Asymptotics of the optimal values of potentials generated by greedy energy sequences on the unit circle

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Joint work with

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Plan of the talk

1) A brief history of the subject

2) On the unit circle: asymptotics

3) Density of limit points

4) Inequalities

5) The logarithmic case

Some history of the subject

Suppose $K \subset \mathbb{C}$ is an infinite compact set. For each $n \geq 2$, let

$$v_n(K) = \sup\{ \prod_{1 \leq i < j \leq n} |z_i - z_j| : z_1, \ldots, z_n \in K \}. \tag{1}$$

The value $\delta_n(K) = v_n(K)^{2/n(n-1)}$ is called the nth diameter of K. A classical theorem of Fekete (1923) and Pólya-Szegő (1931) asserts that

$$\lim_{n \to \infty} \delta_n(K) = \mathrm{cap}(K).$$

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The limit is also given by

$$\mathrm{cap}(K) = e^{-\mathcal{E}_0(K)}$$

where

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If $\mathrm{cap}(K) > 0$, there is a unique minimizer μ_K for (2), and if $(z_{1,n}, \ldots, z_{n,n}) \in K^n$ is optimal for (1) for each n, then $\frac{1}{n} \sum_{k=1}^{n} \delta_{z_{k,n}} \longrightarrow \mu_K$.

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4) In general, take $a_n \in K$ such that $\prod_{j=0}^{n-1} |a_n - a_j| = \max_{z\in K} \prod_{j=0}^{n-1} |z - a_j|$. The condition is equivalent to

$$\sum_{j=0}^{n-1} \log \frac{1}{|a_n - a_j|} = \min_{z\in K} \sum_{j=0}^{n-1} \log \frac{1}{|z - a_j|}.$$

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Edrei proved that for any infinite sequence $(a_n)_{n=0}^{\infty} \subset K$ constructed this way,

$$\lim_{n\to\infty} \prod_{0\le i<j\le n-1} |a_i - a_j|^{2/n(n-1)} = \mathrm{cap}(K).$$

F. Leja (1957), not aware of the work of Edrei, introduced the same construction and proved that if $\mathrm{cap}(K) > 0$, then

$$\lim_{n\to\infty} \|P_n\|_K^{1/n} = \mathrm{cap}(K)$$

where $P_n(z) = \prod_{j=0}^{n-1}(z - a_j)$. Equivalently, $\frac{1}{n}\sum_{j=0}^{n-1} \log \frac{1}{|a_n - a_j|}$ converges to $\mathcal{E}_0(K)$.

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J. Górski (1957) considered a similar construction for the Newtonian potential in $\mathbb{R}^3$:

$$\sum_{j=0}^{n-1} \frac{1}{|a_n - a_j|} = \inf_{x\in K} \sum_{j=0}^{n-1} \frac{1}{|x - a_j|}.$$

Another well known property: If $K \subset \mathbb{C}$ satisfies $\mathrm{cap}(K) > 0$, then

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{a_j} \longrightarrow \mu_K$$

where μ_K is the unique probability measure on K that minimizes the continuous logarithmic energy.

Notation and some more history

Suppose $K \subset \mathbb{C}$ is an infinite compact. For $s > 0$, a sequence $(a_n)_{n=0}^{\infty} \subset K$ is called a greedy s-energy sequence if for all $n \geq 1$,

$$\sum_{k=0}^{n-1} \frac{1}{|a_n - a_k|^s} = \inf_{z \in K} \sum_{k=0}^{n-1} \frac{1}{|z - a_k|^s}$$

and $(a_n)_{n=0}^{\infty} \subset K$ is an Edrei-Leja sequence if for all $n \geq 1$,

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The Riesz s-potential generated by $a_0, \ldots, a_{N-1}$ is denoted

$$U_{N,s}(z) = \begin{cases} \sum_{k=0}^{N-1} \frac{1}{|z-a_k|^s}, & s > 0, \\ \sum_{k=0}^{N-1} \log \frac{1}{|z-a_k|}, & s = 0, \end{cases} \qquad z \in \mathbb{C}.$$

So the condition is

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If $\mu \in \mathcal{P}(K)$, the Riesz s-energy of μ is denoted

$$I_s(\mu) = \begin{cases} \iint \frac{1}{|z-t|^s} d\mu(z) d\mu(t), & s > 0, \\ \iint \log \frac{1}{|z-t|} d\mu(z) d\mu(t), & s = 0. \end{cases}$$

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Theorem (J. Siciak 1970)

Suppose that the compact $K \subset \mathbb{C}$ has positive Hausdorff dimension $\dim K$ and $0 \leq s < \dim K$. If $(a_n)_{n=0}^{\infty} \subset K$ is a greedy s-energy sequence on K, then

$$\lim_{N \to \infty} \frac{U_{N,s}(a_N)}{N} = \mathcal{E}_s(K),$$

where

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On the unit circle

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In a work studying the Lagrange interpolation properties of Edrei-Leja sequences on the unit circle, Bialas-Ciez and Calvi showed that such sequences are intimately related to the binary decomposition of natural numbers:

Theorem (Calvi and Van Manh, 2011)

Suppose that $(a_n)_{n=0}^{\infty} \subset S^1$ is an Edrei-Leja sequence on the unit circle. Let $N \geq 1$ be an integer, and let

$$N = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_p}, \qquad n_1 > n_2 > \cdots > n_p \geq 0.$$

Then

$$\prod_{k=0}^{N-1} |a_N - a_k| = 2^p,$$

equivalently, $U_{N,0}(a_N) = -p \log 2$.

The goal is to describe the asymptotic behavior of the sequence

$$U_{N,s}(a_N)$$

for all values of $s > 0$.

Let

$$\mathcal{V}_s(N) = \sum_{k=1}^{N} \frac{1}{|e^{\pi i/N} - e^{2\pi ki/N}|^s}, \qquad s > 0,$$

be the minimum value of the Riesz potential on the unit circle generated by the N-th roots of unity.

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In joint work with Ryan McCleary we showed that

$$U_{N,s}(a_N) = \sum_{k=1}^{p} \mathcal{V}_s(2^{n_k})$$

if $N = 2^{n_1} + \cdots + 2^{n_p}$, $n_1 > \cdots > n_p \geq 0$.

Different asymptotic regimes

The study of the asymptotics of $U_{N,s}(a_N)$ is subdivided into three cases:

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Theorem (with R. McCleary)

Let $s > 0$ and $(a_n) \subset S^1$ be a greedy s-energy sequence. Then the following sequences are bounded and divergent:

$$\frac{U_{N,s}(a_N) - N\, I_s(\sigma)}{N^s}, \qquad 0 < s < 1,$$

$$\frac{U_{N,1}(a_N) - \pi^{-1} N \log N}{N}, \qquad s = 1,$$

$$\frac{U_{N,s}(a_N)}{N^s}, \qquad s > 1,$$

where σ is the normalized arclength measure on S^1.

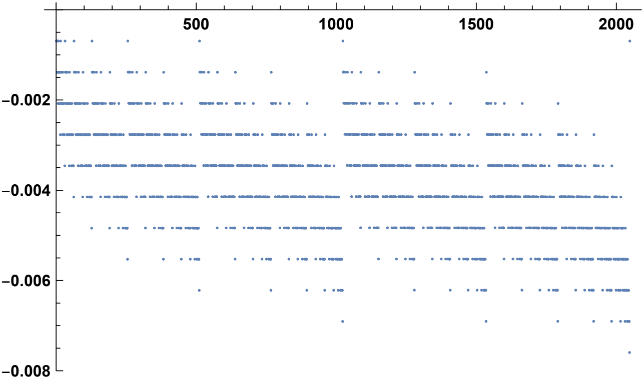

Figure: Plot of the sequence $\left(\frac{U_{N,s}(a_N) - N\, I_s(\sigma)}{N^s}\right)$ for $s = 0.001$, $1 \le N \le 2048$

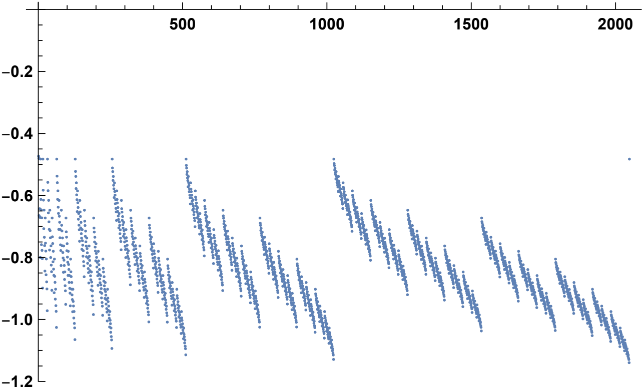

Figure: Plot of the sequence $\left(\frac{U_{N,s}(a_N) - N\, I_s(\sigma)}{N^s}\right)$ for $s = 0.5$

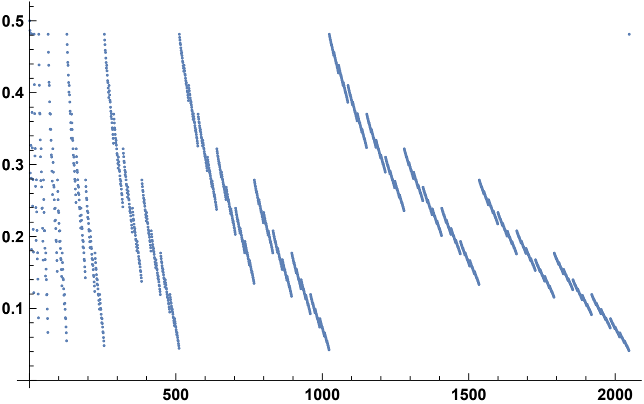

Figure: Plot of the sequence $\left(\frac{U_{N,1}(a_N)-\pi^{-1}N\,\log N}{N}\right)$ for $s=1$

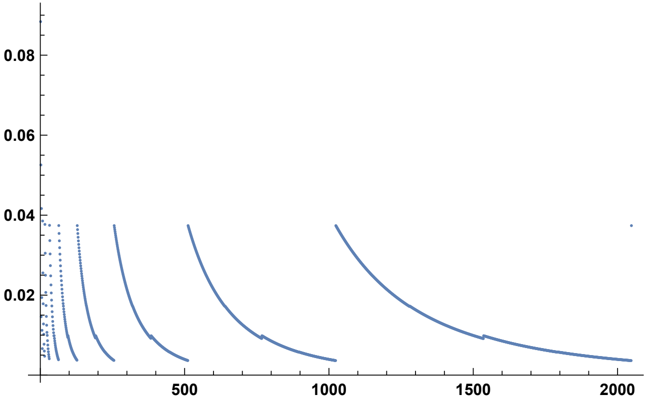

Figure: Plot of the sequence $\left(\frac{U_{N,s}(a_N)}{N^s}\right)$ for $s = 3.5$

Arithmetic functions

For $N \in \mathbb{N}$, let $\eta(N)$ denote the vector

$$\eta(N) = \left(\frac{2^{n_1}}{N}, \frac{2^{n_2}}{N}, \ldots, \frac{2^{n_p}}{N}\right)$$

if $N = 2^{n_1} + \cdots + 2^{n_p}$, $n_1 > \cdots > n_p \geq 0$. So note that

$$\eta(N) = \eta(2N), \qquad N \in \mathbb{N}.$$

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For a vector $\vec{\theta} = (\theta_1, \dots, \theta_r)$ with positive entries, let

$$G(\vec{\theta}, s) = \sum_{k=1}^{r} \theta_k^s,$$

$$\Lambda(\vec{\theta}) = \sum_{k=1}^{r} \theta_k \log \theta_k.$$

The sequences $G(\eta(N), s)$ and $\Lambda(\eta(N))$ play an essential role in the description of the asymptotics of $U_{N,s}(a_N)$.

So for all $N \in \mathbb{N}$

$$G(\eta(N), s) = G(\eta(2N), s), \qquad \Lambda(\eta(N)) = \Lambda(\eta(2N)).$$

The sequences $(\Lambda(\eta(N)))_{N\in\mathbb{N}}$ and $(G(\eta(N), s))_{N\in\mathbb{N}}$ are bounded for any $s > 0$.

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Let

$$F_{N,s} := \begin{cases} \frac{U_{N,s}(a_N) - N\, I_s(\sigma)}{N^s}, & 0 < s < 1, \\ \frac{U_{N,1}(a_N) - \pi^{-1} N \log N}{N}, & s = 1, \\ \frac{U_{N,s}(a_N)}{N^s}, & s > 1. \end{cases}$$

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In joint work with Miña-Díaz we showed that the following asymptotics holds as $N \to \infty$: If $s > 0$ and $s \neq 1$, then

$$F_{N,s} = (2^s - 1)\frac{2\zeta(s)}{(2\pi)^s} G(\eta(N); s) + o(1),$$

where $\zeta(s)$ is the Riemann zeta function. If $s = 1$, then

$$F_{N,1} = \frac{1}{\pi}(\gamma + \log(8/\pi) + \Lambda(\eta(N))) + O(N^{-1}).$$

Limsup, liminf, and density

If $s = 1$, then

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where γ is the Euler-Mascheroni constant $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right)$.
In particular, for all $s > 0$ we have

$$\lim_{N\to\infty}(F_{2N,s} - F_{N,s}) = 0.$$

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If $T_s = [\liminf F_{N,s}, \limsup F_{N,s}]$, then for every $s > 0$, T_s coincides with the set of all limit points of the sequence $(F_{N,s})$.

Inequalities

For all $N \in \mathbb{N}$,

$$1 \leq G(\eta(N), s) < \frac{1}{2^s - 1}, \qquad 0 < s < 1, \tag{3}$$

$$\frac{1}{2^s - 1} < G(\eta(N), s) \leq 1, \qquad s > 1, \tag{4}$$

$$-2 \log 2 < \Lambda(\eta(N)) \leq 0, \tag{5}$$

and the set of limit points of the sequences $G(\eta(N), s)$ and $\Lambda(\eta(N))$ above are the intervals

$$[1, (2^s - 1)^{-1}], \qquad 0 < s < 1,$$
$$[(2^s - 1)^{-1}, 1], \qquad s > 1,$$
$$[-2 \log 2, 0].$$

We used Karamata's inequality to obtain the upper bound in (3) and the lower bounds in (4) and (5).

Finite asymptotic expansion

Using asymptotic formulas for the Riesz s-energy $\mathcal{L}_s(N)$ of the Nth roots of unity by Brauchart, Hardin, and Saff, we can give a finite expansion for $U_{N,s}(a_N)$.

$$\mathcal{V}_s(N) = \frac{\mathcal{L}_s(2N)}{2N} - \frac{\mathcal{L}_s(N)}{N}.$$

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The Riesz s-energy $I_s(\sigma)$ of the normalized arclength measure has the analytic continuation

$$V_s = \frac{2^{-s}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)}, \qquad s \in \mathbb{C} \setminus \{1, 3, 5, \ldots\}.$$

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For the sinc function $\operatorname{sinc} z = \frac{\sin \pi z}{\pi z}$, let

$$\operatorname{sinc}^{-s} z = \sum_{n=0}^{\infty} \alpha_n(s) z^{2n}, \qquad |z| < 1.$$

Theorem (with Miña-Díaz)

For every $s > 0$, s not an odd integer,

$$U_{N,s}(a_N) = v_s N + \frac{2}{(2\pi)^s} \sum_{j=0}^{\lfloor s/2 \rfloor} \alpha_j(s)\zeta(s-2j)(2^{s-2j}-1)G(\eta(N); s-2j)N^{s-2j} + H_{N,s}\,,$$

where the sequence $(H_{N,s})_{N=1}^{\infty}$ is uniformly bounded in N. In the special case when s is a positive even integer, $v_s = 0$ and $H_{N,s} = 0$ for all $N \geq 1$.

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where the sequence $(H_{N,s})_{N=1}^{\infty}$ is uniformly bounded in N. In the special case when s is a positive even integer, $v_s = 0$ and $H_{N,s} = 0$ for all $N \geq 1$.

If $s = 2M+1$, $M \geq 0$ integer,

$$\begin{aligned} U_{N,s}(a_N) = {} & \frac{(\frac{1}{2})_M}{\pi\, 2^{2M} M!} N \log N + \frac{(\frac{1}{2})_M}{\pi\, 2^{2M} M!}(\gamma + \log(4/\pi) + C_M + \Lambda(\eta(N)))N \\ & + \frac{2}{(2\pi)^s} \sum_{j=0}^{M-1} \alpha_j(s)\zeta(s-2j)(2^{s-2j}-1)G(\eta(N); s-2j)N^{s-2j} + H_{N,s}, \end{aligned}$$

where the sequence $(H_{N,s})_{N=1}^{\infty}$ is uniformly bounded in N, C_M is a constant, $(\cdot)_n$ denotes the Pochhammer symbol.

Logarithmic case $s = 0$

If

$$N = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_p}, \qquad n_1 > n_2 > \cdots > n_p \geq 0,$$

we write

$$\tau_b(N) = p.$$

This arithmetic function has the doubling periodicity

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By Calvi and Van Manh we have

$$U_{N,0}(a_N) = -\log(2)\tau_b(N).$$

The following holds:

$$0 < -\frac{U_{N,0}(a_N)}{\log(N+1)} \leq 1, \qquad \text{for all } N \in \mathbb{N}.$$

The set of limit points of this sequence is the interval $[0, 1]$.

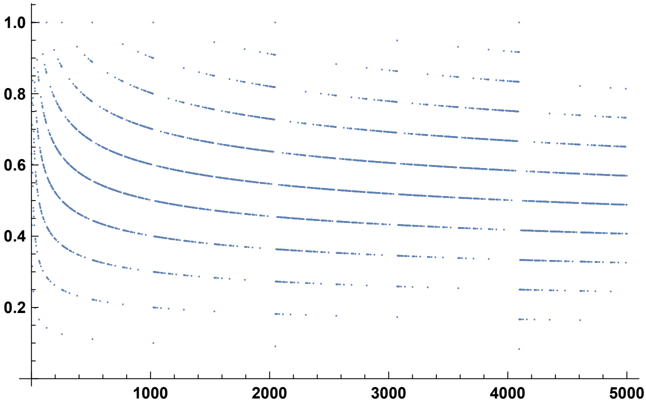

Figure: Plot of the sequence $-\frac{U_{N,0}(a_N)}{\log(N+1)}$ for $1 \leq N \leq 5000$.

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