Spectral properties of random banded Hessenberg matrices

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Banded lower Hessenberg matrices

In this talk we consider **banded lower Hessenberg matrices**

$$H_n = \begin{pmatrix} a_1^{(0)} & 1 & & & & 0 \\ a_1^{(1)} & a_2^{(0)} & 1 & & & \\ \vdots & \ddots & \ddots & \ddots & & \\ a_1^{(p)} & & \ddots & \ddots & \ddots & \\ & \ddots & & \ddots & \ddots & 1 \\ 0 & & a_{n-p}^{(p)} & \cdots & a_{n-1}^{(1)} & a_n^{(0)} \end{pmatrix}$$

with 1's in the first superdiagonal.

The number of subdiagonals is $p \geq 1$, arbitrary fixed positive integer.

$a_j^{(k)}$ is located in the kth subdiagonal ($k = 0$ for the main diagonal) and the jth column.

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$a_j^{(k)}$ is located in the kth subdiagonal ($k = 0$ for the main diagonal) and the jth column.

The characteristic polynomials $Q_n(z) = \det(zI_n - H_n)$ satisfy the $(p+2)$**-term recurrence relation**

$$zQ_n(z) = Q_{n+1}(z) + a_{n+1}^{(0)} Q_n(z) + a_n^{(1)} Q_{n-1}(z) + \cdots + a_{n-p+1}^{(p)} Q_{n-p}(z).$$

The roots of Q_n (eigenvalues of H_n) are in general spread in the complex plane in a rather arbitrary way.

Some notable examples

Examples of such polynomials are **multiple orthogonal polynomials** with respect to **Angelesco systems** or **Nikishin systems** of p measures on the real line, or on a symmetric star with center at the origin in the complex plane with rotationally symmetric generating measures.

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In the case of Angelesco systems on the real line: Take $\mu_1, \ldots, \mu_p$ positive measures with infinite support $\mathrm{supp}(\mu_k) \subset [a_k, b_k]$, such that $[a_k, b_k] \cap [a_j, b_j] = \emptyset$ for $k \neq j$. Then Q_n is the nth-degree monic polynomial satisfying

$$\int Q_n(x)\, x^k \, d\mu_j(x) = 0, \qquad k = 0, \ldots, \left\lfloor \frac{n-j}{p} \right\rfloor, \quad 1 \leq j \leq p.$$

The roots of Q_n are on the real line in this case.

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Then the sequence $(Q_n(z))_{n=0}^{\infty}$ satisfies

$$zQ_n(z) = Q_{n+1}(z) + a_{n+1}^{(0)} Q_n(z) + a_n^{(1)} Q_{n-1}(z) + \cdots + a_{n-p+1}^{(p)} Q_{n-p}(z), \quad n \geq 0,$$

with initial conditions $Q_0 \equiv 1$, $Q_{-1} \equiv \cdots \equiv Q_{-p} \equiv 0$.

Banded Hessenberg operators and resolvent functions

Let $a^{(k)} = (a_n^{(k)})_{n=1}^{\infty}$, $0 \le k \le p$, be bounded sequences of complex numbers, and construct the bounded operator H on $\ell^2(\mathbb{N})$ with matrix representation

$$H = \begin{pmatrix} a_1^{(0)} & 1 & & & & 0 \\ \vdots & \ddots & \ddots & & & \\ a_1^{(p)} & \ddots & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & 1 & \\ & & a_{n-p}^{(p)} & \ddots & a_n^{(0)} & \ddots \\ 0 & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

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Let $\{e_j\}_{j=1}^{\infty}$ be the standard basis in $\ell^2(\mathbb{N})$, with inner product $\langle \cdot, \cdot \rangle$. Let

$$\phi_j(z) = \langle (z - H)^{-1} e_j, e_1 \rangle = \sum_{n=0}^{\infty} \frac{\langle H^n e_j, e_1 \rangle}{z^{n+1}}, \qquad 1 \leq j \leq p.$$

We also define $\phi_0 \equiv 1$. We have defined a map

$$\mathcal{A} = (a^{(0)}, a^{(1)}, \ldots, a^{(p)}) \longmapsto \Phi = (\phi_0, \phi_1, \ldots, \phi_p)$$

which we indicate by writing $\Phi = \Phi(\mathcal{A})$.

Random matrices

Assumptions:

1) Let μ_k, $0 \leq k \leq p$, be a collection of $p + 1$ Borel probability measures with compact support in $\mathbb{C}$.

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Construct the infinite banded Hessenberg matrix

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Let $\{\lambda_{i,n}\}_{i=1}^{n}$ denote the eigenvalues of H_n, counting multiplicities, and let

$$\sigma_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_{i,n}}. \qquad \text{(ESD of } H_n\text{)}$$

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$\mathbb{E}\sigma_n$ is the deterministic probability measure defined via duality by

$$\int f\, d\mathbb{E}\sigma_n = \mathbb{E}\left(\int f\, d\sigma_n\right).$$

Main questions of interest:

1) Is the sequence of average measures $\mathbb{E}\sigma_n$ weakly convergent (in the weak-star topology)?
2) If so, what is the limit, and how is it related to the distributions $\{\mu_k\}_{k=0}^{p}$?

We have a partial answer to these questions, proving the existence of the limits of the average moments

$$\lim_{n\to\infty} \int z^\ell d\mathbb{E}\sigma_n(z) = \lim_{n\to\infty} \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} \lambda_{i,n}^\ell\right) =: \omega_\ell, \qquad \ell \geq 0, \tag{1}$$

and we can describe the **moment generating function**

$$\sum_{\ell=0}^{\infty} \frac{\omega_\ell}{z^{\ell+1}}$$

in terms of resolvent functions of the operator H and an extension of this operator to $\ell^2(\mathbb{Z})$.

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As a consequence of (1), if $\mathrm{sp}(H_n) \subset \mathbb{R}$ for all n, then $\mathbb{E}\sigma_n$ is weakly convergent to a probability measure on $\mathbb{R}$.

In certain models of bi-diagonal random Hessenberg matrices with eigenvalues on a symmetric starlike set, we can also prove weak convergence of $\mathbb{E}\sigma_n$ to a probability measure on the set.

Recall the map

$$\mathcal{A} = (a^{(0)}, a^{(1)}, \dots, a^{(p)}) \longmapsto \Phi = (\phi_0, \phi_1, \dots, \phi_p)$$

which we indicate by writing $\Phi = \Phi(\mathcal{A})$, where $\phi_0 \equiv 1$ and

$$\phi_j(z) := \langle (z - H_{\mathcal{A}})^{-1} e_j, e_1 \rangle, \qquad 1 \leq j \leq p.$$

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Theorem

Let $\mathcal{A} = (a^{(0)}, \ldots, a^{(p)})$ and $\mathcal{B} = (b^{(0)}, \ldots, b^{(p)})$ be two independent collections of random sequences with corresponding distributions $(\mu_0, \ldots, \mu_p)$. Let $\Phi(\mathcal{A}) = (\phi_0, \ldots, \phi_p)$ and $\Psi(\mathcal{B}) = (\psi_0, \ldots, \psi_p)$.

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$$\lim_{n\to\infty} \int z^\ell \, d\mathbb{E}\sigma_n(z) = \mathbb{E}([W]_{\ell+1}),$$

where $[W]_{\ell+1}$ is the coefficient of $z^{-\ell-1}$ in the Laurent series expansion at ∞ of

$$W(z) = \frac{1}{z - \sum_{k=0}^{p} \sum_{j=0}^{k} \alpha_j^{(k)} \, \phi_{k-j}(z) \, \psi_j(z)}.$$

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The expression $\sum_{k=0}^{p} \sum_{j=0}^{k} \alpha_j^{(k)} \phi_{k-j}(z)\, \psi_j(z)$ is the triangular bilinear form:

$$\begin{pmatrix} \psi_0 & \psi_1 & \psi_2 & \dots & \psi_p \end{pmatrix} \begin{pmatrix} \alpha_0^{(0)} & \alpha_0^{(1)} & \alpha_0^{(2)} & \dots & \alpha_0^{(p)} \\ \alpha_1^{(1)} & \alpha_1^{(2)} & \dots & \alpha_1^{(p)} & \\ \alpha_2^{(2)} & \dots & \alpha_2^{(p)} & & \\ \vdots & \iddots & & & \\ \alpha_p^{(p)} & & & & 0 \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{pmatrix}.$$

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The function $W(z)$, analytic in a nbhd of ∞, is a resolvent function of a two-sided operator on $\ell^2(\mathbb{Z})$.

The operator in $\ell^2(\mathbb{Z})$ is obtained extending the sequences in $\mathcal{A} = (a^{(0)}, a^{(1)}, \ldots, a^{(p)})$ from $\mathbb{N}$ to $\mathbb{Z}$ using the sequences in $\mathcal{B} = (b^{(0)}, \ldots, b^{(p)})$ and the array $\alpha = (\alpha_j^{(k)})_{0 \le j \le k \le p}$: For $n \le 0$, we define

$$a_n^{(k)} = \begin{cases} \alpha_{-n}^{(k)} & \text{if } -k \le n \le 0, \\ b_{-n-k}^{(k)} & \text{if } n \le -k-1. \end{cases}$$

Let $\{e_n\}_{n\in\mathbb{Z}}$ be the standard basis in $\ell^2(\mathbb{Z})$, and let M be the operator on $\ell^2(\mathbb{Z})$ that acts on the basis vectors as follows:

$$Me_n = e_{n-1} + \sum_{k=0}^{p} a_n^{(k)} e_{n+k}, \qquad n \in \mathbb{Z}$$

$$M = \begin{pmatrix} \ddots & \ddots & & & & & 0 \\ \ddots & a_1^{(0)} & 1 & & & & \\ \ddots & \ddots & \ddots & \ddots & & & \\ & a_1^{(p)} & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & 1 & \\ & & & a_{n-p}^{(p)} & \ddots & a_n^{(0)} & \ddots \\ 0 & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

Then

$$W(z) = \frac{1}{z - \sum_{k=0}^{p} \sum_{j=0}^{k} \alpha_j^{(k)} \, \phi_{k-j}(z) \, \psi_j(z)} = \langle (z-M)^{-1} e_0, e_0 \rangle .$$

Some key ideas in the proof

1) **Hermite-Padé property**: Let

$$Q_n(z) = \det(zI_n - H_n),$$
$$Q_{n,j}(z) = \det((zI_n - H_n)^{[j]}), \qquad 1 \leq j \leq p, \quad n \geq 0,$$

where $(zI_n - H_n)^{[j]}$ is the submatrix of $zI_n - H_n$ obtained deleting the first j rows and columns.

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Kalyagin (1995) proved that

$$\left(\frac{Q_{n,1}}{Q_n}, \frac{Q_{n,2}}{Q_n}, \ldots, \frac{Q_{n,p}}{Q_n} \right)$$

is a Hermite-Padé approximant at infinity for the system of resolvent functions $(\phi_1, \phi_2, \ldots, \phi_p)$ associated with the operator H:

$$Q_n(z)\phi_j(z) - Q_{n,j}(z) = O\left(\frac{1}{z^{n_j+2}}\right), \quad z \to \infty, \quad 1 \le j \le p,$$

where $n_j = \lfloor (n-j)/p \rfloor$.

2) Let $M = (m_{i,j})_{i,j\in\mathbb{Z}}$ be the bi-infinite banded Hessenberg matrix defined before. For $n \geq 0$, let

$$M_{2n+1} = (m_{i,j})_{-n\leq i,j\leq n}.$$

Then for every $-n \leq j \leq n$,

$$(zI_{2n+1} - M_{2n+1})^{-1}(j,j) - w_j(z) = O\left(\frac{1}{z^{n-|j|+3+\lfloor (n-|j|)/p\rfloor}}\right), \qquad z \to \infty \qquad (2)$$

where

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(2) follows from the Hermite-Padé property and the identity

$$(zI_{2n+1} - M_{2n+1})^{-1}(j,j) = \frac{q^{+}_{n-j}(z)q^{-}_{n+j}(z)}{q_{2n+1}(z)}$$

where

$$q_{2n+1}(z) = \det(zI_{2n+1} - M_{2n+1})$$
$$q^{+}_{\ell}(z) = \det((zI_{2n+1} - M_{2n+1})^{[2n+1-\ell]})$$
$$q^{-}_{\ell}(z) = \det((zI_{2n+1} - M_{2n+1})_{[2n+1-\ell]})$$

In fact, the key relations are

$$\begin{aligned}
(zI_{2n+1}-M_{2n+1})^{-1}(j,j) &= \frac{q^{+}_{n-j}(z)q^{-}_{n+j}(z)}{q_{2n+1}(z)} \\
&= \frac{1}{z-a_j^{(0)}-\sum_{t=1}^{p}\sum_{k=0}^{t} a_{j-k}^{(t)}\,\frac{q^{+}_{n-j+k-t}(z)}{q^{+}_{n-j}(z)}\,\frac{q^{-}_{n+j-k}(z)}{q^{-}_{n+j}(z)}} \\
&\approx \frac{1}{z-a_j^{(0)}-\sum_{t=1}^{p}\sum_{k=0}^{t} a_{j-k}^{(t)}\,\phi^{+}_{j,t-k}(z)\,\phi^{-}_{j,k}(z)} \\
&= \langle (zI-M)^{-1}\,e_j, e_j\rangle \\
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where $\phi^{+}_{j,t-k}(z)$ and $\phi^{-}_{j,k}(z)$ are the resolvent functions of certain restrictions of the operator M on $\ell^2(\mathbb{N})$.

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3) The connection with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{2n+1}$ of M_{2n+1} is

$$
\sum_{j=-n}^{n} (zI_{2n+1} - M_{2n+1})^{-1}(j,j) = \sum_{k=1}^{2n+1} \frac{1}{z - \lambda_k}
$$

From the independence between the elements $\alpha_j^{(k)}$, $\phi_{k-j}(z)$, and $\psi_j(z)$ in the formula

$$W(z) = \frac{1}{z - \sum_{k=0}^{p} \sum_{j=0}^{k} \alpha_j^{(k)} \phi_{k-j}(z) \psi_j(z)}$$

one can easily obtain a relation (in the form of a series) between $\mathbb{E}(W(z))$, moments of the distributions $\mu_0, \mu_1, \ldots, \mu_p$, and joint moments of the random vector $(\phi_1(z), \ldots, \phi_p(z))$:

$$g_{(n_1,\ldots,n_p)}(z) = \mathbb{E}\left(\prod_{k=1}^{p} \phi_k(z)^{n_k}\right).$$

The distribution of $(\phi_1(z), \ldots, \phi_p(z))$

For z large enough, the random vector $(\phi_1(z), \ldots, \phi_p(z))$ is well-defined (resolvent functions of the infinite matrix H on $\ell^2(\mathbb{N})$). Let σ_z be the distribution of this vector (probability measure on $\mathbb{C}^p$). We don't know what it is, but it satisfies an **invariance principle**.

The distribution of $(\phi_1(z), \ldots, \phi_p(z))$

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Let H_1 be the infinite one-sided matrix obtained by removing the first row and the first column of the matrix H, and let

$$\phi_j(z) = \langle (z - H)^{-1} e_j, e_1 \rangle \qquad 1 \leq j \leq p$$
$$\phi_{1,j}(z) = \langle (z - H_1)^{-1} e_j, e_1 \rangle \qquad 1 \leq j \leq p$$

then for z large enough

$$\phi_1(z) = \frac{1}{z - a_1^{(0)} - \sum_{k=1}^{p} a_1^{(k)} \phi_{1,k}(z)}, \tag{3}$$

$$\phi_j(z) = \frac{\phi_{1,j-1}(z)}{z - a_1^{(0)} - \sum_{k=1}^{p} a_1^{(k)} \phi_{1,k}(z)}, \qquad 2 \leq j \leq p, \tag{4}$$

which when iterated gives a **vector continued fraction** expansion of $(\phi_1(z), \ldots, \phi_p(z))$ (vector analogue of the Jacobi continued fraction for OPs).

By the i.i.d. assumptions, the vectors $\Phi(z) := (\phi_1(z), \phi_2(z), \dots, \phi_p(z))$ and $\Phi_1(z) := (\phi_{1,1}(z), \phi_{1,2}(z), \dots, \phi_{1,p}(z))$ clearly have the same distribution σ_z, and the vectors $(a_1^{(0)}, \dots, a_1^{(p)})$ and $\Phi_1(z)$ are independent.

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We will write $\mathbf{t} = (t_0, \ldots, t_p)$, $\mathbf{x} = (x_1, \ldots, x_p)$. Let $\lambda_z : \mathcal{D}_z \longrightarrow \mathbb{C}^p$ be the function

$$\lambda_z(\mathbf{t}, \mathbf{x}) := \left(\frac{1}{z - t_0 - \sum_{k=1}^p t_k\, x_k}, \frac{x_1}{z - t_0 - \sum_{k=1}^p t_k\, x_k}, \ldots, \frac{x_{p-1}}{z - t_0 - \sum_{k=1}^p t_k\, x_k} \right).$$

Then the relations (3)–(4) mean $\Phi(z) = \lambda_z((a_1^{(0)}, \ldots, a_1^{(p)}), \Phi_1(z))$, and the distribution of $((a_1^{(0)}, \ldots, a_1^{(p)}), \Phi_1(z))$ is $\mu \times \sigma_z$, so

$$\sigma_z = (\lambda_z)_*(\mu \times \sigma_z)$$

(push-forward of $\mu \times \sigma_z$ under λ_z).
This means that for any $f : \mathbb{C}^p \longrightarrow \mathbb{C}$ in $L^1(\sigma_z)$, we have

$$\int f\, d\sigma_z = \iint f \circ \lambda_z\, d(\mu \times \sigma_z).$$

Literature on random matrices

Most of the literature on random matrices is devoted to the study of ensembles of Hermitian/real-symmetric or unitary/orthogonal matrices. The great advantage of such ensembles is that the eigenvalues are located on the real line or the unit circle. Important examples are GUE, GOE, GSE, CUE.

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Random banded Hessenberg matrices are on the contrary non-symmetric, and so it is difficult in general to locate the eigenvalues, so the spectral analysis presents new challenges. We have found that Hermite-Padé approximation can be an effective tool for this analysis.

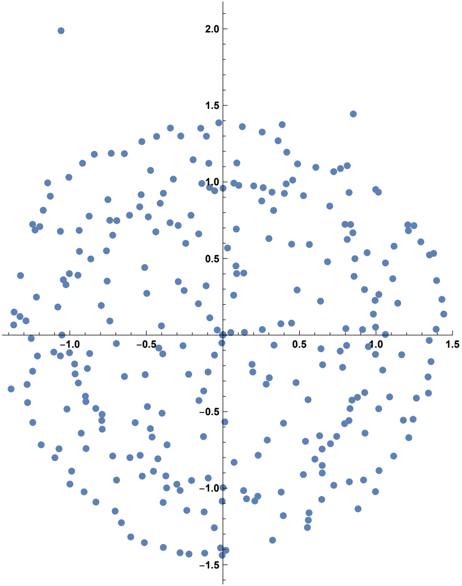

Figure: Eigenvalues of H_n, $n = 300$, $p = 2$, $a_n^{(0)} = a_n^{(1)} = 0$ for all n, $(a_n^{(2)})_{n=1}^{\infty}$ is i.i.d. with uniform distribution on the unit circle.

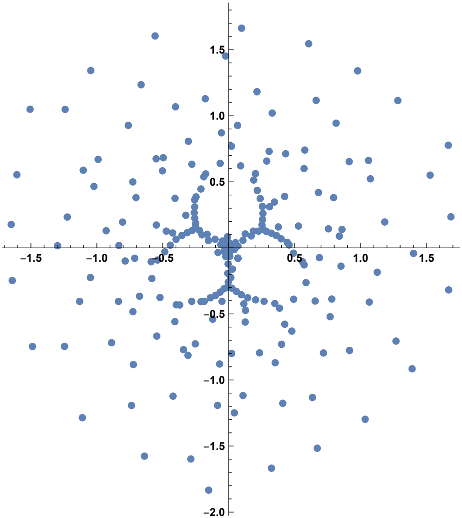

Figure: Eigenvalues of H_n, $n = 300$, $p = 2$, $a_n^{(0)} = a_n^{(1)} = 0$ for all n, $(a_n^{(2)})_{n=1}^{\infty}$ is i.i.d. with uniform distribution on the arc of the unit circle $\{e^{i\theta} : \frac{\pi}{3} \leq \theta \leq \frac{2\pi}{3}\}$.

References

1) A. López-García and V.A. Prokhorov, *Characteristic polynomials of random banded Hessenberg matrices and Hermite-Padé approximation*, Complex Anal. Oper. Theory 15:113, 2021.

2) V.A. Kalyagin, *Hermite-Padé approximants and spectral analysis of nonsymmetric operators*, Sbornik: Math. 82 (1995), 199–216.