

Measures

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1 Introduction

We present here a construction of a complete measure space. We do not follow the standard Carathéodory construction. The approach used here is based on [1]. The integral with respect to a measure is defined as a Daniell integral. The proofs of the Radon-Nikodym Theorem and the Lebesgue Decomposition Theorem follow the ideas of von Neumann. The proof of Fubini's Theorem is based on [2], where it is presented for the Lebesgue measure in $\mathbb{R}^N$.

2 Measure spaces

Let X be a nonempty set.

Definition 2.1. A collection $\mathcal{R}$ of subsets of X is called a *ring of subsets* of X if

$$A, B \in \mathcal{R} \text{ implies } A \cup B, A \setminus B \in \mathcal{R}.$$

Definition 2.2. A map $\mu : \mathcal{R} \rightarrow [0, \infty)$ is called *σ -additive* if for any sequence of disjoint sets $A_1, A_2, \dots \in \mathcal{R}$ such that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$ we have

$$\mu \left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Note that σ -additivity implies finite additivity.

In this note we use the same symbol to denote a subset of X and the characteristic function of that set, that is, if $A \subset X$ we will write

$$A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}.$$

Definition 2.3. A set $S \subset X$ is called *summable* if there are $A_1, A_2, \dots \in \mathcal{R}$ and $\alpha_1, \alpha_2, \dots \in \{-1, 1\}$ such that

$$\text{I } \sum_{n=1}^{\infty} \mu(A_n) < \infty$$

$$\text{II } S(x) = \sum_{n=1}^{\infty} \alpha_n A_n(x) \text{ for every } x \in X \text{ for which } \sum_{n=1}^{\infty} A_n(x) < \infty$$

If conditions I and II are satisfied, we write $S \simeq \sum_{n=1}^{\infty} \alpha_n A_n$.

Definition 2.4. A function $f : X \rightarrow \mathbb{Z}$ is called a *simple function* if

$$f(x) = \alpha_1 A_1(x) + \dots + \alpha_n A_n(x) \quad (1)$$

for some $A_1, \dots, A_n \in \mathcal{R}$ and $\alpha_1, \dots, \alpha_n \in \mathbb{Z}$. For the simple function (1) we define

$$I(f) = \alpha_1 \mu(A_1) + \dots + \alpha_n \mu(A_n).$$

Exercise 2.5. Show that I is a well-defined additive functional on the space of simple functions.

Lemma 2.6. If (f_n) is a non-increasing sequence of simple functions convergent to 0 at every point of X , then $\lim_{n \rightarrow \infty} I(f_n) = 0$.

Proof. Let $A_n = \text{supp } f_n = \{x \in X : f_n(x) \neq 0\}$, $n \in \mathbb{N}$. Since (f_n) is a non-increasing sequence, we have $A_1 \supset A_2 \supset \dots$. Define $B_n = A_n \setminus A_{n+1}$. Since $\lim_{n \rightarrow \infty} f_n(x) = 0$ for all $x \in X$, we have $A_1 = \cup_{n=1}^{\infty} B_n$. Then $\mu(A_1) = \sum_{n=1}^{\infty} \mu(B_n)$, by σ -additivity of μ on $\mathcal{R}$. Hence $\sum_{k=n}^{\infty} \mu(B_k) \rightarrow 0$ as $n \rightarrow \infty$ and consequently

$$I(f_n) \leq \max |f_n| \mu(A_n) = \max |f_n| \sum_{k=n}^{\infty} \mu(B_k) \rightarrow 0$$

as $n \rightarrow \infty$. □

Lemma 2.7. Let (g_n) and (h_n) be non-decreasing sequences of simple functions such that

$$\lim_{n \rightarrow \infty} h_n(x) \leq \lim_{n \rightarrow \infty} g_n(x)$$

for every $x \in X$. Then

$$\lim_{n \rightarrow \infty} I(h_n) \leq \lim_{n \rightarrow \infty} I(g_n).$$

(In both inequalities we allow ∞ .)

Proof. Fix $k \in \mathbb{N}$ and define

$$f_n = h_k - \min(h_k, g_n).$$

Note that (f_n) is a non-increasing sequence convergent to 0 at every $x \in X$. By Lemma 2.6,

$$\lim_{n \rightarrow \infty} I(f_n) = \lim_{n \rightarrow \infty} I(h_k - \min(h_k, g_n)) = 0.$$

Hence

$$I(h_k) = \lim_{n \rightarrow \infty} I(\min(h_k, g_n)) \leq \lim_{n \rightarrow \infty} I(g_n).$$

We finish the proof by letting $k \rightarrow \infty$. □

Lemma 2.8. *If $U \simeq \sum_{n=1}^{\infty} \alpha_n A_n$, then $\sum_{n=1}^{\infty} \alpha_n \mu(A_n) \geq 0$.*

Proof. Let $\varepsilon > 0$. The $\sum_{n=n_0+1}^{\infty} \mu(A_n) < \varepsilon$ for some $n_0 \in \mathbb{N}$. Define

$$g_n = \alpha_1 A_1 + \cdots + \alpha_{n_0} A_{n_0} + A_{n_0+1} + A_{n_0+2} + \cdots$$

and

$$h_n = \max(g_n, 0).$$

Note that (g_n) and (h_n) satisfy the assumptions of Lemma 2.7. Hence

$$0 \leq \lim_{n \rightarrow \infty} I(h_n) \leq \lim_{n \rightarrow \infty} I(g_n)$$

and consequently

$$0 \leq \alpha_1 \mu(A_1) + \cdots + \alpha_{n_0} \mu(A_{n_0}) + \mu(A_{n_0+1}) + \mu(A_{n_0+2}) + \cdots$$

Since $\sum_{n=n_0}^{\infty} \mu(A_n) < \varepsilon$, we must have

$$-2\varepsilon < \sum_{n=1}^{\infty} \alpha_n \mu(A_n),$$

which shows that $0 \leq \sum_{n=1}^{\infty} \alpha_n \mu(A_n)$, because ε is an arbitrary positive number. □

Lemma 2.9. *If $S \simeq \sum_{n=1}^{\infty} \alpha_n A_n$ and $S \simeq \sum_{n=1}^{\infty} \beta_n B_n$, then $\sum_{n=1}^{\infty} \alpha_n \mu(A_n) = \sum_{n=1}^{\infty} \beta_n \mu(B_n)$.*

Proof. Note that

$$\emptyset \simeq \alpha_1 A_1 - \beta_1 B_1 + \alpha_2 A_2 - \beta_2 B_2 + \dots$$

By Lemma 2.8

$$0 \leq \alpha_1 \mu(A_1) - \beta_1 \mu(B_1) + \alpha_2 \mu(A_2) - \beta_2 \mu(B_2) + \dots$$

and consequently

$$\sum_{n=1}^{\infty} \beta_n \mu(B_n) \leq \sum_{n=1}^{\infty} \alpha_n \mu(A_n).$$

By switching the two representations of S , we obtain the inequality in the opposite directions. $\square$

The above lemma allows us to define the measure of a summable set.

Definition 2.10. Let S be a summable set and let $S \simeq \sum_{n=1}^{\infty} \alpha_n A_n$. By the *measure* of S we mean the number

$$\mu(S) = \sum_{n=1}^{\infty} \alpha_n \mu(A_n).$$

Definition 2.11. A set $U \subset X$ is called *measurable* if $U \cap S$ is summable for all $S \in \mathcal{R}$. If U is measurable but not summable, we define $\mu(U) = \infty$. The collection of all measurable subsets of X will be denoted by Σ .

Lemma 2.12. If $S \subset X$ is summable, then for every $\varepsilon > 0$ there is a representation $S \simeq \sum_{n=1}^{\infty} \alpha_n A_n$ such that

$$\sum_{n=1}^{\infty} \mu(A_n) \leq \mu(S) + \varepsilon.$$

Proof. Let $S \simeq \sum_{n=1}^{\infty} \beta_n B_n$ be an arbitrary representation of S . There is $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0+1}^{\infty} \mu(B_n) < \frac{1}{4}\varepsilon.$$

Let $C_1, \dots, C_{n_1}, D_1, \dots, D_{n_2} \in \mathcal{R}$ be such that $C_j \cap D_k = \emptyset$ for all j and k and such that

$$\beta_1 B_1 + \dots + \beta_{n_0} B_{n_0} = C_1 + \dots + C_{n_1} - D_1 - \dots - D_{n_2}.$$

Note that

$$S \simeq C_1 + \cdots + C_{n_1} - D_1 - \cdots - D_{n_2} + \beta_{n_0+1}B_{n_0+1} + \beta_{n_0+2}B_{n_0+2} + \cdots$$

Let $V = \text{supp}(D_1 \cup \cdots \cup D_{n_2})$. Then

$$S \cap V \simeq -D_1 - \cdots - D_{n_2} + \beta_{n_0+1}(B_{n_0+1} \cap V) + \beta_{n_0+2}(B_{n_0+2} \cap V) + \cdots$$

and hence, by Lemma 2.8,

$$\begin{aligned} 0 &\leq -\mu(D_1) - \cdots - \mu(D_{n_2}) + \beta_{n_0+1}\mu(B_{n_0+1} \cap V) + \beta_{n_0+2}\mu(B_{n_0+2} \cap V) + \cdots \\ &< -\mu(D_1) - \cdots - \mu(D_{n_2}) + \frac{1}{4}\varepsilon. \end{aligned}$$

Consequently $\mu(D_1) + \cdots + \mu(D_{n_2}) < \frac{1}{4}\varepsilon$. Since

$$\sum_{n=1}^{n_1} \mu(C_n) - \sum_{n=1}^{n_2} \mu(D_n) - \sum_{n=n_0+1}^{\infty} \mu(B_n) \leq \mu(S) \leq \sum_{n=1}^{n_1} \mu(C_n) + \sum_{n=1}^{n_2} \mu(D_n) + \sum_{n=n_0+1}^{\infty} \mu(B_n)$$

and

$$\left(\sum_{n=1}^{n_1} \mu(C_n) + \sum_{n=1}^{n_2} \mu(D_n) + \sum_{n=n_0+1}^{\infty} \mu(B_n) \right) - \left(\sum_{n=1}^{n_1} \mu(C_n) - \sum_{n=1}^{n_2} \mu(D_n) - \sum_{n=n_0+1}^{\infty} \mu(B_n) \right) < \varepsilon,$$

we conclude

$$\sum_{n=1}^{n_1} \mu(C_n) + \sum_{n=1}^{n_2} \mu(D_n) + \sum_{n=n_0+1}^{\infty} \mu(B_n) < \mu(S) + \varepsilon.$$

□

Now we extend the use of the symbol $\simeq$ to series of summable sets.

Definition 2.13. If $U \subset X$ and there are summable sets $S_1, S_2, \cdots \subset X$ and $\alpha_1, \alpha_2, \cdots \in \{-1, 1\}$ such that

$$\text{I } \sum_{n=1}^{\infty} \mu(S_n) < \infty \text{ and}$$

$$\text{II } U(x) = \sum_{n=1}^{\infty} \alpha_n S_n(x) \text{ for every } x \in X \text{ for which } \sum_{n=1}^{\infty} S_n(x) < \infty,$$

then we write $U \simeq \sum_{n=1}^{\infty} \alpha_n S_n$.

Lemma 2.14. *If $U \simeq \sum_{n=1}^{\infty} \alpha_n S_n$ and $S_1, S_2, \dots$ are summable sets, then U is summable and $\mu(U) = \sum_{n=1}^{\infty} \alpha_n \mu(S_n)$.*

Proof. Let $B_{k,n} \in \mathcal{R}$ and $\beta_{k,n} \in \{-1, 1\}$, for $k, n \in \mathbb{N}$, be such that

$$S_k \simeq \sum_{n=1}^{\infty} \beta_{k,n} B_{k,n} \quad \text{and} \quad \sum_{n=1}^{\infty} \mu(B_{k,n}) < \mu(S_k) + 2^{-k}$$

for all $k \in \mathbb{N}$. Let $\sum_{n=1}^{\infty} \gamma_n A_n$ be a series arranged from all series $\sum_{n=1}^{\infty} \alpha_k \beta_{k,n} B_{k,n}$, $k \in \mathbb{N}$. It is easy to verify that $U \simeq \sum_{n=1}^{\infty} \gamma_n A_n$. Consequently, U is summable and

$$\mu(U) = \sum_{n=1}^{\infty} \gamma_n \mu(A_n) = \sum_{n=1}^{\infty} \alpha_n \mu(S_n).$$

□

Theorem 2.15. Σ is a σ -ring.

Proof. Let $U, V \in \Sigma$ and let $S \in \mathcal{R}$. If

$$U \cap S \simeq \sum_{n=1}^{\infty} \alpha_n A_n \quad \text{and} \quad V \cap S \simeq \sum_{n=1}^{\infty} \beta_n B_n,$$

then

$$(U \setminus V) \cap S \simeq \sum_{n=1}^{\infty} \alpha_n A_n - \left(\sum_{n=1}^{\infty} \alpha_n A_n \right) \left(\sum_{n=1}^{\infty} \beta_n B_n \right)$$

and

$$(U \cup V) \cap S \simeq \sum_{n=1}^{\infty} \alpha_n A_n + \sum_{n=1}^{\infty} \beta_n B_n - \left(\sum_{n=1}^{\infty} \alpha_n A_n \right) \left(\sum_{n=1}^{\infty} \beta_n B_n \right).$$

If $U_1, U_2, \dots \in \Sigma$, then the sets

$$V_1 = U_1, V_n = U_n \setminus (U_1 \cup \dots \cup U_{n-1}) \text{ for } n > 1$$

are measurable in view of the above. If $S \in \mathcal{R}$, then

$$\left(\bigcup_{n=1}^{\infty} U_n \right) \cap S = \left(\bigcup_{n=1}^{\infty} V_n \right) \cap S \simeq \sum_{n=1}^{\infty} (V_n \cap S)$$

and thus $\bigcup_{n=1}^{\infty} U_n \in \Sigma$.

□

Theorem 2.16. μ is a measure on Σ .

Proof. If $U_1, U_2, \dots \in \Sigma$ are disjoint and $\bigcup_{n=1}^{\infty} \mu(U_n) < \infty$, then $\bigcup_{n=1}^{\infty} U_n \simeq \sum_{n=1}^{\infty} U_n$ and hence $\mu(\bigcup_{n=1}^{\infty} U_n) = \sum_{n=1}^{\infty} \mu(U_n)$. If $U_1, U_2, \dots \in \Sigma$ are disjoint and $\bigcup_{n=1}^{\infty} \mu(U_n) = \infty$, then it is easy to see that $\mu(\bigcup_{n=1}^{\infty} U_n) = \infty$. $\square$

Theorem 2.17. μ is complete.

Proof. If $\mu(U) = 0$ and $V \subset U$, then $V \simeq U + U + \dots$ and thus $V \in \Sigma$ and $\mu(V) = \mu(U) + \mu(U) + \dots = 0$. $\square$

3 Regularity of the Lebesgue measure in $\mathbb{R}^N$

Theorem 3.1. *Let $S \subset \mathbb{R}^N$ be a summable set and let $\varepsilon > 0$. Then there exist an open set U and a compact set K such that*

- (a) $K \subset S \subset U$,
- (b) $\mu(U \setminus S) < \varepsilon$,
- (c) $\mu(S \setminus K) < \varepsilon$.

Proof. If $S \subset \mathbb{R}^N$ is summable and $\varepsilon > 0$, then by Lemma 2.12 there is a representation

$$S \simeq \sum_{n=1}^{\infty} \alpha_n A_n$$

where $A_n \in \mathcal{R}$ and

$$\sum_{n=1}^{\infty} \mu(A_n) \leq \mu(S) + \frac{\varepsilon}{2}.$$

Let $U_1, U_2, \dots \subset \mathbb{R}^N$ be open sets such that $A_n \subset U_n$ for every $n \in \mathbb{N}$ and

$$\sum_{n=1}^{\infty} \mu(U_n) < \sum_{n=1}^{\infty} \mu(A_n) + \frac{\varepsilon}{2}.$$

If we define $U = \cup_{n=1}^{\infty} U_n$, then U is an open set such that $S \subset U$ and

$$\begin{aligned} \mu(U \setminus S) &= \mu(U) - \mu(S) \\ &\leq \sum_{n=1}^{\infty} \mu(U_n) - \mu(S) \\ &< \sum_{n=1}^{\infty} \mu(A_n) + \frac{\varepsilon}{2} - \mu(S) \\ &\leq \mu(S) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} - \mu(S) = \varepsilon. \end{aligned}$$

Now let $A \in \mathcal{R}$ be a closed set such that

$$\mu(S) \leq \mu(S \cap A) + \frac{\varepsilon}{2},$$

so that we have

$$\mu(S \setminus A) \leq \frac{\varepsilon}{2}.$$

Since $A \setminus S$ is summable, by the first part of the proof, there exists an open set V such that $A \setminus S \subset V$ and

$$\mu(V \setminus (A \setminus S)) < \frac{\varepsilon}{2}.$$

If we define $K = A \setminus V$, then K is a compact set and we have

$$K = A \setminus V \subset A \setminus (A \setminus S) = S.$$

Moreover, since

$$S \setminus K = S \setminus (A \setminus V) = (S \setminus A) \cup (S \cap A \cap V)$$

and

$$S \cap A \cap V \subset V \setminus (A \setminus S),$$

we have

$$\mu(S \setminus K) = \mu(S \setminus A) + \mu(V \setminus (A \setminus S)) < \varepsilon.$$

□

4 Integration on measure spaces

Let (X, Σ, μ) be a measure space. A function $f : X \rightarrow \mathbb{R}$ is called a simple function if it has the form

$$f = \alpha_1 A_1 + \cdots + \alpha_n A_n$$

where $n \in \mathbb{N}$, $A_1, \dots, A_n \in \Sigma$, and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. The space of all simple function on the measure space (X, Σ, μ) will be denoted by $\mathcal{S}(X, \Sigma, \mu)$. On $\mathcal{S}(X, \Sigma, \mu)$ we define a functional $\int$:

$$\int_X f d\mu = \int_X (\alpha_1 A_1 + \cdots + \alpha_n A_n) d\mu = \alpha_1 \mu(A_1) + \cdots + \alpha_n \mu(A_n).$$

Exercise 4.1. Show that the integral of a simple function is well defined.

Theorem 4.2. $(X, \mathcal{S}(X, \Sigma, \mu), \int)$ is a Daniell space satisfying the Stone condition.

Proof. We will only prove that $\lim_{n \rightarrow \infty} \int_X f_n d\mu = 0$ for any non-increasing sequence of simple functions (f_n) convergent to 0 at every point of X .

Let (f_n) be a non-increasing sequence of simple functions convergent to 0 at every point of X and let $\varepsilon > 0$. Let

$$A_n = \left\{ x \in X : f_n(x) > \frac{\varepsilon}{2\mu(\text{supp } f_1)} \right\}.$$

Then $A_n \in \Sigma$ and, since (f_n) is a non-increasing sequence, we have $A_1 \supset A_2 \supset \dots$. Define $B_n = A_n \setminus A_{n+1}$. Since $\lim_{n \rightarrow \infty} f_n(x) = 0$ for all $x \in X$, we have $A_1 = \bigcup_{n=1}^{\infty} B_n$. Then $\mu(A_1) = \sum_{n=1}^{\infty} \mu(B_n)$, by σ -additivity of μ . Hence $\sum_{k=n}^{\infty} \mu(B_k) \rightarrow 0$ as $n \rightarrow \infty$. Let $n_0 \in \mathbb{N}$ be such that

$$\mu(A_n) = \sum_{k=n}^{\infty} \mu(B_k) < \frac{\varepsilon}{2\|f_1\|_{\infty}}$$

for all $n > n_0$. Then, for $n > n_0$, we have

$$\begin{aligned} \int_X f_n d\mu &= \int_{A_n} f_n d\mu + \int_{X \setminus A_n} f_n d\mu \\ &\leq \|f_n\|_{\infty} \mu(A_n) + \frac{\varepsilon}{2\mu(\text{supp } f_1)} \mu(\text{supp } f_n) < \varepsilon. \end{aligned}$$

□

Corollary 4.3. $L^1(X, \Sigma, \mu)$ can be defined as the Daniell extension of $(X, \mathcal{S}(X, \Sigma, \mu), \int)$.

5 The Radon-Nikodym Theorem and the Lebesgue Decomposition Theorem

Definition 5.1 (Absolute continuity of measures). Let μ and ν be two measures on (X, Σ) . We say that ν is absolutely continuous with respect to μ , denoted by $\nu \ll \mu$, if $\mu(A) = 0$ implies $\nu(A) = 0$ for all $A \in \Sigma$.

Exercise 5.2. Let μ and ν be two measures on (X, Σ) . If ν is finite, then $\nu \ll \mu$ if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mu(A) < \delta$ implies $\nu(A) < \varepsilon$ for every $A \in \Sigma$.

Exercise 5.3. Let μ be a measures on (X, Σ) and let $h \in L^1(X, \mu)$ be a non-negative function. Then

$$\nu(A) = \int_A h d\mu$$

defines a measure on (X, Σ) and $\nu \ll \mu$.

Definition 5.4 (Measure support). Let μ be a measures on (X, Σ) . We say that μ is supported on $S \in \Sigma$ if $\mu(A) = \mu(A \cap S)$ for all $A \in \Sigma$.

Definition 5.5 (Singularity of measures). Let μ and ν be two measures on (X, Σ) . We say that ν is singular with respect to μ , denoted by $\nu \perp \mu$, if there is $A \in \Sigma$ such that ν is supported on A and $\mu(A) = 0$. We say that ν and μ are mutually singular, denoted by $\nu \perp \mu$, if there are disjoint $A, B \in \Sigma$ such that ν is supported on A and μ is supported on B .

Exercise 5.6. Let μ and ν be two measures on (X, Σ) . Show that ν is singular with respect to μ if and only if ν and μ are mutually singular.

Theorem 5.7. If μ and ν are two finite measures on (X, Σ) , then there exists a non-negative measurable function $g : X \rightarrow \mathbb{R}$ and $B \in \Sigma$ such that $\mu(B) = 0$ and

$$\nu(A) = \int_A g d\mu + \nu(A \cap B)$$

for all $A \in \Sigma$.

Proof. The functional

$$\Lambda(f) = \int f d\nu$$

is a bounded linear functional on the Hilbert space $L^2(X, \mu + \nu)$. By the Riesz representation theorem there exists $h \in L^2(X, \mu + \nu)$ such that

$$\Lambda(f) = \int f d\nu = \int f h d(\mu + \nu) = \int f h d\mu + \int f h d\nu$$

for all $f \in L^2(X, \mu + \nu)$. Let

$$\begin{aligned} N &= \{x \in X : h(x) < 0\}, \\ M &= \{x \in X : 0 \leq h(x) < 1\}, \\ B &= \{x \in X : h(x) \geq 1\}. \end{aligned}$$

Since

$$0 \geq \int_N h d(\mu + \nu) = \int N h d(\mu + \nu) = \int N d\nu = \nu(N),$$

we have $\nu(N) = 0$. On the other hand,

$$0 = \nu(N) = \int_N h d(\mu + \nu) = \int_N h d\mu + \int_N h d\nu,$$

so $\mu(N) = 0$. Moreover, since

$$\nu(B) = \int B d\nu = \int_B h d\mu + \int_B h d\nu \geq \nu(B) + \mu(B),$$

we have $\mu(B) = 0$.

For $n \in \mathbb{N}$ we define

$$M_n = \left\{ x \in X : 0 \leq h(x) \leq 1 - \frac{1}{n} \right\}.$$

Note that

$$\int f h d\mu = \int f(1 - h) d\nu$$

for any $f \in L^2(X, \mu + \nu)$. Hence, for any $A \in \Sigma$, we have

$$\nu(A \cap M_n) = \int_A \frac{M_n}{1 - h} (1 - h) d\nu = \int_A \frac{M_n}{1 - h} h d\mu.$$

By the monotone convergence theorem we obtain

$$\nu(A \cap M) = \int_A \frac{M}{1 - h} h d\mu = \int_{A \cap M} \frac{h}{1 - h} d\mu.$$

But $\mu(N) = 0$ and $\mu(B) = 0$, so

$$\nu(A \cap M) = \int_{A \cap M} \frac{h}{1-h} d\mu = \int_A \frac{h}{1-h} d\mu.$$

Consequently

$$\nu(A) = \nu(A \cap N) + \nu(A \cap M) + \nu(A \cap B) = \int_A g d\mu + \nu(A \cap B),$$

where $g = \frac{h}{1-h}$. □

Corollary 5.8. *If μ and ν are two σ -finite measures on (X, Σ) , then there exists a non-negative measurable function $g : X \rightarrow \mathbb{R}$ and $B \in \Sigma$ such that $\mu(B) = 0$ and*

$$\nu(A) = \int_A g d\mu + \nu(A \cap B)$$

for all $A \in \Sigma$.

Corollary 5.9 (Radon-Nikodym Theorem). *If μ and ν are two σ -finite measures on (X, Σ) and $\nu \ll \mu$, then there exists a non-negative measurable function $g : X \rightarrow \mathbb{R}$ such that*

$$\nu(A) = \int_A g d\mu$$

for all $A \in \Sigma$.

Corollary 5.10 (Lebesgue Decomposition Theorem). *If μ and ν are two σ -finite measures on (X, Σ) , then there exist two measures ν_a and ν_s on (X, Σ) such that $\nu = \nu_a + \nu_s$, $\nu_a \ll \mu$, and $\nu_s \perp \mu$.*

Proof. Define

$$\nu_a(A) = \int_A g d\mu \quad \text{and} \quad \nu_s(A) = \nu(A \cap B).$$

□

6 Product measures and Fubini's Theorem

Let (X, Σ, μ) and (Y, Ξ, ν) be measure spaces. We are interested in construction of a measure γ on $X \times Y$ such that

$$\gamma(A \times B) = \mu(A)\nu(B)$$

for all $A \in \Sigma$ and $B \in \Xi$.

First we note that the set

$$\Theta_0 = \{A \times B : A \in \Sigma \text{ and } B \in \Xi\}$$

is not a σ -ring. Let Θ_1 be the ring of generated by Θ_0 , that is, the smallest ring containing Θ_0 . Note that

$$\Theta_1 = \{A_1 \times B_1 \cup \cdots \cup A_n \times B_n : A_j \in \Sigma, B_j \in \Xi, j = 1, \dots, n, n \in \mathbb{N}\}$$

and that we can always assume that the sets $A_1 \times B_1, \dots, A_n \times B_n$ disjoint. We define

$$\gamma(A_1 \times B_1 \cup \cdots \cup A_n \times B_n) = \mu(A_1)\nu(B_1) + \cdots + \mu(A_n)\nu(B_n)$$

whenever $A_1 \times B_1, \dots, A_n \times B_n$ are disjoint.

Lemma 6.1. *γ is a σ -additive measure on Θ_1 .*

Proof. It suffices to show that $\bigcup_{n=1}^{\infty} A_n \times B_n = A \times B \in \Theta_0$ for some disjoint $A_1 \times B_1, A_2 \times B_2, \dots \in \Theta_0$ implies

$$\sum_{n=1}^{\infty} \mu(A_n)\nu(B_n) = \mu(A)\nu(B).$$

Since $A_1 \times B_1, A_2 \times B_2, \dots$ are disjoint,

$$\sum_{n=1}^{\infty} A_n(x)B_n(y) = A(x)B(y).$$

For a fixed $x \in X$, the monotone convergence theorem on Y gives us

$$\sum_{n=1}^{\infty} A_n(x)\nu(B_n) = A(x)\nu(B).$$

Now, the monotone convergence theorem on X gives us

$$\sum_{n=1}^{\infty} \mu(A_n) \nu(B_n) = \mu(A) \nu(B).$$

□

The ring Θ_1 can be extended to a σ -ring Θ and γ can be extended to a complete σ -additive measure on Θ . Clearly, $\gamma(A \times B) = \mu(A) \nu(B)$ for all $A \in \Sigma$ and $B \in \Xi$.

For the rest of this section we adopt the following notation: if $f : X \times Y \rightarrow \mathbb{R}$, then for $x \in X$ by f_x we mean the function $f_x : Y \rightarrow \mathbb{R}$ defined by $f_x(y) = f(x, y)$ and for $y \in Y$ by f_y we mean the function $f_y : X \rightarrow \mathbb{R}$ defined by $f_y(x) = f(x, y)$.

Lemma 6.2. *Let (X, Σ, μ) and (Y, Ξ, ν) be measure spaces and let $f : X \times Y \rightarrow \mathbb{R}$. If*

$$f \simeq \sum_{k=1}^{\infty} \alpha_k A_k \times B_k$$

where $A_k \in \Sigma$ and $B_k \in \Xi$, then

$$f_x \simeq \sum_{k=1}^{\infty} \alpha_k A_k(x) B_k$$

except possibly for a set of x of μ -measure zero and

$$f_y \simeq \sum_{k=1}^{\infty} \alpha_k B_k(y) A_k$$

except possibly for a set of y of ν -measure zero.

Proof. In view of full symmetry, we will only prove the part about f_x . Let

$$h_n(x) = \sum_{k=1}^n |\alpha_k| A_k(x) \nu(B_k)$$

for $n \in \mathbb{N}$. Since the sequence (h_n) is monotone and

$$\left| \int_X h_n d\mu \right| \leq \sum_{k=1}^{\infty} |\alpha_k| \mu(A_k) \nu(B_k) < \infty,$$

we can apply the monotone convergence theorem and conclude that there is $h \in L^1(X, \Sigma, \mu)$ such that $h_n \rightarrow h$ a.e. and

$$\int_X h d\mu = \sum_{k=1}^{\infty} |\alpha_k| \mu(A_k) \nu(B_k).$$

For $p = 1, 2, 3, \dots$ and $q \in \mathbb{R}$ let

$$Z_{pq} = \left\{ x \in X : \sum_{k=1}^p |\alpha_k| A_k(x) \nu(B_k) \geq q \right\},$$

$$Z_q = \left\{ x \in X : \sum_{k=1}^{\infty} |\alpha_k| A_k(x) \nu(B_k) \geq q \right\},$$

and

$$Z = \left\{ x \in X : \sum_{k=1}^{\infty} |\alpha_k| A_k(x) \nu(B_k) = \infty \right\}.$$

We see that

$$Z_q = \bigcap_{r=1}^{\infty} \left(\bigcup_{p=1}^{\infty} Z_{p, q - \frac{1}{r}} \right) \quad \text{and} \quad Z = \bigcap_{q=1}^{\infty} Z_q.$$

Notice that $Z_{pq}, Z_q, Z \in \Sigma$ for all p and q . For $q > 0$ since

$$qZ_q(x) = \begin{cases} 0 & \text{if } x \notin Z_q \\ q & \text{if } x \in Z_q \end{cases},$$

we see that $qZ_q \leq h$ μ -a.e.. Thus $q\mu(Z_q) \leq \int_X h d\mu$. Since $Z \subseteq Z_q$, we see that $\mu(Z) \leq \frac{1}{q} \int_X h d\mu$ for all $q > 0$. Hence Z is a null set.

We now want to show that for all $x \notin Z$ we have

$$f_x \simeq \sum_{k=1}^{\infty} \alpha_k A_k(x) B_k.$$

Choose $x \notin Z$. We see that

$$\sum_{k=1}^{\infty} |\alpha_k| A_k(x) \int_Y B_k d\nu = \sum_{k=1}^{\infty} |\alpha_k| A_k(x) \nu(B_k) < \infty.$$

If $y \in Y$ is such that

$$\sum_{k=1}^{\infty} |\alpha_k| A_k(x) B_k(y) < \infty,$$

then

$$\sum_{k=1}^{\infty} |\alpha_k| (A_k \times B_k)(x, y) = \sum_{k=1}^{\infty} |\alpha_k| A_k(x) B_k(y) < \infty$$

and hence

$$\sum_{k=1}^{\infty} \alpha_k A_k(x) B_k(y) = \sum_{k=1}^{\infty} \alpha_k (A_k \times B_k)(x, y) = f(x, y) = f_x(y),$$

which completes the proof. $\square$

Theorem 6.3 (Fubini's theorem). *If $f \in L^1(X \times Y, \Theta, \gamma)$ is integrable, then*

$$\int_X \left(\int_Y f_x d\nu \right) d\mu = \int_{X \times Y} f d\gamma = \int_Y \left(\int_X f_y d\mu \right) d\nu.$$

Proof. We prove only the first equality. If

$$f \simeq \sum_{k=1}^{\infty} \alpha_k A_k \times B_k$$

where $A_k \in \Sigma$ and $B_k \in \Xi$, then

$$f_x \simeq \sum_{k=1}^{\infty} \alpha_k A_k(x) B_k$$

for all $x \in X$ except possibly a set Z of μ -measure zero. Then

$$\int_Y f_x d\nu = \sum_{k=1}^{\infty} \alpha_k A_k(x) \nu(B_k)$$

whenever $x \notin Z$. Set

$$h_n = \sum_{k=1}^n \alpha_k \nu(B_k) A_k$$

and

$$h(x) = \begin{cases} \sum_{k=1}^{\infty} \alpha_k \nu(B_k) A_k(x) & \text{if } \sum_{k=1}^{\infty} |\alpha_k| \nu(B_k) A_k(x) < \infty \\ 0 & \text{otherwise} \end{cases}.$$

Each h_n is integrable and $h_n \rightarrow h$ μ -a.e.. In the proof of Lemma 6.2 we showed that

$$\sum_{k=1}^{\infty} |\alpha_k| \nu(B_k) A_k$$

is integrable, and clearly

$$|h_n| \leq \sum_{k=1}^{\infty} |\alpha_k| \nu(B_k) A_k$$

for each n , so the Lebesgue dominated convergence theorem implies that h is integrable over X and that

$$\int_X h d\mu = \sum_{k=1}^{\infty} \alpha_k \mu(A_k) \nu(B_k) = \sum_{k=1}^{\infty} \alpha_k \gamma(A_k \times B_k).$$

But this amounts to

$$\int_X \left(\int_Y f_x d\nu \right) d\mu = \int_{X \times Y} f d\gamma.$$

□

References

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