Asymptotic properties of the Riesz energy of sections of greedy sequences on the unit circle

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Joint work with

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On the unit circle and notation

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Let $s \in \mathbb{R}$ be fixed. We say that $(a_n)_{n=0}^{\infty} \subset S^1$ is a **greedy s-energy sequence** if for every $n \geq 1$,

$$\sum_{j=0}^{n-1} \frac{1}{|a_n - a_j|^s} = \min_{z \in S^1} \sum_{j=0}^{n-1} \frac{1}{|z - a_j|^s}, \qquad \text{if } s > 0,$$

$$\sum_{j=0}^{n-1} \log \frac{1}{|a_n - a_j|} = \min_{z \in S^1} \sum_{j=0}^{n-1} \log \frac{1}{|z - a_j|}, \qquad \text{if } s = 0,$$

$$\sum_{j=0}^{n-1} |a_n - a_j|^{-s} = \max_{z \in S^1} \sum_{j=0}^{n-1} |z - a_j|^{-s}, \qquad \text{if } s < 0.$$

If $\omega = \{z_1, \ldots, z_N\} \subset S^1$ is a multiset of $N \geq 2$ points on the unit circle and $s \in \mathbb{R}$, the **Riesz s-energy of** ω is by defnition

$$E_s(\omega) = \begin{cases} 2\sum_{i<j} |z_i - z_j|^{-s} & \text{if } s \neq 0, \\ 2\sum_{i<j} \log \frac{1}{|z_i - z_j|} & \text{if } s = 0. \end{cases}$$

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So greedy sequences $(a_n)_{n=0}^{\infty} \subset S^1$ are equivalently defined as those satisfying that for every $n \geq 1$,

$$E_s(\{a_0, \ldots, a_n\}) = \min_{z \in S^1} E_s(\{a_0, \ldots, a_{n-1}, z\}) \quad \text{if } s \geq 0,$$
$$E_s(\{a_0, \ldots, a_n\}) = \max_{z \in S^1} E_s(\{a_0, \ldots, a_{n-1}, z\}) \quad \text{if } s < 0.$$

Let

$$\alpha_{N,s} := \{a_0, \ldots, a_{N-1}\}, \qquad N \geq 1.$$

The case $s \leq -2$ is degenerate: If $s < -2$, greedy sequences are concentrated on antipodal points:

$$\{a_n\}_{n=0}^{\infty} \subset \{a_0, a_1\}, \qquad a_1 = -a_0.$$

If $s = -2$, greedy sequences are characterized as those satisfying $a_{2k+1} = -a_{2k}$ for each $k \geq 0$.

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So if $s \leq -2$, then

$$E_s(\alpha_{N,s}) = \begin{cases} 2^{1-s} n^2, & N = 2n, \\ 2^{1-s}(n^2 + n), & N = 2n+1. \end{cases}$$

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So the behavior of $E_s(\alpha_{N,s})$ is trivial if $s \leq -2$.

The interesting case to analyze is when $s > -2$, which we assume to hold in the rest of this talk.

In a work studying the Lagrange interpolation properties of Leja sequences on the unit circle, Bialas-Ciez and Calvi showed that such sequences are intimately related to the binary decomposition of natural numbers.

Theorem (Calvi and Van Manh, 2011)

Suppose that $(a_n)_{n=0}^{\infty} \subset S^1$ *is a Leja sequence on the unit circle (i.e., greedy sequence for* $s = 0$*). Let* $N \geq 1$ *be an integer, and let*

$$N = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_p}, \qquad n_1 > n_2 > \cdots > n_p \geq 0.$$

Then

$$\prod_{k=0}^{N-1} |a_N - a_k| = 2^p,$$

equivalently

$$\min_{z \in S^1} \sum_{k=0}^{N-1} \log \frac{1}{|z - a_k|} = \sum_{k=0}^{N-1} \log \frac{1}{|a_N - a_k|} = -p \log 2.$$

Geometry

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Bialas-Ciez and Calvi showed that if $(a_n)_{n=0}^{\infty} \subset S^1$ is a Leja sequence with $a_0 = 1$ and

$$N = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_p}, \qquad n_1 > n_2 > \cdots > n_p \ge 0,$$

then the set

$$\alpha_{N,0} = \{a_0, \ldots, a_{N-1}\}$$

is the union of p disjoint configurations which are rotated copies of the sets $\omega_{2^{n_i}}$:

$$\alpha_{N,0} = \omega_{2^{n_1}} \cup \rho_1\, \omega_{2^{n_2}} \cup \rho_1 \rho_2\, \omega_{2^{n_3}} \cup \cdots \cup (\rho_1 \cdots \rho_p)\, \omega_{2^{n_p}}$$

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where ρ_i are 2^{n_i}-th roots of -1.

For any $s > -2$, greedy s-energy sequences on S^1 have the same geometric structure present in the case $s = 0$, so in what follows we will write $\alpha_{N,s} = \alpha_N$.

The goal is to describe the asymptotic behavior of the sequence

$$E_s(\alpha_N), \qquad \alpha_N = \{a_0, \ldots, a_{N-1}\},$$

for all values of $s > -2$.

Let $\omega_N = \{e^{2\pi ik/N} : 1 \leq k \leq N\}$, and

$$\mathcal{L}_s(N) := E_s(\omega_N).$$

Then

$$\mathcal{L}_s(N) = \begin{cases} 2^{-s} N \sum_{k=1}^{N-1} \left(\sin \frac{k\pi}{N}\right)^{-s}, & s \neq 0, \\ -N \log N, & s = 0. \end{cases}$$

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In joint works with Ryan McCleary and Douglas Wagner we showed that

$$E_s(\alpha_N) = \sum_{k=1}^{p-1} \sum_{j=k+1}^{p} 2^{n_j - n_k} \mathcal{L}_s(2^{n_k+1}) + \sum_{k=1}^{p} \left(1 - \sum_{j=k+1}^{p} 2^{n_j - n_k + 1}\right) \mathcal{L}_s(2^{n_k})$$

if $N = 2^{n_1} + \cdots + 2^{n_p}$, $n_1 > \cdots > n_p \geq 0$.

Different asymptotic regimes

The study of the asymptotics of $E_s(\alpha_N)$ is subdivided into six cases:

$$-2 < s < -1, \quad s = -1, \quad -1 < s < 1, s \neq 0, \quad s = 0, \quad s = 1, \quad s > 1.$$

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In the logarithmic case $s = 0$, a result of Albert Edrei implies that

$$\lim_{N\to\infty} \frac{E_0(\alpha_N)}{N^2} = \lim_{N\to\infty} \frac{2}{N^2} \sum_{0\leq i<j\leq N-1} \log \frac{1}{|a_i - a_j|} = 0.$$

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We know that $E_0(\alpha_N) \geq \mathcal{L}_0(N) = -N\log N$. In joint work with Wagner (2015) we showed that

$$\lim_{N\to\infty} \frac{E_0(\alpha_N)}{N\log N} = -1.$$

In fact we proved that

$$0 \leq \frac{E_0(\alpha_N) + N\log N}{N} < \log(4/3), \qquad N \geq 2.$$

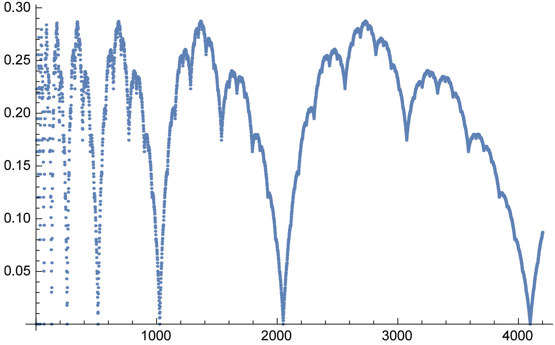

Figure: Plot of the sequence $\frac{E_0(\alpha_N)+N\log N}{N}$, $2 \leq N \leq 4200$

This sequence has the doubling periodicity property

$$\frac{E_0(\alpha_{2N}) + 2N\log(2N)}{2N} = \frac{E_0(\alpha_N) + N\log N}{N}, \qquad N \geq 1.$$

We can show now that the set of all limit points of the sequence $(E_0(\alpha_N) + N\log N)/N$ fills out the interval $[0, \log(4/3)]$.

Using essentially the same argument of Edrei, it can be shown that for any $-2 < s < 1$, $s \neq 0$, we have

$$\lim_{N\to\infty} \frac{E_s(\alpha_N)}{N^2} = I_s(\sigma) = \iint_{S^1\times S^1} |x-y|^{-s}\, d\sigma(x)\, d\sigma(y),$$

where σ is the normalized Lebesgue measure on S^1 ($\sigma(S^1) = 1$). In joint work with E. Saff we proved that in the case $s = 1$ we have

$$\lim_{N\to\infty} \frac{E_1(\alpha_N)}{N^2 \log N} = \frac{1}{\pi}.$$

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Somewhat surprisingly, in the same work it was shown that the sequence

$$\frac{E_s(\alpha_N)}{N^{1+s}}, \qquad s > 1,$$

is bounded but divergent!

Theorem (with E. Saff, D. Wagner, and R. McCleary)

Let $s > -2$ and $(a_n) \subset S^1$ be a greedy s-energy sequence. Then the following sequences are bounded and divergent:

$$\frac{E_0(\alpha_N) + N \log N}{N}, \quad s = 0, \qquad \frac{E_s(\alpha_N) - N^2 I_s(\sigma)}{N^{1+s}}, \quad -1 < s < 1, \ s \neq 0,$$

$$\frac{E_1(\alpha_N) - \pi^{-1} N^2 \log N}{N^2}, \quad s = 1, \qquad \frac{E_s(\alpha_N)}{N^{1+s}}, \quad s > 1,$$

$$E_s(\alpha_N) - N^2 I_s(\sigma), \quad -2 < s < -1, \qquad \frac{E_{-1}(\alpha_N) - I_{-1}(\sigma) N^2}{\log N}, \quad s = -1,$$

where σ is the normalized Lebesgue measure on S^1.

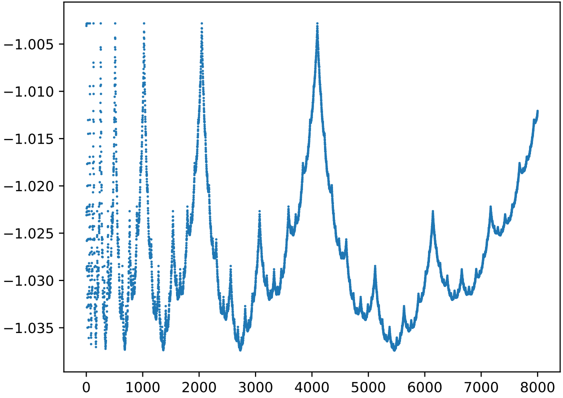

Figure: Plot in the case $s = -0.1$ of the sequence $\frac{E_s(\alpha_N) - I_s(\sigma)\, N^2}{N^{1+s}}$, $2 \leq N \leq 8000$

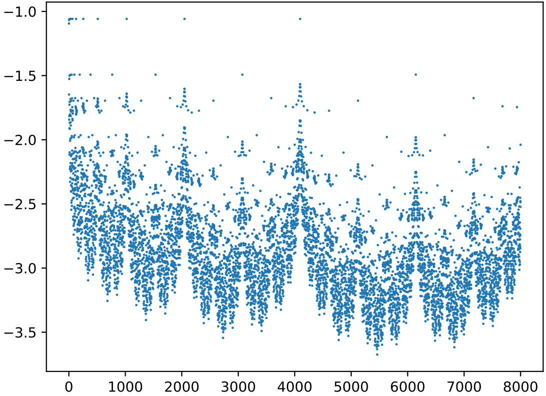

Figure: Plot in the case $s = -0.9$ of the sequence $\frac{E_s(\alpha_N) - I_s(\sigma)\, N^2}{N^{1+s}}$, $2 \leq N \leq 8000$

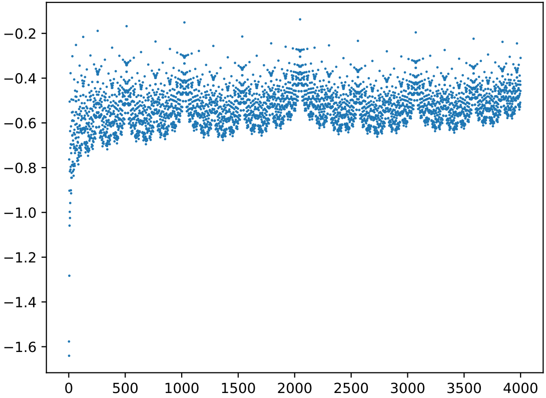

Figure: Plot of the sequence $\frac{E_{-1}(\alpha_N) - I_{-1}(\sigma)\,N^2}{\log N}$, $2 \leq N \leq 4000$

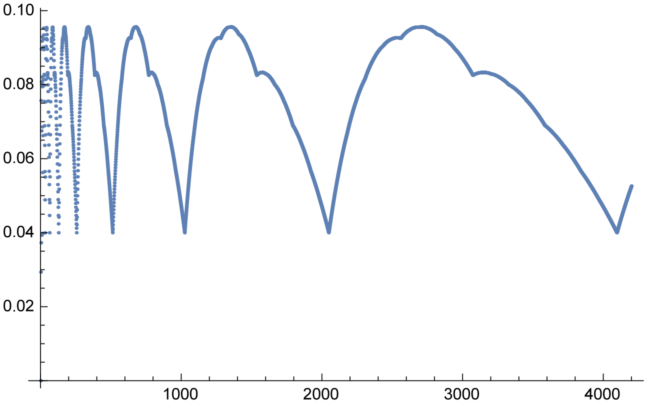

Figure: Plot of the sequence $\frac{E_1(\alpha_N)-\pi^{-1}N^2\log N}{N^2}$, $2 \leq N \leq 4200$

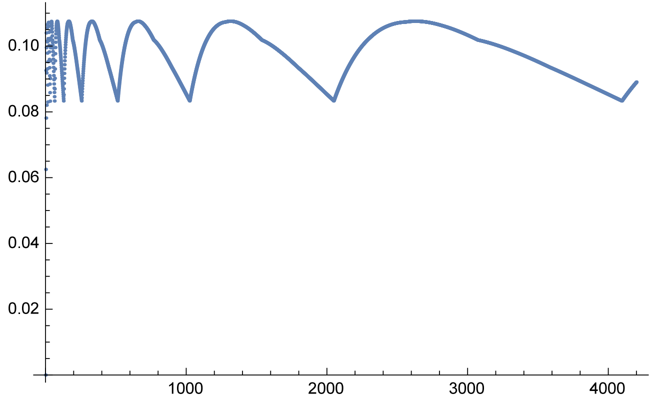

Figure: Plot in the case $s = 2$ of the sequence $\frac{E_s(\alpha_N)}{N^{1+s}}$, $2 \leq N \leq 4200$

Arithmetic functions

For $N \in \mathbb{N}$, let $\eta(N)$ denote the vector

$$\eta(N) = \left(\frac{2^{n_1}}{N}, \frac{2^{n_2}}{N}, \ldots, \frac{2^{n_p}}{N} \right)$$

if $N = 2^{n_1} + \cdots + 2^{n_p}$, $n_1 > \cdots > n_p \geq 0$. So note that $\eta(N) = \eta(2N)$ for all $N \in \mathbb{N}$.

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For a vector $\vec{\theta} = (\theta_1, \dots, \theta_p)$ with positive entries, let

$$H(\vec{\theta}, s) = \sum_{k=1}^{p} \theta_k^s \left(2(2^s - 1)\left(\sum_{j=k+1}^{p} \theta_j \right) + \theta_k\right),$$

$$K(\vec{\theta}) = \log(4) + \sum_{k=1}^{p} \theta_k^2 \log(\theta_k/4) + 2 \sum_{k=1}^{p-1} \sum_{j=k+1}^{p} \theta_j \, \theta_k \log \theta_k,$$

$$R(\vec{\theta}) = -\sum_{k=1}^{p} (\log(4)(k-1) + \log \theta_k) \, \theta_k.$$

The sequences $H(\eta(N), s), K(\eta(N)), R(\eta(N))$ play an essential role in the description of the asymptotics of $E_s(\alpha_N)$.

It can be proved that the sequences $K(\eta(N)), R(\eta(N))$ are bounded and divergent, and for any $s > -1$, the sequence $H(\eta(N), s)$ is also bounded and divergent.

In the case $s \leq -1$, the sequence $H(\eta(N), s)$ is unbounded, and we have $H(\eta(N), -1) = O(\log N)$ and $H(\eta(N), s) = O(N^{-(s+1)})$ for $s < -1$.

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For any $N \in \mathbb{N}$, the function $s \mapsto H(\eta(N), s)$ is convex on the whole real line, and we have $H(\eta(N), 0) = H(\eta(N), 1) = 1$.

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Let

$$T_{N,s} := \begin{cases} E_s(\alpha_N) - N^2 I_s(\sigma), & -2 < s < -1, \\ \frac{E_{-1}(\alpha_N) - N^2 I_{-1}(\sigma)}{\log N}, & s = -1, \\ \frac{E_0(\alpha_N) + N \log N}{N}, & s = 0, \\ \frac{E_s(\alpha_N) - N^2 I_s(\sigma)}{N^{1+s}}, & -1 < s < 1, \ s \neq 0, \\ \frac{E_1(\alpha_N) - \pi^{-1} N^2 \log N}{N^2}, & s = 1, \\ \frac{E_s(\alpha_N)}{N^{1+s}}, & s > 1. \end{cases}$$

Theorem (with Miña-Díaz)

The following asymptotic formulae hold true as $N \to \infty$: If $s > -1$ and $s \neq 0, 1$, then

$$T_{N,s} = \frac{2\zeta(s)}{(2\pi)^s} H(\eta(N), s) + \begin{cases} O(N^{-1-s}), & -1 < s < 1,\ s \neq 0, \\ O(N^{1-s}), & 1 < s < 3,\ s \neq 2, \\ O(N^{-2}), & s = 2,\ \textit{or } s > 3, \\ O(N^{-2}\log N), & s = 3, \end{cases}$$

where $\zeta(s)$ is the classical Riemann zeta function, or its analytic continuation. If $s = 1$, then

$$T_{N,1} = \frac{E_1(\alpha_N) - \pi^{-1} N^2 \log N}{N^2} = \frac{1}{\pi}(\gamma + \log(2/\pi) + K(\eta(N))) + O(N^{-2}).$$

where γ is the Euler constant. If $s = -1$, then

$$T_{N,-1} = \frac{E_{-1}(\alpha_N) - N^2 I_{-1}(\sigma)}{\log N} = -\frac{\pi}{3} \frac{H(\eta(N), -1)}{\log N} + O((\log N)^{-1}).$$

For each integer $N \geq 2$ we have $T_{N,0} = \frac{E_0(\alpha_N) + N \log N}{N} = R(\eta(N))$.

Finite asymptotic expansion

As a corollary of the previous result, for every $s \geq -1$ we have the asymptotic doubling periodicity property

$$\lim_{N\to\infty}(T_{2N,s} - T_{N,s}) = 0.$$

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The Riesz s-energy $I_s(\sigma)$ of the normalized Lebesgue measure has the analytic continuation

$$V_s = \frac{2^{-s}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(1 - \frac{s}{2}\right)}, \qquad s \in \mathbb{C} \setminus \{1, 3, 5, \ldots\}.$$

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For the sinc function $\operatorname{sinc} z = \frac{\sin \pi z}{\pi z}$, let

$$\operatorname{sinc}^{-s} z = \sum_{n=0}^{\infty} \beta_n(s) z^{2n}, \qquad |z| < 1,$$

with $\beta_0(s) = 1$.

Theorem (with Miña-Díaz)

For every $s \geq -1$ such that $s \neq 0$ and s is not a positive odd integer, we have

$$E_s(\alpha_N) = v_s N^2 + \frac{2}{(2\pi)^s} \sum_{j=0}^{\lfloor (s+1)/2 \rfloor} \beta_j(s)\zeta(s-2j)H(\eta(N), s-2j)N^{s-2j+1} + O_s(1).$$

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If $s = 2M+1$, $M \geq 0$ integer,

$$\begin{aligned} E_s(\alpha_N) = {} & \frac{(\frac{1}{2})_M}{\pi\, 2^{2M} M!} N^2 \log N + \frac{(\frac{1}{2})_M}{\pi\, 2^{2M} M!}(\gamma - \log(\pi) + C_M + K(\eta(N)))N^2 \\ & + \frac{2}{(2\pi)^s} \sum_{j=0}^{M-1} \beta_j(s)\zeta(s-2j)H(\eta(N), s-2j)N^{s-2j+1} + O_M(1), \end{aligned}$$

where C_M is a constant and $(\cdot)_n$ denotes the Pochhammer symbol.

$$C_M = \frac{\beta_M'(2M+1)}{\beta_M(2M+1)} + \frac{1}{2}\psi(M+1) - \frac{1}{2}\psi(M+1/2), \qquad \psi = \Gamma'/\Gamma.$$

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As mentioned earlier, this interval equals $[0, \log(4/3)]$ in the case $s = 0$.

The theorem follows from the fact that the set of all limit points of the sequences $H(\eta(N), s)$, $s > -1$, $K(\eta(N))$, $R(\eta(N))$ form an interval.

In the case of the function H, the interval is of the form

$$\begin{array}{ll} [d_s, 1], & \text{if } 0 < s < 1, \\ [1, d_s], & \text{if } -1 < s < 0 \ \text{ or } \ s > 1. \end{array}$$

We have estimates for d_s but we don't know its value.

Theorem (with Miña-Díaz)

For any $s > -1$, the set of all limit points of the sequence $(T_{N,s})_N$ forms a compact interval.

As mentioned earlier, this interval equals $[0, \log(4/3)]$ in the case $s = 0$.

The theorem follows from the fact that the set of all limit points of the sequences $H(\eta(N), s)$, $s > -1$, $K(\eta(N))$, $R(\eta(N))$ form an interval.

In the case of the function H, the interval is of the form

$$\begin{aligned} &[d_s, 1], && \text{if } 0 < s < 1, \\ &[1, d_s], && \text{if } -1 < s < 0 \ \text{ or } \ s > 1. \end{aligned}$$

We have estimates for d_s but we don't know its value.

In the case of the function K the interval is of the form $[0, \kappa]$, for some $\kappa > 0$. We don't know the value of κ.

Thank you!