Lattice paths, vector continued fractions, and resolvents of banded Hessenberg operators

Abey López-García

University of Central Florida

Joint work with

Vasiliy A. Prokhorov (University of South Alabama)

Baylor Analysis Fest

May 23 - 27, 2022

Plan of the talk

1) Lattice paths and associated formal Laurent series.

2) Algebraic relations between the Laurent series.

3) Vector continued fractions, Hermite-Padé approximation, and banded Hessenberg operators.

Plan of the talk

1) Lattice paths and associated formal Laurent series.

2) Algebraic relations between the Laurent series.

3) Vector continued fractions, Hermite-Padé approximation, and banded Hessenberg operators.

The goal is to describe a combinatorial interpretation of certain classes of vector continued fractions. In the scalar case, P. Flajolet (1980) described the connection between standard continued fractions and Motzkin/Dyck paths.

Let

$$\mathcal{V} = \mathbb{Z}_{\geq 0} \times \mathbb{Z}.$$

If $v, v' \in \mathcal{V}$, then $v \to v'$ denotes the segment from v to v'.

Let

$$\mathcal{V} = \mathbb{Z}_{\geq 0} \times \mathbb{Z}.$$

If $v, v' \in \mathcal{V}$, then $v \to v'$ denotes the segment from v to v'.

Fix an integer $p \geq 1$. A **lattice path** is a concatenation of finitely many segments (called **steps**)

$$\gamma = e_1 e_2 \cdots e_k.$$

The segments belong to any of the following collections:

$$\mathcal{E}_u := \{(n,m) \to (n+1, m+1) : (n,m) \in \mathcal{V}\} \qquad \text{(upsteps)}$$

$$\mathcal{E}_\ell := \{(n,m) \to (n+1, m) : (n,m) \in \mathcal{V}\} \qquad \text{(level steps)}$$

$$\mathcal{E}_d := \{(n,m) \to (n+1, m-j) : (n,m) \in \mathcal{V}, \ \ 1 \leq j \leq p\} \qquad \text{(downsteps)}$$

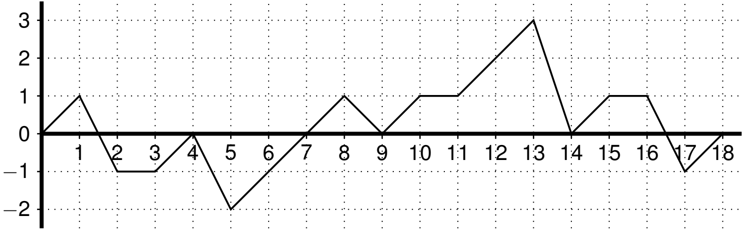


Figure: Example of a lattice path in the case $p = 3$.

Some definitions

We say that the path

$$\gamma = e_1 e_2 \cdots e_k$$

has **length** k.

A path of length zero is just a point in $\mathcal{V}$.

Some definitions

We say that the path

$$\gamma = e_1 e_2 \cdots e_k$$

has **length** k.

A path of length zero is just a point in $\mathcal{V}$.

If $(n, m) \in \mathcal{V}$ is a vertex in γ, we say that γ has height m at n.

We define $\mathtt{max}(\gamma)$ to be the maximum of the heights of all the vertices in γ, and $\mathtt{min}(\gamma)$ to be the minimum of those heights.

Some definitions

We say that the path

$$\gamma = e_1 e_2 \cdots e_k$$

has **length** k.

A path of length zero is just a point in $\mathcal{V}$.

If $(n, m) \in \mathcal{V}$ is a vertex in γ, we say that γ has height m at n.

We define $\max(\gamma)$ to be the maximum of the heights of all the vertices in γ, and $\min(\gamma)$ to be the minimum of those heights.

If $q \geq 0$ is an integer, then $\gamma + q$ denotes the path obtained by shifting γ upwards q units, and $\gamma - q$ is the path obtained by shifting γ downwards q units.

Weights

Paths will be given a weight (or label) as follows.

First, fix a collection of $p+1$ bi-infinite sequences of complex numbers $(a_n^{(j)})_{n\in\mathbb{Z}}$, $0 \le j \le p$.

Weights

Paths will be given a weight (or label) as follows.

First, fix a collection of $p+1$ bi-infinite sequences of complex numbers $(a_n^{(j)})_{n\in\mathbb{Z}}$, $0 \le j \le p$.

The **weight of a segment** is:

$$\begin{aligned} w((n,m) \to (n+1, m+1)) &= 1, \\ w((n,m) \to (n+1, m-j)) &= a_{m-j}^{(j)}, \qquad 0 \le j \le p. \end{aligned}$$

Weights

Paths will be given a weight (or label) as follows.

First, fix a collection of $p+1$ bi-infinite sequences of complex numbers $(a_n^{(j)})_{n\in\mathbb{Z}}$, $0 \le j \le p$.

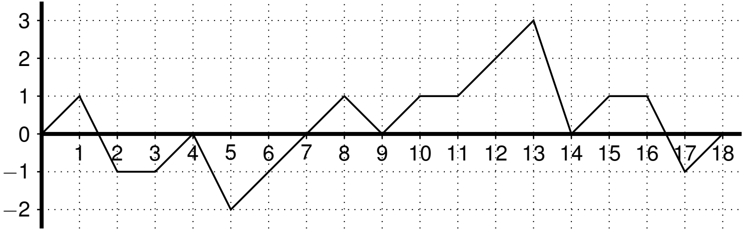
The **weight of a segment** is:

$$w((n,m) \to (n+1, m+1)) = 1,$$
$$w((n,m) \to (n+1, m-j)) = a_{m-j}^{(j)}, \qquad 0 \le j \le p.$$

The **weight of a path** is

$$w(\gamma) = \prod_{e \subset \gamma} w(e)$$

where the product runs over the different steps of γ. If γ has length zero then the weight of γ is by definition 1.



The path γ above has length 18, $\max(\gamma) = 3$, $\min(\gamma) = -2$, and

$$w(\gamma) = (a_{-1}^{(2)})^2\, a_{-1}^{(0)}\, a_{-2}^{(2)}\, a_0^{(1)}\, (a_1^{(0)})^2\, a_0^{(3)}$$

Families of lattice paths

For each $n \in \mathbb{Z}_{\geq 0}$ and $0 \leq j \leq p$, let $\mathcal{P}_{[n,j]}$ denote the collection of all paths of length n, with starting point $(0,0)$ and terminal point (n,j).

Families of lattice paths

For each $n \in \mathbb{Z}_{\geq 0}$ and $0 \leq j \leq p$, let $\mathcal{P}_{[n,j]}$ denote the collection of all paths of length n, with starting point $(0,0)$ and terminal point (n,j).

For $n \in \mathbb{Z}_{\geq 0}$ and $0 \leq j \leq p$, let

$$\mathcal{D}_{[n,j]} := \{\gamma \in \mathcal{P}_{[n,j]} : \min(\gamma) = 0\}.$$

The paths in $\mathcal{D}_{[n,j]}$ are called **partial p-Łukasiewicz paths**.

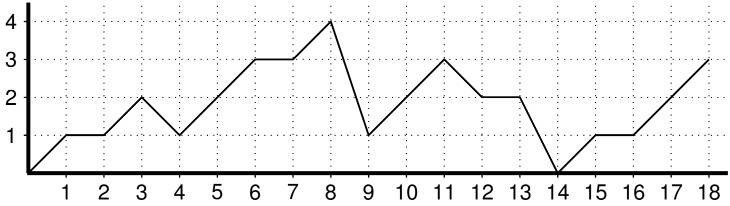


Figure: Example, in the case $p = 3$, of a path in the collection $\mathcal{D}_{[18,3]}$.

For $n \in \mathbb{Z}_{\geq 0}$ and $0 \leq j \leq p$, let $\widehat{\mathcal{D}}_{[n,j]}$ denote the collection of all paths γ of length n, with initial point $(0, -j)$, final point $(n, 0)$, and satisfying $\mathtt{max}(\gamma) = 0$.

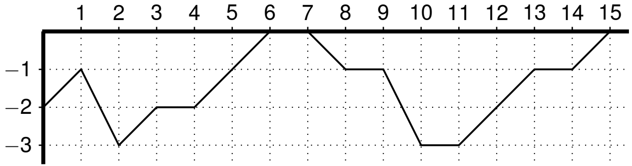


Figure: Example, in the case $p = 2$, of a path in the collection $\widehat{\mathcal{D}}_{[15,2]}$.

Weight polynomials and formal Laurent series

If $\mathcal{L}$ is a finite collection of lattice paths, the expression

$$\sum_{\gamma \in \mathcal{L}} w(\gamma)$$

is called the **weight polynomial associated with** $\mathcal{L}$.

Weight polynomials and formal Laurent series

If $\mathcal{L}$ is a finite collection of lattice paths, the expression

$$\sum_{\gamma \in \mathcal{L}} w(\gamma)$$

is called the **weight polynomial associated with** $\mathcal{L}$.

If $\mathcal{L} = \emptyset$, then the weight polynomial for $\mathcal{L}$ is by definition 0.

Weight polynomials and formal Laurent series

If $\mathcal{L}$ is a finite collection of lattice paths, the expression

$$\sum_{\gamma \in \mathcal{L}} w(\gamma)$$

is called the **weight polynomial associated with** $\mathcal{L}$.

If $\mathcal{L} = \emptyset$, then the weight polynomial for $\mathcal{L}$ is by definition 0.

We define the weight polynomials

$$W_{[n,j]} := \sum_{\gamma \in \mathcal{P}_{[n,j]}} w(\gamma)$$

$$A_{[n,j]} := \sum_{\gamma \in \mathcal{D}_{[n,j]}} w(\gamma)$$

$$B_{[n,j]} := \sum_{\gamma \in \widehat{\mathcal{D}}_{[n,j]}} w(\gamma)$$

We also need to define the following weight polynomials. For an integer $q \geq 0$, let

$$A_{[n,j]}^{(q)} := \sum_{\gamma \in \mathcal{D}_{[n,j]}} w(\gamma + q),$$

$$B_{[n,j]}^{(q)} := \sum_{\gamma \in \widehat{\mathcal{D}}_{[n,j]}} w(\gamma - q).$$

We also need to define the following weight polynomials. For an integer $q \geq 0$, let

$$
\begin{aligned}
A_{[n,j]}^{(q)} &:= \sum_{\gamma \in \mathcal{D}_{[n,j]}} w(\gamma + q), \\
B_{[n,j]}^{(q)} &:= \sum_{\gamma \in \widehat{\mathcal{D}}_{[n,j]}} w(\gamma - q).
\end{aligned}
$$

Let $\mathbb{C}((z^{-1}))$ be the algebraic field of all formal Laurent series

$$a(z) = \sum_{k \in \mathbb{Z}} \frac{a_k}{z^k}$$

with complex coefficients such that only finitely many a_k with $k < 0$ are non-zero.

We put the weight polynomials as coefficients of the following formal Laurent series: For each $0 \leq j \leq p$, let

$$A_j(z) := \sum_{n=0}^{\infty} \frac{A_{[n,j]}}{z^{n+1}}$$

$$B_j(z) := \sum_{n=0}^{\infty} \frac{B_{[n,j]}}{z^{n+1}}$$

$$W_j(z) := \sum_{n=0}^{\infty} \frac{W_{[n,j]}}{z^{n+1}}$$

We also define for an integer $q \geq 0$ the series

$$A_j^{(q)}(z) := \sum_{n=0}^{\infty} \frac{A_{[n,j]}^{(q)}}{z^{n+1}}$$

$$B_j^{(q)}(z) := \sum_{n=0}^{\infty} \frac{B_{[n,j]}^{(q)}}{z^{n+1}}$$

It is easy to see that

$$A_j(z) = \frac{1}{z^{j+1}} + O\left(\frac{1}{z^{j+2}}\right)$$

and the same can be said about the other Laurent series with subindex j.

It is easy to see that

$$A_j(z) = \frac{1}{z^{j+1}} + O\left(\frac{1}{z^{j+2}}\right)$$

and the same can be said about the other Laurent series with subindex j.

Theorem

The following relations hold:

$$A_0(z) = \frac{1}{z - a_0^{(0)} - \sum_{j=1}^{p} a_0^{(j)} A_{j-1}^{(1)}(z)} \tag{1}$$

$$A_j(z) = A_0(z)\, A_{j-1}^{(1)}(z) \qquad 1 \leq j \leq p. \tag{2}$$

It is easy to see that

$$A_j(z) = \frac{1}{z^{j+1}} + O\left(\frac{1}{z^{j+2}}\right)$$

and the same can be said about the other Laurent series with subindex j.

Theorem

The following relations hold:

$$A_0(z) = \frac{1}{z - a_0^{(0)} - \sum_{j=1}^{p} a_0^{(j)} A_{j-1}^{(1)}(z)} \tag{1}$$

$$A_j(z) = A_0(z) A_{j-1}^{(1)}(z) \qquad 1 \leq j \leq p. \tag{2}$$

Analogous relations hold between the series $A_j^{(q)}(z)$ and $A_j^{(q+1)}(z)$ for every integer q:

$$\begin{aligned} A_0^{(q)}(z) &= \frac{1}{z - a_q^{(0)} - \sum_{j=1}^{p} a_q^{(j)} A_{j-1}^{(q+1)}(z)} \\ A_j^{(q)}(z) &= A_0^{(q)}(z) A_{j-1}^{(q+1)}(z) \qquad 1 \leq j \leq p. \end{aligned} \tag{3}$$

The relations (1)–(3) allow us to obtain a vector continued fraction expansion for the vector $(A_0(z), A_1(z), \ldots, A_{p-1}(z))$.

The relations (1)–(3) allow us to obtain a vector continued fraction expansion for the vector $(A_0(z), A_1(z), \ldots, A_{p-1}(z))$.

Other relations one can prove are:

$$zA_0(z) - 1 = \sum_{j=0}^{p} a_0^{(j)} A_j(z)$$

$$A_j(z) = A_i(z)\, A_{j-i-1}^{(i+1)}(z) \qquad 0 \leq i < j \leq p.$$

The relations (1)–(3) allow us to obtain a vector continued fraction expansion for the vector $(A_0(z), A_1(z), \ldots, A_{p-1}(z))$.

Other relations one can prove are:

$$zA_0(z) - 1 = \sum_{j=0}^{p} a_0^{(j)} A_j(z)$$

$$A_j(z) = A_i(z)\, A_{j-i-1}^{(i+1)}(z) \qquad 0 \le i < j \le p.$$

Theorem

The following relations hold:

$$W_0(z) = \frac{1}{z - a_0^{(0)} - \sum_{j=1}^{p} \sum_{k=0}^{j} a_{-k}^{(j)}\, A_{j-k-1}^{(1)}(z) B_{k-1}^{(1)}(z)}$$

$$W_j(z) = W_i(z)\, A_{j-i-1}^{(i+1)}(z) \qquad 0 \le i < j \le p,$$

where in the first formula we understand that $A_{-1}^{(1)}(z) \equiv B_{-1}^{(1)}(z) \equiv 1$.

Vector continued fractions

Vector continued fractions appeared first, at least disguised in the form of equations, in a number theory work of Jacobi (published posthumously in 1868), concerning the problem of simultaneous rational approximation of numbers.

Vector continued fractions

Vector continued fractions appeared first, at least disguised in the form of equations, in a number theory work of Jacobi (published posthumously in 1868), concerning the problem of simultaneous rational approximation of numbers.

He proposed an algorithm that was further studied by Perron (1907) and others.

Vector continued fractions

Vector continued fractions appeared first, at least disguised in the form of equations, in a number theory work of Jacobi (published posthumously in 1868), concerning the problem of simultaneous rational approximation of numbers.

He proposed an algorithm that was further studied by Perron (1907) and others.

In function theory, the **Jacobi-Perron algorithm** is an algorithm to expand a vector of power series in a continued fraction where the partial numerators and denominators are vectors of polynomials.

Let $\mathbf{F} := \mathbb{C}((z^{-1}))$. In $\mathbf{F}^p$ we define the following **division** operation: If $y_p \neq 0$,

$$\frac{(x_1, \ldots, x_p)}{(y_1, \ldots, y_p)} := \left(\frac{x_1}{y_p}, \frac{x_2\, y_1}{y_p}, \frac{x_3\, y_2}{y_p}, \ldots, \frac{x_p\, y_{p-1}}{y_p} \right).$$

Let $\mathbf{F} := \mathbb{C}((z^{-1}))$. In $\mathbf{F}^p$ we define the following **division** operation: If $y_p \neq 0$,

$$\frac{(x_1, \ldots, x_p)}{(y_1, \ldots, y_p)} := \left(\frac{x_1}{y_p}, \frac{x_2\, y_1}{y_p}, \frac{x_3\, y_2}{y_p}, \ldots, \frac{x_p\, y_{p-1}}{y_p} \right).$$

In particular

$$\frac{(1, \ldots, 1)}{(y_1, \ldots, y_p)} := \left(\frac{1}{y_p}, \frac{y_1}{y_p}, \frac{y_2}{y_p}, \ldots, \frac{y_{p-1}}{y_p} \right).$$

Let $\mathbf{F} := \mathbb{C}((z^{-1}))$. In $\mathbf{F}^p$ we define the following **division** operation: If $y_p \neq 0$,

$$\frac{(x_1, \ldots, x_p)}{(y_1, \ldots, y_p)} := \left(\frac{x_1}{y_p}, \frac{x_2\, y_1}{y_p}, \frac{x_3\, y_2}{y_p}, \ldots, \frac{x_p\, y_{p-1}}{y_p} \right).$$

In particular

$$\frac{(1, \ldots, 1)}{(y_1, \ldots, y_p)} := \left(\frac{1}{y_p}, \frac{y_1}{y_p}, \frac{y_2}{y_p}, \ldots, \frac{y_{p-1}}{y_p} \right).$$

If $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are now vectors of formal Laurent series, then we can form the finite continued fraction

$$\mathop{\mathbf{K}}_{m=1}^{n} \left(\frac{\mathbf{a}_m}{\mathbf{b}_m} \right) := \cfrac{\mathbf{a}_1}{\mathbf{b}_1 + \cfrac{\mathbf{a}_2}{\mathbf{b}_2 + \cfrac{\mathbf{a}_3}{\ddots + \cfrac{\mathbf{a}_n}{\mathbf{b}_n}}}}$$

provided that each division can be performed.

Vector continued fraction for $(A_0(z), \ldots, A_{p-1}(z))$

We put the $p+1$ sequences of weights

$$(a_n^{(j)})_{n\geq 0}, \qquad 0 \leq j \leq p,$$

as diagonals of the infinite banded Hessenberg matrix

$$H = \begin{pmatrix} a_0^{(0)} & 1 & & & & 0 \\ a_0^{(1)} & a_1^{(0)} & 1 & & & \\ a_0^{(2)} & a_1^{(1)} & a_2^{(0)} & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ a_0^{(p)} & & \ddots & \ddots & \ddots & \ddots \\ & a_1^{(p)} & & \ddots & \ddots & \ddots \\ & & a_2^{(p)} & & \ddots & \ddots \\ 0 & & & \ddots & & \ddots \end{pmatrix}$$

The relations in Theorem 1 were

$$A_0(z) = \frac{1}{z - \sum_{j=0}^{p} a_0^{(j)} A_{j-1}^{(1)}(z)} \qquad (A_{-1}^{(1)}(z) \equiv 1)$$

$$A_j(z) = A_0(z)\, A_{j-1}^{(1)}(z) \qquad 1 \leq j \leq p.$$

The relations in Theorem 1 were

$$A_0(z) = \frac{1}{z - \sum_{j=0}^{p} a_0^{(j)}\, A_{j-1}^{(1)}(z)} \qquad (A_{-1}^{(1)}(z) \equiv 1)$$

$$A_j(z) = A_0(z)\, A_{j-1}^{(1)}(z) \qquad 1 \le j \le p.$$

Equivalently,

$$(A_0, \ldots, A_{p-1}) = \left(\frac{1}{z - \sum_{j=0}^{p} a_0^{(j)} A_{j-1}^{(1)}}, \frac{A_0^{(1)}}{z - \sum_{j=0}^{p} a_0^{(j)} A_{j-1}^{(1)}}, \ldots, \frac{A_{p-2}^{(1)}}{z - \sum_{j=0}^{p} a_0^{(j)} A_{j-1}^{(1)}} \right)$$

$$= \frac{(1, 1, \ldots, 1)}{(A_0^{(1)}, \ldots, A_{p-2}^{(1)}, z - \sum_{j=0}^{p} a_0^{(j)} A_{j-1}^{(1)})}$$

$$= \frac{(1, 1, \ldots, 1)}{(0, \ldots, 0, z - a_0^{(0)}) + (A_0^{(1)}, \ldots, A_{p-2}^{(1)}, -\sum_{j=1}^{p} a_0^{(j)} A_{j-1}^{(1)})}.$$

Now repeat the same procedure to the vector
$\mathbf{v}_1 = (A_0^{(1)}, \ldots, A_{p-2}^{(1)}, -\sum_{j=1}^{p} a_0^{(j)} A_{j-1}^{(1)})$.

Using the relations

$$A_0^{(q)}(z) = \frac{1}{z - a_q^{(0)} - \sum_{j=1}^{p} a_q^{(j)} \, A_{j-1}^{(q+1)}(z)}$$

$$A_j^{(q)}(z) = A_0^{(q)}(z) \, A_{j-1}^{(q+1)}(z) \qquad 1 \leq j \leq p$$

for $q = 1, 2$, we get

$$\mathbf{v}_1 = \frac{(1, 1, \ldots, 1)}{(0, \ldots, 0, -a_0^{(1)}, z - a_1^{(0)}) + \mathbf{v}_2}$$

$$\mathbf{v}_2 = (A_0^{(2)}, \ldots, A_{p-3}^{(2)}, -\sum_{j=2}^{p} a_0^{(j)} A_{j-2}^{(2)}, -\sum_{j=1}^{p} a_1^{(j)} A_{j-1}^{(2)})$$

Using the relations

$$A_0^{(q)}(z) = \frac{1}{z - a_q^{(0)} - \sum_{j=1}^{p} a_q^{(j)}\, A_{j-1}^{(q+1)}(z)}$$

$$A_j^{(q)}(z) = A_0^{(q)}(z)\, A_{j-1}^{(q+1)}(z) \qquad 1 \le j \le p$$

for $q = 1, 2$, we get

$$\mathbf{v}_1 = \frac{(1, 1, \ldots, 1)}{(0, \ldots, 0, -a_0^{(1)}, z - a_1^{(0)}) + \mathbf{v}_2}$$

$$\mathbf{v}_2 = (A_0^{(2)}, \ldots, A_{p-3}^{(2)}, -\sum_{j=2}^{p} a_0^{(j)} A_{j-2}^{(2)}, -\sum_{j=1}^{p} a_1^{(j)} A_{j-1}^{(2)})$$

$$\mathbf{v}_2 = \frac{(1, 1, \ldots, 1)}{(0, \ldots, 0, -a_0^{(2)}, -a_1^{(1)}, z - a_2^{(0)}) + \mathbf{v}_3}$$

$$\mathbf{v}_3 = (A_0^{(3)}, \ldots, A_{p-4}^{(3)}, -\sum_{j=3}^{p} a_0^{(j)} A_{j-3}^{(3)}, -\sum_{j=2}^{p} a_1^{(j)} A_{j-2}^{(3)}, -\sum_{j=1}^{p} a_2^{(j)} A_{j-1}^{(3)}).$$

Continuing in this fashion we get finite vector continued fractions for $(A_0(z), \ldots, A_{p-1}(z))$. Let

$$\mathbf{c}_j := \begin{cases} (1, \ldots, 1), & 1 \leq j \leq p, \\ (-a_{j-p-1}^{(p)}, 0, \ldots, 0), & j \geq p+1, \end{cases}$$

$$\mathbf{d}_j(z) := \begin{cases} (0, \ldots, 0, -a_0^{(j-1)}, -a_1^{(j-2)}, \ldots, -a_{j-2}^{(1)}, z - a_{j-1}^{(0)}), & 1 \leq j \leq p, \\ (-a_{j-p}^{(p-1)}, -a_{j-p+1}^{(p-2)}, -a_{j-p+2}^{(p-3)}, \ldots, -a_{j-2}^{(1)}, z - a_{j-1}^{(0)}), & j \geq p+1. \end{cases}$$

Continuing in this fashion we get finite vector continued fractions for $(A_0(z), \ldots, A_{p-1}(z))$. Let

$$\mathbf{c}_j := \begin{cases} (1, \ldots, 1), & 1 \le j \le p, \\ (-a_{j-p-1}^{(p)}, 0, \ldots, 0), & j \ge p+1, \end{cases}$$

$$\mathbf{d}_j(z) := \begin{cases} (0, \ldots, 0, -a_0^{(j-1)}, -a_1^{(j-2)}, \ldots, -a_{j-2}^{(1)}, z - a_{j-1}^{(0)}), & 1 \le j \le p, \\ (-a_{j-p}^{(p-1)}, -a_{j-p+1}^{(p-2)}, -a_{j-p+2}^{(p-3)}, \ldots, -a_{j-2}^{(1)}, z - a_{j-1}^{(0)}), & j \ge p+1. \end{cases}$$

Then for every $n \ge p+1$,

$$(A_0(z), \ldots, A_{p-1}(z)) = \mathop{\mathbf{K}}_{j=1}^{n} \left(\frac{\mathbf{c}_j}{\widetilde{\mathbf{d}}_j(z)} \right)$$

where $\widetilde{\mathbf{d}}_j(z) = \mathbf{d}_j(z)$ if $j \le n-1$ and $\widetilde{\mathbf{d}}_n(z) = \mathbf{d}_n(z) + \mathbf{v}_n(z)$,

$$\mathbf{v}_n(z) = (-a_{n-p}^{(p)} A_0^{(n)}, -\sum_{j=p-1}^{p} a_{n-p+1}^{(j)} A_{j-p+1}^{(n)}, \ldots, -\sum_{j=1}^{p} a_{n-1}^{(j)} A_{j-1}^{(n)}).$$

Since formula

$$(A_0(z), \ldots, A_{p-1}(z)) = \mathop{\mathbf{K}}_{j=1}^{n} \left(\frac{\mathbf{c}_j}{\widetilde{\mathbf{d}}_j(z)} \right)$$

is valid for every $n \geq p+1$, we have the formal identity

$$(A_0(z), \ldots, A_{p-1}(z)) = \mathop{\mathbf{K}}_{j=1}^{\infty} \left(\frac{\mathbf{c}_j}{\mathbf{d}_j(z)} \right).$$

Since formula

$$(A_0(z), \ldots, A_{p-1}(z)) = \mathop{\mathbf{K}}_{j=1}^{n} \left(\frac{\mathbf{c}_j}{\widetilde{\mathbf{d}}_j(z)} \right)$$

is valid for every $n \geq p+1$, we have the formal identity

$$(A_0(z), \ldots, A_{p-1}(z)) = \mathop{\mathbf{K}}_{j=1}^{\infty} \left(\frac{\mathbf{c}_j}{\mathbf{d}_j(z)} \right).$$

The same formula was obtained by Valery A. Kalyagin in 1995 for vectors of resolvent functions of a banded Hessenberg operator.

J. Van Iseghem has several works extending this formula for banded operators with any number of superdiagonals using matrix continued fractions.

Kalyagin's approach and Hermite-Padé approximants

Let H be the infinite banded Hessenberg matrix

$$H = \begin{pmatrix} a_0^{(0)} & 1 & & & & 0 \\ a_0^{(1)} & a_1^{(0)} & 1 & & & \\ a_0^{(2)} & a_1^{(1)} & a_2^{(0)} & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ a_0^{(p)} & & \ddots & \ddots & \ddots & \ddots \\ & a_1^{(p)} & & \ddots & \ddots & \ddots \\ & & a_2^{(p)} & & \ddots & \ddots \\ 0 & & & \ddots & & \ddots \end{pmatrix}$$

Let $\{e_j\}_{j=0}^{\infty}$ denote the standard basis in $\ell^2(\mathbb{Z}_{\geq 0})$. The band structure of H allows us to define the formal Laurent series

$$\phi_j(z) := \langle (zI - H)^{-1} e_j, e_0 \rangle = \sum_{n=0}^{\infty} \frac{\langle H^n e_j, e_0 \rangle}{z^{n+1}}, \qquad 0 \leq j \leq p.$$

Let $\{e_j\}_{j=0}^{\infty}$ denote the standard basis in $\ell^2(\mathbb{Z}_{\geq 0})$. The band structure of H allows us to define the formal Laurent series

$$\phi_j(z) := \langle (zI - H)^{-1} e_j, e_0 \rangle = \sum_{n=0}^{\infty} \frac{\langle H^n e_j, e_0 \rangle}{z^{n+1}}, \qquad 0 \leq j \leq p.$$

It is easy to prove that

$$\phi_j(z) = A_j(z) = \sum_{n=0}^{\infty} \frac{A_{[n,j]}}{z^{n+1}}, \qquad 0 \leq j \leq p$$

so

$$(\phi_0(z), \ldots, \phi_{p-1}(z)) = \mathop{\mathbf{K}}_{j=1}^{\infty} \left(\frac{\mathbf{c}_j}{\mathbf{d}_j(z)} \right).$$

$$(\phi_0(z), \ldots, \phi_{p-1}(z)) = \mathop{\mathbf{K}}_{j=1}^{\infty} \left(\frac{\mathbf{c}_j}{\mathbf{d}_j(z)} \right).$$

Theorem (Kalyagin 1995)

The convergents (finite truncations) of the vector continued fraction are Hermite-Padé approximants at infinity (of type II) for the vector of resolvent functions $(\phi_0(z), \ldots, \phi_{p-1}(z))$.

$$(\phi_0(z), \ldots, \phi_{p-1}(z)) = \mathop{\mathbf{K}}_{j=1}^{\infty} \left(\frac{\mathbf{c}_j}{\mathbf{d}_j(z)} \right).$$

Theorem (Kalyagin 1995)

The convergents (finite truncations) of the vector continued fraction are Hermite-Padé approximants at infinity (of type II) for the vector of resolvent functions $(\phi_0(z), \ldots, \phi_{p-1}(z))$.

This means that the vectors of rational functions

$$\mathop{\mathbf{K}}_{j=1}^{n} \left(\frac{\mathbf{c}_j}{\mathbf{d}_j(z)} \right) = \left(\frac{q_{n,1}(z)}{q_n(z)}, \frac{q_{n,2}(z)}{q_n(z)}, \ldots, \frac{q_{n,p}(z)}{q_n(z)} \right)$$

satisfy the condition

$$q_n(z)\phi_j(z) - q_{n,j}(z) = O\left(\frac{1}{z^{n_j+1}} \right), \qquad z \to \infty,$$

where $n_j = \lfloor (n-j)/p \rfloor + 1$, for every $1 \leq j \leq p$.

References

For more details, see

1) A. López-García and V.A. Prokhorov, *Lattice paths, vector continued fractions, and resolvents of banded Hessenberg operators*, preprint arXiv:2203.00243.
2) V.A. Kalyagin, *Hermite-Padé approximants and spectral analysis of nonsymmetric operators*, Russian Acad. Sci. Sb. Math. 82 (1995), 199–216.
3) J. Van Iseghem, *Matrix continued fraction for the resolvent function of the band operator*, Acta Appl. Math. 61 (2000), 351–365.