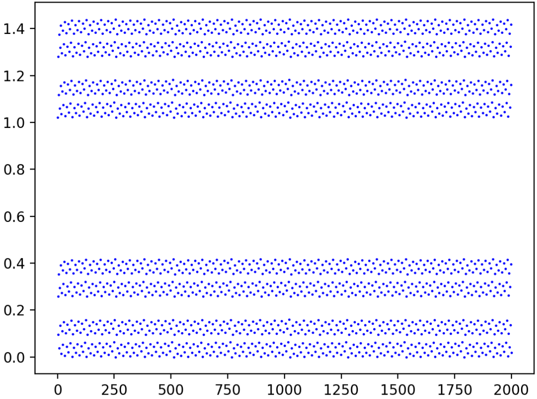


Figure: The first 2000 points of the sequence $(U_{N,\lambda}(a_N) - I_\lambda(\sigma_1)N)$ for $\lambda = 1.3$

# The sequence $(U_{N,\lambda}(a_N) - I_\lambda(\sigma_1)N)$

## Theorem (López, McCleary)

*Let $0 < \lambda < 2$, and let $(a_n)_{n=0}^\infty \subset S^1$ be a greedy $\lambda$-energy sequence. Then, the sequence $(U_{N,\lambda}(a_N) - I_\lambda(\sigma_1)N)_{N=1}^\infty$ is bounded and divergent. For every $N \geq 1$,*

$$0 < U_{N,\lambda}(a_N) - I_\lambda(\sigma_1)N < I_\lambda(\sigma_1)$$

*and we have*

$$\liminf_{N\to\infty} (U_{N,\lambda}(a_N) - I_\lambda(\sigma_1)N) = 0,$$
$$\limsup_{N\to\infty} (U_{N,\lambda}(a_N) - I_\lambda(\sigma_1)N) = I_\lambda(\sigma_1).$$

## The sequence $(H_\lambda(\alpha_{N,\lambda}) - I_\lambda(\sigma_1)N^2)$

Let

$$\kappa_\lambda(N) = \begin{cases} N^{1-\lambda} & 0 < \lambda < 1, \\ \log N & \lambda = 1, \\ 1 & 1 < \lambda < 2. \end{cases}$$

# The sequence $(H_\lambda(\alpha_{N,\lambda}) - I_\lambda(\sigma_1)N^2)$

Let

$$\kappa_\lambda(N) = \begin{cases} N^{1-\lambda} & 0 < \lambda < 1, \\ \log N & \lambda = 1, \\ 1 & 1 < \lambda < 2. \end{cases}$$

For any $0 < \lambda < 2$, the sequence

$$\left( \frac{H_\lambda(\alpha_{N,\lambda}) - I_\lambda(\sigma_1)N^2}{\kappa_\lambda(N)} \right)_{N=2}^{\infty}$$

is bounded and divergent.

## The sequence $(H_\lambda(\alpha_{N,\lambda}) - I_\lambda(\sigma_1)N^2)$

Let

$$\kappa_\lambda(N) = \begin{cases} N^{1-\lambda} & 0 < \lambda < 1, \\ \log N & \lambda = 1, \\ 1 & 1 < \lambda < 2. \end{cases}$$

For any $0 < \lambda < 2$, the sequence

$$\left( \frac{H_\lambda(\alpha_{N,\lambda}) - I_\lambda(\sigma_1)N^2}{\kappa_\lambda(N)} \right)_{N=2}^{\infty}$$

is bounded and divergent.

In contrast, for the $N$-th roots of unity, its energy $\mathcal{L}_\lambda(N)$ satisfies, for all $0 < \lambda < 2$,

$$\lim_{N\to\infty} \frac{\mathcal{L}_\lambda(N) - I_\lambda(\sigma_1)N^2}{N^{1-\lambda}} = (2\pi)^\lambda \, 2\zeta(-\lambda),$$

as shown by Brauchart-Hardin-Saff, where $\zeta(s)$ is the Riemann zeta function.

The sequence

$$\left(\frac{H_\lambda(\alpha_{N,\lambda}) - I_\lambda(\sigma_1)N^2}{N^{1-\lambda}}\right)_{N=2}^{\infty}$$

is bounded and divergent.

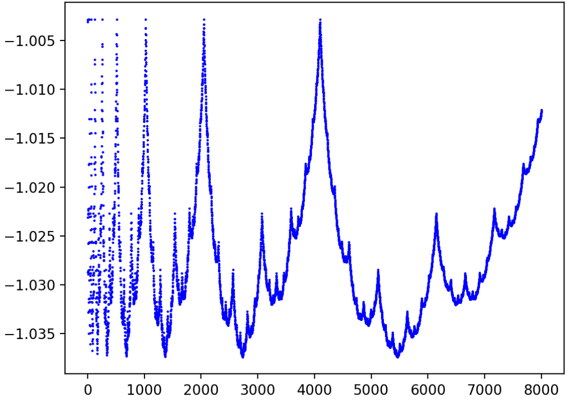


Figure: The first 8000 points of the sequence $\left(\frac{H_\lambda(\alpha_{N,\lambda}) - I_\lambda(\sigma_1)N^2}{N^{1-\lambda}}\right)$ for $\lambda = 0.1$

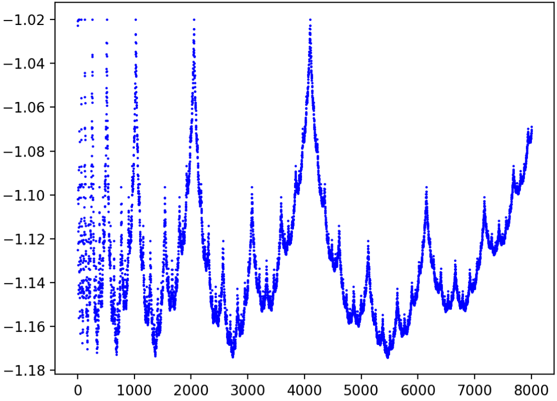


Figure: The first 8000 points of the sequence $\left(\frac{H_\lambda(\alpha_{N,\lambda})-I_\lambda(\sigma_1)N^2}{N^{1-\lambda}}\right)$ for $\lambda = 0.3$

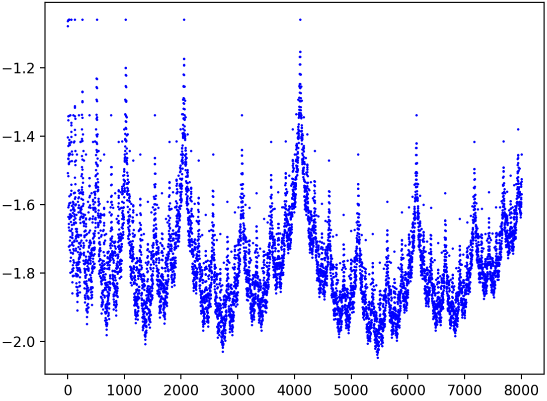


Figure: The first 8000 points of the sequence $\left(\frac{H_{\lambda}(\alpha_{N,\lambda})-I_{\lambda}(\sigma_1)N^2}{N^{1-\lambda}}\right)$ for $\lambda = 0.7$

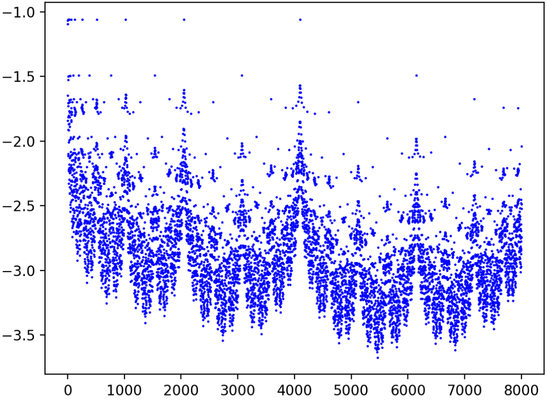


Figure: The first 8000 points of the sequence $\left(\frac{H_\lambda(\alpha_{N,\lambda}) - I_\lambda(\sigma_1)N^2}{N^{1-\lambda}}\right)$ for $\lambda = 0.9$

# The sequence $\left(\frac{H_\lambda(\alpha_{N,\lambda})-I_\lambda(\sigma_1)N^2}{N^{1-\lambda}}\right)$ for $0<\lambda<1$

**Class of functions $G(\vec{\theta},\lambda)$:**

Given an odd integer $M=2^{t_1}+2^{t_2}+\cdots+2^{t_{p-1}}+1$, $t_1>t_2>\cdots>t_{p-1}>0$, we construct the vector

$$\vec{\theta}=\vec{\theta}_M=\left(\frac{2^{t_1}}{M},\frac{2^{t_2}}{M},\ldots,\frac{2^{t_{p-1}}}{M},\frac{1}{M}\right).$$

Let $\Theta$ be the collection of all such vectors (of any length).

Given $\vec{\theta}=(\theta_1,\ldots,\theta_p)\in\Theta$, let

$$G(\vec{\theta},\lambda):=\sum_{k=1}^{p}\theta_k^{-\lambda}(2(2^{-\lambda}-1)\Big(\sum_{j=k+1}^{p}\theta_j\Big)+\theta_k).$$

## The sequence $\left(\frac{H_\lambda(\alpha_{N,\lambda})-I_\lambda(\sigma_1)N^2}{N^{1-\lambda}}\right)$ for $0<\lambda<1$

**Class of functions $G(\vec{\theta},\lambda)$:**

Given an odd integer $M=2^{t_1}+2^{t_2}+\cdots+2^{t_{p-1}}+1$, $t_1>t_2>\cdots>t_{p-1}>0$, we construct the vector

$$\vec{\theta}=\vec{\theta}_M=\left(\frac{2^{t_1}}{M},\frac{2^{t_2}}{M},\ldots,\frac{2^{t_{p-1}}}{M},\frac{1}{M}\right).$$

Let $\Theta$ be the collection of all such vectors (of any length).

Given $\vec{\theta}=(\theta_1,\ldots,\theta_p)\in\Theta$, let

$$G(\vec{\theta},\lambda):=\sum_{k=1}^{p}\theta_k^{-\lambda}(2(2^{-\lambda}-1)\Big(\sum_{j=k+1}^{p}\theta_j\Big)+\theta_k).$$

For $0<\lambda<1$, let

$$g(\lambda):=\sup_{\vec{\theta}\in\Theta}G(\vec{\theta},\lambda).$$

This function takes values $>1$.

# The sequence $\left(\frac{H_\lambda(\alpha_{N,\lambda})-I_\lambda(\sigma_1)N^2}{N^{1-\lambda}}\right)$ for $0<\lambda<1$

## Theorem

*Let $0<\lambda<1$, and let $(a_n)_{n=0}^{\infty}\subset S^1$ be a greedy $\lambda$-energy sequence. Then, the sequence*

$$\left(\frac{H_\lambda(\alpha_{N,\lambda})-I_\lambda(\sigma_1)N^2}{N^{1-\lambda}}\right)_{N=2}^{\infty} \tag{4}$$

*is bounded and divergent, and we have*

$$\limsup_{N\to\infty}\frac{H_\lambda(\alpha_{N,\lambda})-I_\lambda(\sigma_1)N^2}{N^{1-\lambda}}=(2\pi)^\lambda\,2\zeta(-\lambda),$$

$$\liminf_{N\to\infty}\frac{H_\lambda(\alpha_{N,\lambda})-I_\lambda(\sigma_1)N^2}{N^{1-\lambda}}=g(\lambda)\,(2\pi)^\lambda\,2\zeta(-\lambda).$$

# The sequence $\left(\frac{H_\lambda(\alpha_{N,\lambda})-I_\lambda(\sigma_1)N^2}{N^{1-\lambda}}\right)$ for $0 < \lambda < 1$

## Theorem

*Let $0 < \lambda < 1$, and let $(a_n)_{n=0}^{\infty} \subset S^1$ be a greedy $\lambda$-energy sequence. Then, the sequence*

$$\left(\frac{H_\lambda(\alpha_{N,\lambda}) - I_\lambda(\sigma_1)N^2}{N^{1-\lambda}}\right)_{N=2}^{\infty} \tag{4}$$

*is bounded and divergent, and we have*

$$\limsup_{N\to\infty} \frac{H_\lambda(\alpha_{N,\lambda}) - I_\lambda(\sigma_1)N^2}{N^{1-\lambda}} = (2\pi)^\lambda \, 2\zeta(-\lambda),$$

$$\liminf_{N\to\infty} \frac{H_\lambda(\alpha_{N,\lambda}) - I_\lambda(\sigma_1)N^2}{N^{1-\lambda}} = g(\lambda)\,(2\pi)^\lambda \, 2\zeta(-\lambda).$$

*For every $\vec{\theta} \in \Theta$, the value $G(\vec{\theta}, \lambda)(2\pi)^\lambda \, 2\zeta(-\lambda)$ is a limit point of (4).*

# The sequence $\left(\frac{H_\lambda(\alpha_{N,\lambda})-I_\lambda(\sigma_1)N^2}{N^{1-\lambda}}\right)$ for $0<\lambda<1$

**Theorem**

*Let $0<\lambda<1$, and let $(a_n)_{n=0}^\infty \subset S^1$ be a greedy $\lambda$-energy sequence. Then, the sequence*

$$\left(\frac{H_\lambda(\alpha_{N,\lambda}) - I_\lambda(\sigma_1)N^2}{N^{1-\lambda}}\right)_{N=2}^\infty \tag{4}$$

*is bounded and divergent, and we have*

$$\limsup_{N\to\infty} \frac{H_\lambda(\alpha_{N,\lambda}) - I_\lambda(\sigma_1)N^2}{N^{1-\lambda}} = (2\pi)^\lambda\, 2\zeta(-\lambda),$$
$$\liminf_{N\to\infty} \frac{H_\lambda(\alpha_{N,\lambda}) - I_\lambda(\sigma_1)N^2}{N^{1-\lambda}} = g(\lambda)\,(2\pi)^\lambda\, 2\zeta(-\lambda).$$

*For every $\vec{\theta}\in\Theta$, the value $G(\vec{\theta},\lambda)(2\pi)^\lambda\, 2\zeta(-\lambda)$ is a limit point of (4).*

Note that for $0<\lambda<1$, we have $\zeta(-\lambda)<0$ and $g(\lambda)>1$, so indeed $\liminf < \limsup$. We don't have an explicit expression for $g(\lambda)$.

# References


1) G. Björck, Distributions of positive mass, which maximize a certain generalized energy integral, Ark. Mat. 3 (1956), 255–269.

2) J.S. Brauchart, D.P. Hardin, and E.B. Saff, The Riesz energy of the $N$-th roots of unity: an asymptotic expansion for large $N$, Bull. London Math. Soc. 41 (2009), 621–633.

3) A. Edrei, Sur les déterminants récurrents et les singularités d'une fonction donnée par son développement de Taylor, Compositio Math. 7 (1939), 20–88.

4) F. Leja, Sur certaines suites liées aux ensembles plans et leur application à la représentation conforme, Ann. Polon. Math. 4 (1957), 8–13.

5) A. López-García and R.E. McCleary, Asymptotics of greedy energy sequences on the unit circle and the sphere, preprint arXiv:2007.06109.

6) A. López-García and D.A. Wagner, Asymptotics of the energy of sections of greedy energy sequences on the unit circle, and some conjectures for general sequences, Comput. Methods Funct. Theory 15 (2015), 721-750.