Lattice paths, vector continued fractions, and resolvents of banded Hessenberg operators

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Joint work with

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Plan of the talk

1) Lattice paths and associated formal Laurent series.

2) Algebraic relations between the Laurent series.

3) Vector continued fractions, Hermite-Padé approximation, and banded Hessenberg operators.

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1) Lattice paths and associated formal Laurent series.
2) Algebraic relations between the Laurent series.
3) Vector continued fractions, Hermite-Padé approximation, and banded Hessenberg operators.

The goal is to describe a combinatorial interpretation of certain classes of vector continued fractions. In the scalar case, P. Flajolet (1980) described the connection between standard continued fractions and Motzkin/Dyck paths.

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Fix an integer $p \geq 1$. A **lattice path** is a concatenation of finitely many segments (called **steps**)

$$\gamma = e_1 e_2 \cdots e_k.$$

The segments belong to any of the following collections:

$$\mathcal{E}_u := \{(n,m) \to (n+1, m+1) : (n,m) \in \mathcal{V}\} \qquad \text{(upsteps)}$$

$$\mathcal{E}_\ell := \{(n,m) \to (n+1, m) : (n,m) \in \mathcal{V}\} \qquad \text{(level steps)}$$

$$\mathcal{E}_d := \{(n,m) \to (n+1, m-j) : (n,m) \in \mathcal{V}, \;\; 1 \leq j \leq p\} \qquad \text{(downsteps)}$$

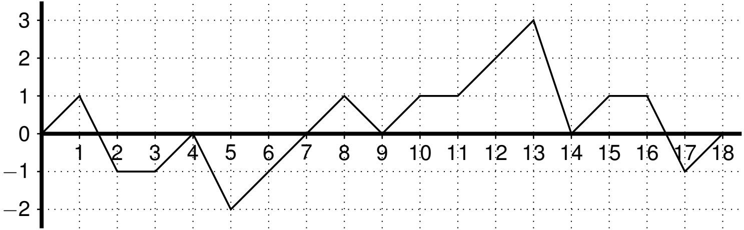

Figure: Example of a lattice path in the case $p = 3$.

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has **length** k.

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We define $\mathsf{max}(\gamma)$ to be the maximum of the heights of all the vertices in γ, and $\mathsf{min}(\gamma)$ to be the minimum of those heights.

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If $q \geq 0$ is an integer, then $\gamma + q$ denotes the path obtained by shifting γ upwards q units, and $\gamma - q$ is the path obtained by shifting γ downwards q units.

Weights

Paths will be given a weight (or label) as follows.

First, fix a collection of $p+1$ bi-infinite sequences of complex numbers $(a_n^{(j)})_{n\in\mathbb{Z}}$, $0 \leq j \leq p$.

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The **weight of a segment** is:

$$w((n,m) \to (n+1, m+1)) = 1,$$
$$w((n,m) \to (n+1, m-j)) = a_{m-j}^{(j)}, \qquad 0 \le j \le p.$$

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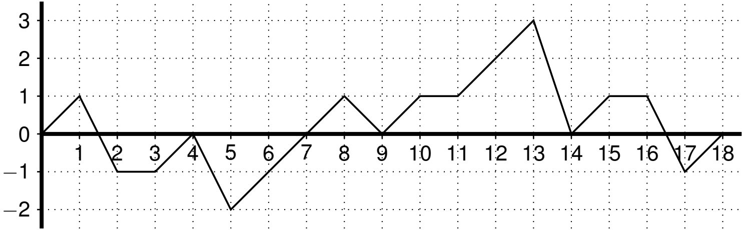
The **weight of a path** is

$$w(\gamma) = \prod_{e \subset \gamma} w(e)$$

where the product runs over the different steps of γ. If γ has length zero then the weight of γ is by definition 1.



The path γ above has length 18, $\max(\gamma) = 3$, $\min(\gamma) = -2$, and

$$w(\gamma) = (a_{-1}^{(2)})^2\, a_{-1}^{(0)}\, a_{-2}^{(2)}\, a_0^{(1)}\, (a_1^{(0)})^2\, a_0^{(3)}$$

Families of lattice paths

For each $n \in \mathbb{Z}_{\geq 0}$ and $0 \leq j \leq p$, let $\mathcal{P}_{[n,j]}$ denote the collection of all paths of length n, with starting point $(0,0)$ and terminal point (n,j).

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For $n \in \mathbb{Z}_{\geq 0}$ and $0 \leq j \leq p$, let

$$\mathcal{D}_{[n,j]} := \{\gamma \in \mathcal{P}_{[n,j]} : \min(\gamma) = 0\}.$$

The paths in $\mathcal{D}_{[n,j]}$ are called **partial p-Łukasiewicz paths**.

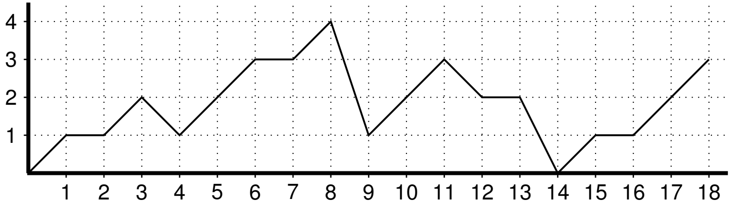

Figure: Example, in the case $p = 3$, of a path in the collection $\mathcal{D}_{[18,3]}$.

For $n \in \mathbb{Z}_{\geq 0}$ and $0 \leq j \leq p$, let $\widehat{\mathcal{D}}_{[n,j]}$ denote the collection of all paths γ of length n, with initial point $(0, -j)$, final point $(n, 0)$, and satisfying $\max(\gamma) = 0$.

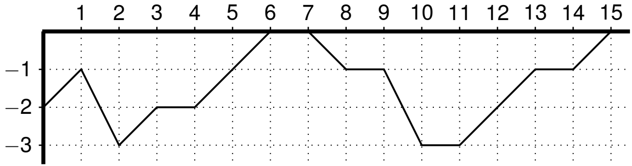

Figure: Example, in the case $p = 2$, of a path in the collection $\widehat{\mathcal{D}}_{[15,2]}$.

Weight polynomials and formal Laurent series

If $\mathcal{L}$ is a finite collection of lattice paths, the expression

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We define the weight polynomials

$$W_{[n,j]} := \sum_{\gamma \in \mathcal{P}_{[n,j]}} w(\gamma)$$

$$A_{[n,j]} := \sum_{\gamma \in \mathcal{D}_{[n,j]}} w(\gamma)$$

$$B_{[n,j]} := \sum_{\gamma \in \widehat{\mathcal{D}}_{[n,j]}} w(\gamma)$$

We also need to define the following weight polynomials. For an integer $q \geq 0$, let

$$A^{(q)}_{[n,j]} := \sum_{\gamma \in \mathcal{D}_{[n,j]}} w(\gamma + q),$$

$$B^{(q)}_{[n,j]} := \sum_{\gamma \in \widehat{\mathcal{D}}_{[n,j]}} w(\gamma - q).$$

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$$B_{[n,j]}^{(q)} := \sum_{\gamma \in \widehat{\mathcal{D}}_{[n,j]}} w(\gamma - q).$$

Let $\mathbb{C}((z^{-1}))$ be the algebraic field of all formal Laurent series

$$a(z) = \sum_{k \in \mathbb{Z}} \frac{a_k}{z^k}$$

with complex coefficients such that only finitely many a_k with $k < 0$ are non-zero.

We put the weight polynomials as coefficients of the following formal Laurent series: For each $0 \le j \le p$, let

$$A_j(z) := \sum_{n=0}^{\infty} \frac{A_{[n,j]}}{z^{n+1}}$$

$$B_j(z) := \sum_{n=0}^{\infty} \frac{B_{[n,j]}}{z^{n+1}}$$

$$W_j(z) := \sum_{n=0}^{\infty} \frac{W_{[n,j]}}{z^{n+1}}$$

We also define for an integer $q \ge 0$ the series

$$A_j^{(q)}(z) := \sum_{n=0}^{\infty} \frac{A_{[n,j]}^{(q)}}{z^{n+1}}$$

$$B_j^{(q)}(z) := \sum_{n=0}^{\infty} \frac{B_{[n,j]}^{(q)}}{z^{n+1}}$$

It is easy to see that

$$A_j(z) = \frac{1}{z^{j+1}} + O\left(\frac{1}{z^{j+2}}\right)$$

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Theorem

The following relations hold:

$$A_0(z) = \frac{1}{z - a_0^{(0)} - \sum_{j=1}^{p} a_0^{(j)} A_{j-1}^{(1)}(z)} \tag{1}$$

$$A_j(z) = A_0(z)\, A_{j-1}^{(1)}(z) \qquad 1 \leq j \leq p. \tag{2}$$

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Analogous relations hold between the series $A_j^{(q)}(z)$ and $A_j^{(q+1)}(z)$ for every integer q:

$$A_0^{(q)}(z) = \frac{1}{z - a_q^{(0)} - \sum_{j=1}^{p} a_q^{(j)} A_{j-1}^{(q+1)}(z)}$$
$$A_j^{(q)}(z) = A_0^{(q)}(z)\, A_{j-1}^{(q+1)}(z) \qquad 1 \le j \le p. \tag{3}$$

The relations (1)–(3) allow us to obtain a vector continued fraction expansion for the vector $(A_0(z), A_1(z), \ldots, A_{p-1}(z))$.

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Other relations one can prove are:

$$zA_0(z) - 1 = \sum_{j=0}^{p} a_0^{(j)} A_j(z)$$

$$A_j(z) = A_i(z)\, A_{j-i-1}^{(i+1)}(z) \qquad 0 \leq i < j \leq p.$$

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The following relations hold:

$$W_0(z) = \frac{1}{z - a_0^{(0)} - \sum_{j=1}^{p} \sum_{k=0}^{j} a_{-k}^{(j)}\, A_{j-k-1}^{(1)}(z) B_{k-1}^{(1)}(z)}$$

$$W_j(z) = W_i(z)\, A_{j-i-1}^{(i+1)}(z) \qquad 0 \leq i < j \leq p,$$

where in the first formula we understand that $A_{-1}^{(1)}(z) \equiv B_{-1}^{(1)}(z) \equiv 1$.

Vector continued fractions

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In function theory, the **Jacobi-Perron algorithm** is an algorithm to expand a vector of power series in a continued fraction where the partial numerators and denominators are vectors of polynomials.

Let $\mathbf{F} := \mathbb{C}((z^{-1}))$. In $\mathbf{F}^p$ we define the following **division** operation: If $y_p \neq 0$,

$$\frac{(x_1, \ldots, x_p)}{(y_1, \ldots, y_p)} := \left(\frac{x_1}{y_p}, \frac{x_2\, y_1}{y_p}, \frac{x_3\, y_2}{y_p}, \ldots, \frac{x_p\, y_{p-1}}{y_p} \right).$$

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In particular

$$\frac{(1, \ldots, 1)}{(y_1, \ldots, y_p)} := \left(\frac{1}{y_p}, \frac{y_1}{y_p}, \frac{y_2}{y_p}, \ldots, \frac{y_{p-1}}{y_p} \right).$$

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If $\mathbf{a}_1,\ldots,\mathbf{a}_n$ and $\mathbf{b}_1,\ldots,\mathbf{b}_n$ are now vectors of formal Laurent series, then we can form the finite continued fraction

$$\mathop{\mathbf{K}}_{m=1}^{n}\left(\frac{\mathbf{a}_m}{\mathbf{b}_m}\right) := \cfrac{\mathbf{a}_1}{\mathbf{b}_1 + \cfrac{\mathbf{a}_2}{\mathbf{b}_2 + \cfrac{\mathbf{a}_3}{\ddots + \cfrac{\mathbf{a}_n}{\mathbf{b}_n}}}}$$

provided that each division can be performed.

Vector continued fraction for $(A_0(z), \ldots, A_{p-1}(z))$

We put the $p+1$ sequences of weights

$$(a_n^{(j)})_{n\geq 0}, \qquad 0 \leq j \leq p,$$

as diagonals of the infinite banded Hessenberg matrix

$$H = \begin{pmatrix} a_0^{(0)} & 1 & & & & 0 \\ a_0^{(1)} & a_1^{(0)} & 1 & & & \\ a_0^{(2)} & a_1^{(1)} & a_2^{(0)} & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ a_0^{(p)} & & \ddots & \ddots & \ddots & \ddots \\ & a_1^{(p)} & & \ddots & \ddots & \ddots \\ & & a_2^{(p)} & & \ddots & \ddots \\ 0 & & & \ddots & & \ddots \end{pmatrix}$$

The relations in Theorem 1 were

$$A_0(z) = \frac{1}{z - \sum_{j=0}^{p} a_0^{(j)} A_{j-1}^{(1)}(z)} \qquad (A_{-1}^{(1)}(z) \equiv 1)$$

$$A_j(z) = A_0(z)\, A_{j-1}^{(1)}(z) \qquad 1 \leq j \leq p.$$

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Equivalently,

$$(A_0, \ldots, A_{p-1}) = \left(\frac{1}{z - \sum_{j=0}^{p} a_0^{(j)} A_{j-1}^{(1)}}, \frac{A_0^{(1)}}{z - \sum_{j=0}^{p} a_0^{(j)} A_{j-1}^{(1)}}, \ldots, \frac{A_{p-2}^{(1)}}{z - \sum_{j=0}^{p} a_0^{(j)} A_{j-1}^{(1)}} \right)$$

$$= \frac{(1, 1, \ldots, 1)}{(A_0^{(1)}, \ldots, A_{p-2}^{(1)}, z - \sum_{j=0}^{p} a_0^{(j)} A_{j-1}^{(1)})}$$

$$= \frac{(1, 1, \ldots, 1)}{(0, \ldots, 0, z - a_0^{(0)}) + (A_0^{(1)}, \ldots, A_{p-2}^{(1)}, -\sum_{j=1}^{p} a_0^{(j)} A_{j-1}^{(1)})}.$$

Now repeat the same procedure to the vector
$\mathbf{v}_1 = (A_0^{(1)}, \ldots, A_{p-2}^{(1)}, -\sum_{j=1}^{p} a_0^{(j)} A_{j-1}^{(1)})$.

Using the relations

$$A_0^{(q)}(z) = \frac{1}{z - a_q^{(0)} - \sum_{j=1}^{p} a_q^{(j)} A_{j-1}^{(q+1)}(z)}$$

$$A_j^{(q)}(z) = A_0^{(q)}(z)\, A_{j-1}^{(q+1)}(z) \qquad 1 \le j \le p$$

for $q = 1, 2$, we get

$$\mathbf{v}_1 = \frac{(1, 1, \ldots, 1)}{(0, \ldots, 0, -a_0^{(1)}, z - a_1^{(0)}) + \mathbf{v}_2}$$

$$\mathbf{v}_2 = (A_0^{(2)}, \ldots, A_{p-3}^{(2)}, -\sum_{j=2}^{p} a_0^{(j)} A_{j-2}^{(2)}, -\sum_{j=1}^{p} a_1^{(j)} A_{j-1}^{(2)})$$

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$$\mathbf{v}_2 = \frac{(1, 1, \ldots, 1)}{(0, \ldots, 0, -a_0^{(2)}, -a_1^{(1)}, z - a_2^{(0)}) + \mathbf{v}_3}$$

$$\mathbf{v}_3 = (A_0^{(3)}, \ldots, A_{p-4}^{(3)}, -\sum_{j=3}^{p} a_0^{(j)} A_{j-3}^{(3)}, -\sum_{j=2}^{p} a_1^{(j)} A_{j-2}^{(3)}, -\sum_{j=1}^{p} a_2^{(j)} A_{j-1}^{(3)}).$$

Continuing in this fashion we get finite vector continued fractions for $(A_0(z), \ldots, A_{p-1}(z))$. Let

$$\mathbf{c}_j := \begin{cases} (1, \ldots, 1), & 1 \leq j \leq p, \\ (-a^{(p)}_{j-p-1}, 0, \ldots, 0), & j \geq p+1, \end{cases}$$

$$\mathbf{d}_j(z) := \begin{cases} (0, \ldots, 0, -a^{(j-1)}_0, -a^{(j-2)}_1, \ldots, -a^{(1)}_{j-2}, z - a^{(0)}_{j-1}), & 1 \leq j \leq p, \\ (-a^{(p-1)}_{j-p}, -a^{(p-2)}_{j-p+1}, -a^{(p-3)}_{j-p+2}, \ldots, -a^{(1)}_{j-2}, z - a^{(0)}_{j-1}), & j \geq p+1. \end{cases}$$

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Then for every $n \ge p+1$,

$$(A_0(z), \ldots, A_{p-1}(z)) = \mathop{\mathbf{K}}_{j=1}^{n} \left(\frac{\mathbf{c}_j}{\widetilde{\mathbf{d}}_j(z)} \right)$$

where $\widetilde{\mathbf{d}}_j(z) = \mathbf{d}_j(z)$ if $j \le n-1$ and $\widetilde{\mathbf{d}}_n(z) = \mathbf{d}_n(z) + \mathbf{v}_n(z)$,

$$\mathbf{v}_n(z) = (-a^{(p)}_{n-p} A_0^{(n)}, -\sum_{j=p-1}^{p} a^{(j)}_{n-p+1} A^{(n)}_{j-p+1}, \ldots, -\sum_{j=1}^{p} a^{(j)}_{n-1} A^{(n)}_{j-1}).$$

Since formula

$$(A_0(z), \ldots, A_{p-1}(z)) = \mathop{\mathbf{K}}_{j=1}^{n} \Big(\frac{\mathbf{c}_j}{\widetilde{\mathbf{d}}_j(z)} \Big)$$

is valid for every $n \geq p+1$, we have the formal identity

$$(A_0(z), \ldots, A_{p-1}(z)) = \mathop{\mathbf{K}}_{j=1}^{\infty} \Big(\frac{\mathbf{c}_j}{\mathbf{d}_j(z)} \Big).$$

Since formula

$$(A_0(z), \ldots, A_{p-1}(z)) = \overset{n}{\underset{j=1}{\mathbf{K}}} \left(\frac{\mathbf{c}_j}{\widetilde{\mathbf{d}}_j(z)} \right)$$

is valid for every $n \geq p+1$, we have the formal identity

$$(A_0(z), \ldots, A_{p-1}(z)) = \overset{\infty}{\underset{j=1}{\mathbf{K}}} \left(\frac{\mathbf{c}_j}{\mathbf{d}_j(z)} \right).$$

The same formula was obtained by Valery A. Kalyagin in 1995 for vectors of resolvent functions of a banded Hessenberg operator.

J. Van Iseghem has several works extending this formula for banded operators with any number of superdiagonals using matrix continued fractions.

Kalyagin's approach and Hermite-Padé approximants

Let H be the infinite banded Hessenberg matrix

$$H = \begin{pmatrix} a_0^{(0)} & 1 & & & & 0 \\ a_0^{(1)} & a_1^{(0)} & 1 & & & \\ a_0^{(2)} & a_1^{(1)} & a_2^{(0)} & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ a_0^{(p)} & & \ddots & \ddots & \ddots & \ddots \\ & a_1^{(p)} & & \ddots & \ddots & \ddots \\ & & a_2^{(p)} & & \ddots & \ddots \\ 0 & & & \ddots & & \ddots \end{pmatrix}$$

Let $\{e_j\}_{j=0}^{\infty}$ denote the standard basis in $\ell^2(\mathbb{Z}_{\geq 0})$. The band structure of H allows us to define the formal Laurent series

$$\phi_j(z) := \langle (zI - H)^{-1} e_j, e_0 \rangle = \sum_{n=0}^{\infty} \frac{\langle H^n e_j, e_0 \rangle}{z^{n+1}}, \qquad 0 \leq j \leq p.$$

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It is easy to prove that

$$\phi_j(z) = A_j(z) = \sum_{n=0}^{\infty} \frac{A_{[n,j]}}{z^{n+1}}, \qquad 0 \leq j \leq p$$

so

$$(\phi_0(z), \ldots, \phi_{p-1}(z)) = \mathop{\mathbf{K}}_{j=1}^{\infty} \left(\frac{\mathbf{c}_j}{\mathbf{d}_j(z)} \right).$$

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Theorem (Kalyagin 1995)

The convergents (finite truncations) of the vector continued fraction are Hermite-Padé approximants at infinity (of type II) for the vector of resolvent functions $(\phi_0(z), \ldots, \phi_{p-1}(z))$.

$$(\phi_0(z), \ldots, \phi_{p-1}(z)) = \underset{j=1}{\overset{\infty}{\mathbf{K}}} \left(\frac{\mathbf{c}_j}{\mathbf{d}_j(z)} \right).$$

Theorem (Kalyagin 1995)

The convergents (finite truncations) of the vector continued fraction are Hermite-Padé approximants at infinity (of type II) for the vector of resolvent functions $(\phi_0(z), \ldots, \phi_{p-1}(z))$.

This means that the vectors of rational functions

$$\underset{j=1}{\overset{n}{\mathbf{K}}} \left(\frac{\mathbf{c}_j}{\mathbf{d}_j(z)} \right) = \left(\frac{q_{n,1}(z)}{q_n(z)}, \frac{q_{n,2}(z)}{q_n(z)}, \ldots, \frac{q_{n,p}(z)}{q_n(z)} \right)$$

satisfy the condition

$$q_n(z)\phi_j(z) - q_{n,j}(z) = O\left(\frac{1}{z^{n_j+1}}\right), \qquad z \to \infty,$$

where $n_j = \lfloor (n-j)/p \rfloor + 1$, for every $1 \leq j \leq p$.

References

For more details, see

1) A. López-García and V.A. Prokhorov, *Lattice paths, vector continued fractions, and resolvents of banded Hessenberg operators*, preprint arXiv:2203.00243.
2) V.A. Kalyagin, *Hermite-Padé approximants and spectral analysis of nonsymmetric operators*, Russian Acad. Sci. Sb. Math. 82 (1995), 199–216.
3) J. Van Iseghem, *Matrix continued fraction for the resolvent function of the band operator*, Acta Appl. Math. 61 (2000), 351–365.