Positive measures and vector measures

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1 Introduction

We present here a construction of a complete measure space given a triple $(X, \mathcal{R}, \mu)$, where X is a nonempty set, $\mathcal{R}$ is a ring of subsets of X, and μ is a σ-additive set function with values in $[0, \infty)$. Then we extend the construction to vector measures, that is, measures with values in a Banach space $(\mathbb{E}, \|\cdot\|)$. A section on integration of real-valued functions with respect to vector measures is included at the end.

The approach presented here is based on ideas from [3] where the Bochner integral is presented for functions defined on $\mathbb{R}^N$. The method used here seems to be quite flexible and works well for a number of different constructions in analysis: the Lebesgue integral for functions on $\mathbb{R}^N$ (see [3], as well as [2], [4], and [6]), the Bochner integral for functions on $\mathbb{R}^N$ in (see [3]), integrals of functions with values in Banach spaces and locally convex spaces defined on abstract measure spaces (see [5]), the Daniell integral (see [1])).

2 Measure spaces

Let X be a nonempty set. A collection $\mathcal{R}$ of subsets of X is called a *ring of subsets* of X if

$A, B \in \mathcal{R}$ implies $A \cup B, A \setminus B \in \mathcal{R}$.

A map $\mu : \mathcal{R} \to [0, \infty)$ is called *σ-additive* if for any sequence of disjoint sets $A_1, A_2, \dots \in \mathcal{R}$ such that $\cup_{n=1}^{\infty} A_n \in \mathcal{R}$ we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Note that σ-additivity implies finite additivity.

In this note we use the same symbol to denote a subset of X and the characteristic function of that set, that is, if $A \subset X$ we will write

$$A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}.$$

Definition 2.1. A set $S \subset X$ is called *summable* if there are $A_1, A_2, \dots \in \mathcal{R}$ and $\alpha_1, \alpha_2, \dots \in \{-1, 1\}$ such that

$\mathbb{I}$ $\sum_{n=1}^{\infty} \mu(A_n) < \infty$,

$\mathbb{II}$ $S(x) = \sum_{n=1}^{\infty} \alpha_n A_n(x)$ for every $x \in X$ for which $\sum_{n=1}^{\infty} A_n(x) < \infty$.

If conditions $\mathbb{I}$ and $\mathbb{II}$ are satisfied, we write $S \simeq \sum_{n=1}^{\infty} \alpha_n A_n$.

Definition 2.2. A function $f : X \to \mathbb{Z}$ is called a *simple function* if

$$f(x) = \alpha_1 A_1(x) + \cdots + \alpha_n A_n(x) \tag{2.1}$$

for some $A_1, \dots, A_n \in \mathcal{R}$ and $\alpha_1, \dots \alpha_n \in \mathbb{Z}$. For the simple function (2.1) we define

$$I(f) = \alpha_1 \mu(A_1) + \cdots + \alpha_n \mu(A_n).$$

Exercise 2.3. Show that I is a well-defined additive functional on the space of simple functions.

Lemma 2.4. *If (f_n) is a non-increasing sequence of simple functions convergent to 0 at every point of X, then $\lim_{n\to\infty} I(f_n) = 0$.*

Proof. Let $A_n = \operatorname{supp} f_n = \{x \in X : f_n(x) \neq 0\}$, $n \in \mathbb{N}$. Since (f_n) is a non-increasing sequence, we have $A_1 \supset A_2 \supset \dots$. Define $B_n = A_n \setminus A_{n+1}$. Since $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in X$, we have $A_1 = \cup_{n=1}^{\infty} B_n$. Then $\mu(A_1) = \sum_{n=1}^{\infty} \mu(B_n)$, by σ-additivity of μ on $\mathcal{R}$. Hence $\sum_{k=n}^{\infty} \mu(B_k) \to 0$ as $n \to \infty$ and consequently

$$I(f_n) \leq \max |f_1| \mu(A_n) = \max |f_1| \sum_{k=n}^{\infty} \mu(B_k) \to 0$$

as $n \to \infty$. □

Lemma 2.5. *Let (g_n) and (h_n) be non-decreasing sequences of simple functions such that*

$$\lim_{n\to\infty} h_n(x) \leq \lim_{n\to\infty} g_n(x)$$

for every $x \in X$. Then

$$\lim_{n\to\infty} I(h_n) \leq \lim_{n\to\infty} I(g_n).$$

(In both inequalities we allow ∞.)

Proof. Fix $k \in \mathbb{N}$ and define

$$f_n = h_k - \min(h_k, g_n).$$

Note that (f_n) is a non-increasing sequence convergent to 0 at every $x \in X$. By Lemma 2.4,

$$\lim_{n\to\infty} I(f_n) = \lim_{n\to\infty} I(h_k - \min(h_k, g_n)) = 0.$$

Hence

$$I(h_k) = \lim_{n\to\infty} I(\min(h_k, g_n)) \le \lim_{n\to\infty} I(g_n).$$

We finish the proof by letting $k \to \infty$. □

Lemma 2.6. *If* $U \simeq \sum_{n=1}^{\infty} \alpha_n A_n$*, then* $\sum_{n=1}^{\infty} \alpha_n \mu(A_n) \ge 0$.

Proof. Let $\varepsilon > 0$. The $\sum_{n=n_0+1}^{\infty} \mu(A_n) < \varepsilon$ for some $n_0 \in \mathbb{N}$. Define

$$g_n = \alpha_1 A_1 + \dots + \alpha_{n_0} A_{n_0} + A_{n_0+1} + A_{n_0+2} + \dots$$

and

$$h_n = \max(g_n, 0).$$

Note that (g_n) and (h_n) satisfy the assumptions of Lemma 2.5. Hence

$$0 \le \lim_{n\to\infty} I(h_n) \le \lim_{n\to\infty} I(g_n)$$

and consequently

$$0 \le \alpha_1 \mu(A_1) + \dots + \alpha_{n_0} \mu(A_{n_0}) + \mu(A_{n_0+1}) + \mu(A_{n_0+2}) + \dots$$

Since $\sum_{n=n_0}^{\infty} \mu(A_n) < \varepsilon$, we must have

$$-2\varepsilon < \sum_{n=1}^{\infty} \alpha_n \mu(A_n),$$

which shows that $0 \le \sum_{n=1}^{\infty} \alpha_n \mu(A_n)$, because ε is an arbitrary positive number. □

Lemma 2.7. *If* $S \simeq \sum_{n=1}^{\infty} \alpha_n A_n$ *and* $S \simeq \sum_{n=1}^{\infty} \beta_n B_n$*, then* $\sum_{n=1}^{\infty} \alpha_n \mu(A_n) = \sum_{n=1}^{\infty} \beta_n \mu(B_n)$.

Proof. Note that

$$\emptyset \simeq \alpha_1 A_1 - \beta_1 B_1 + \alpha_2 A_2 - \beta_2 B_2 + \dots$$

By Lemma 2.6

$$0 \le \alpha_1 \mu(A_1) - \beta_1 \mu(B_1) + \alpha_2 \mu(A_2) - \beta_2 \mu(B_2) + \dots$$

and consequently

$$\sum_{n=1}^{\infty} \beta_n \mu(B_n) \le \sum_{n=1}^{\infty} \alpha_n \mu(A_n).$$

By switching the two representations of S, we obtain the inequality in the opposite directions. □

The above lemma allows us to define the measure of a summable set.

Definition 2.8. Let S be a summable set and let $S \simeq \sum_{n=1}^{\infty} \alpha_n A_n$. By the *measure* of S we mean the number

$$\mu(S) = \sum_{n=1}^{\infty} \alpha_n \mu(A_n).$$

Definition 2.9. A set $U \subset X$ is called *measurable* if $U \cap S$ is summable for all $S \in \mathcal{R}$. If U is measurable but not summable, we define $\mu(U) = \infty$. The collection of all measurable subsets of X will be denoted by $\Sigma(\mathcal{R}, \mu)$.

Lemma 2.10. *If $S \subset X$ is summable, then for every $\varepsilon > 0$ there is a representation $S \simeq \sum_{n=1}^{\infty} \alpha_n A_n$ such that*

$$\sum_{n=1}^{\infty} \mu(A_n) \leq \mu(S) + \varepsilon.$$

Proof. Let $S \simeq \sum_{n=1}^{\infty} \beta_n B_n$ be an arbitrary representation of S. There is $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0+1}^{\infty} \mu(B_n) < \frac{1}{4}\varepsilon.$$

Let $C_1, \dots, C_{n_1}, D_1, \dots, D_{n_2} \in \mathcal{R}$ be such that $C_j \cap D_k = \emptyset$ for all j and k and such that

$$\beta_1 B_1 + \dots + \beta_{n_0} B_{n_0} = C_1 + \dots + C_{n_1} - D_1 - \dots - D_{n_2}.$$

Note that

$$S \simeq C_1 + \dots + C_{n_1} - D_1 - \dots - D_{n_2} + \beta_{n_0+1} B_{n_0+1} + \beta_{n_0+2} B_{n_0+2} + \dots$$

Let $V = \operatorname{supp}(D_1 \cup \dots \cup D_{n_2})$. Then

$$S \cap V \simeq -D_1 - \dots - D_{n_2} + \beta_{n_0+1}(B_{n_0+1} \cap V) + \beta_{n_0+2}(B_{n_0+2} \cap V) + \dots$$

and hence, by Lemma 2.6,

$$\begin{aligned} 0 &\leq -\mu(D_1) - \dots - \mu(D_{n_2}) + \beta_{n_0+1}\mu(B_{n_0+1} \cap V) + \beta_{n_0+2}\mu(B_{n_0+2} \cap V) + \dots \\ &< -\mu(D_1) - \dots - \mu(D_{n_2}) + \frac{1}{4}\varepsilon. \end{aligned}$$

Consequently $\mu(D_1) + \dots + \mu(D_{n_2}) < \frac{1}{4}\varepsilon$. Since

$$\sum_{n=1}^{n_1} \mu(C_n) - \sum_{n=1}^{n_2} \mu(D_n) - \sum_{n=n_0+1}^{\infty} \mu(B_n) \leq \mu(S) \leq \sum_{n=1}^{n_1} \mu(C_n) + \sum_{n=1}^{n_2} \mu(D_n) + \sum_{n=n_0+1}^{\infty} \mu(B_n)$$

and

$$\left(\sum_{n=1}^{n_1} \mu(C_n) + \sum_{n=1}^{n_2} \mu(D_n) + \sum_{n=n_0+1}^{\infty} \mu(B_n)\right) - \left(\sum_{n=1}^{n_1} \mu(C_n) - \sum_{n=1}^{n_2} \mu(D_n) - \sum_{n=n_0+1}^{\infty} \mu(B_n)\right) < \varepsilon,$$

we conclude

$$\sum_{n=1}^{n_1} \mu(C_n) + \sum_{n=1}^{n_2} \mu(D_n) + \sum_{n=n_0+1}^{\infty} \mu(B_n) < \mu(S) + \varepsilon.$$

□

Now we extend the use of the symbol $\simeq$ to series of summable sets.

Definition 2.11. If $U \subset X$ and there are summable sets $S_1, S_2, \cdots \subset X$ and $\alpha_1, \alpha_2, \cdots \in \{-1, 1\}$ such that

$\mathbb{I}$ $\displaystyle\sum_{n=1}^{\infty} \mu(S_n) < \infty$ and

$\mathbb{II}$ $U(x) = \displaystyle\sum_{n=1}^{\infty} \alpha_n S_n(x)$ for every $x \in X$ for which $\displaystyle\sum_{n=1}^{\infty} S_n(x) < \infty$,

then we write $U \simeq \sum_{n=1}^{\infty} \alpha_n S_n$.

Lemma 2.12. *If $U \simeq \sum_{n=1}^{\infty} \alpha_n S_n$ and $S_1, S_2, \ldots$ are summable sets, then U is summable and $\mu(U) = \sum_{n=1}^{\infty} \alpha_n \mu(S_n)$.*

Proof. Let $B_{k,n} \in \mathcal{R}$ and $\beta_{k,n} \in \{-1, 1\}$, for $k, n \in \mathbb{N}$, be such that

$$S_k \simeq \sum_{n=1}^{\infty} \beta_{k,n} B_{k,n} \quad \text{and} \quad \sum_{n=1}^{\infty} \mu(B_{k,n}) < \mu(S_k) + 2^{-k}$$

for all $k \in \mathbb{N}$. Let $\sum_{n=1}^{\infty} \gamma_n A_n$ be a series arranged from all series $\sum_{n=1}^{\infty} \alpha_k \beta_{k,n} B_{k,n}$, $k \in \mathbb{N}$. It is easy to verify that $U \simeq \sum_{n=1}^{\infty} \gamma_n A_n$. Consequently, U is summable and

$$\mu(U) = \sum_{n=1}^{\infty} \gamma_n \mu(A_n) = \sum_{n=1}^{\infty} \alpha_n \mu(S_n).$$

□

Theorem 2.13. *$\Sigma(\mathcal{R}, \mu)$ is a σ-ring.*

Proof. Let $U, V \in \Sigma(\mathcal{R}, \mu)$ and let $S \in \mathcal{R}$. If

$$U \cap S \simeq \sum_{n=1}^{\infty} \alpha_n A_n \quad \text{and} \quad V \cap S \simeq \sum_{n=1}^{\infty} \beta_n B_n,$$

then

$$(U \setminus V) \cap S \simeq \sum_{n=1}^{\infty} \alpha_n A_n - \left(\sum_{n=1}^{\infty} \alpha_n A_n \right) \left(\sum_{n=1}^{\infty} \beta_n B_n \right)$$

and

$$(U \cup V) \cap S \simeq \sum_{n=1}^{\infty} \alpha_n A_n + \sum_{n=1}^{\infty} \beta_n B_n - \left(\sum_{n=1}^{\infty} \alpha_n A_n \right) \left(\sum_{n=1}^{\infty} \beta_n B_n \right).$$

If $U_1, U_2, \dots \in \Sigma$, then the sets

$$V_1 = U_1, V_n = U_n \setminus (U_1 \cup \dots \cup U_{n-1}) \text{ for } n > 1$$

are measurable in view of the above. If $S \in \mathcal{R}$, then

$$\left(\bigcup_{n=1}^{\infty} U_n \right) \cap S = \left(\bigcup_{n=1}^{\infty} V_n \right) \cap S \simeq \sum_{n=1}^{\infty} (V_n \cap S)$$

and thus $\bigcup_{n=1}^{\infty} U_n \in \Sigma$. $\square$

Theorem 2.14. *μ is a measure on Σ.*

Proof. If $U_1, U_2, \dots \in \Sigma$ are disjoint and $\bigcup_{n=1}^{\infty} \mu(U_n) < \infty$, then $\bigcup_{n=1}^{\infty} U_n \simeq \sum_{n=1}^{\infty} U_n$ and hence $\mu\left(\bigcup_{n=1}^{\infty} U_n\right) = \sum_{n=1}^{\infty} \mu(U_n)$. If $U_1, U_2, \dots \in \Sigma$ are disjoint and $\bigcup_{n=1}^{\infty} \mu(U_n) = \infty$, then it is easy to see that $\mu\left(\bigcup_{n=1}^{\infty} U_n\right) = \infty$. $\square$

Theorem 2.15. *μ is complete.*

Proof. If $\mu(U) = 0$ and $V \subset U$, then $V \simeq U + U + \dots$ and thus $V \in \Sigma$ and $\mu(V) = \mu(U) + \mu(U) + \dots = 0$. $\square$

3 Vector Measures

Let X be a nonempty set, $\mathcal{R}$ a ring of subsets of X, and let $(\mathbb{E}, \|\cdot\|)$ be a Banach space.

A set function $\mu : \mathcal{R} \to \mathbb{E}$ is called *σ-additive* if for any sequence of disjoint sets $A_1, A_2, \dots \in \mathcal{R}$ such that $\cup_{n=1}^{\infty} A_n \in \mathcal{R}$ we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Definition 3.1. By the *variation* a σ-additive set function $\mu : \mathcal{R} \to \mathbb{E}$ we mean the set function $|\mu| : \mathcal{R} \to [0, \infty]$ defined by

$$|\mu|(A) = \sup \left\{ \sum_{B \in \pi} ||\mu(B)|| : \pi \subset \mathcal{R} \text{ is a finite partition of } A \right\}.$$

Note that $|\mu|(A) \geq \|\mu(A)\|$ for any set $A \in \mathcal{R}$. If $|\mu|(X) < \infty$, then we say that μ is of *bounded variation.*

In the reminder of this paper we assume that $\mu : \mathcal{R} \to E$ is a σ-additive set function of bounded variation.

Lemma 3.2. *$|\mu|$ is monotone and finitely additive.*

Proof. Clearly $|\mu|$ is monotone. Now consider disjoint $A_1, \ldots, A_n \in \mathcal{R}$. Since the union of finite partitions of $A_1, \ldots, A_n$ is a finite partition of $A_1 \cup \cdots \cup A_n$, we have

$$|\mu|\left(\bigcup_{k=1}^{n} A_k\right) \geq \sum_{k=1}^{n} |\mu|(A_k).$$

On the other hand, every finite partition π of $\cup_{k=1}^{n} A_k$ can be refined to a partition $\cup_{k=1}^{n} \pi_k$, where π_k is a finite partition of A_k for $k = 1, \ldots, n$. Then

$$\sum_{B\in\pi} ||\mu(B)|| \leq \sum_{B\in\pi_1\cup\cdots\cup\pi_n} ||\mu(B)|| = \sum_{k=1}^{n} \sum_{B\in\pi_k} ||\mu(B)|| \leq \sum_{k=1}^{n} |\mu|(A_k)$$

and consequently

$$|\mu|\left(\bigcup_{k=1}^{n} A_k\right) \leq \sum_{k=1}^{n} |\mu|(A_k).$$

□

Theorem 3.3. *If $\mu : \mathcal{R} \to E$ is a σ-additive set function of bounded variation, then $|\mu|$ is a σ-additive positive measure on $\mathcal{R}$.*

Proof. Let $A_1, A_2, \dots \in \mathcal{R}$ be a sequence of disjoint sets such that $A = \cup_{n=1}^{\infty} A_n \in \mathcal{R}$. By Lemma 3.2, for every $m \in \mathbb{N}$, we have

$$\sum_{n=1}^{m} |\mu|(A_n) \leq |\mu|(A).$$

Consequently

$$\sum_{n=1}^{\infty} |\mu|(A_n) \leq |\mu|(A).$$

Now let ε be an arbitrary positive number. Then

$$|\mu|(A) < \sum_{j=1}^{k} \|\mu(B_j)\| + \varepsilon,$$

for some partition $\{B_1, \ldots, B_k\}$ of A, and we have

$$\begin{aligned}
|\mu|(A) &< \sum_{j=1}^{k} \|\mu(B_j)\| + \varepsilon \\
&= \sum_{j=1}^{k} \left\| \mu\left(\bigcup_{n=1}^{\infty} (A_n \cap B_j) \right) \right\| + \varepsilon \\
&\le \sum_{j=1}^{k} \sum_{n=1}^{\infty} \|\mu(A_n \cap B_j)\| + \varepsilon \\
&= \sum_{n=1}^{\infty} \sum_{j=1}^{k} \|\mu(A_n \cap B_j)\| + \varepsilon \\
&\le \sum_{n=1}^{\infty} |\mu(A_n)| + \varepsilon.
\end{aligned}$$

Since ε is arbitrary, we obtain

$$|\mu|(A) \le \sum_{n=1}^{\infty} |\mu(A_n)|.$$

□

From the construction in Section 2 we obtain the following

Corollary 3.4. *If $\mu : \mathcal{R} \to \mathbb{E}$ is a σ-additive set function of bounded variation, then $|\mu|$ can be extended to a complete σ-additive positive measure on $\Sigma(\mathcal{R}, |\mu|)$.*

We are going to use $|\mu|$ to denote the variation of μ defined on $\mathcal{R}$ as well as its extension to a positive measure on $\Sigma(\mathcal{R}, |\mu|)$.

Our next goal is to define an extension of μ to a σ-additive $\mathbb{E}$-valued measure on $\Sigma(\mathcal{R}, |\mu|)$. We start with the following auxiliary lemma.

Lemma 3.5. *If $\emptyset \simeq \sum_{n=1}^{\infty} \alpha_n A_n$, then $\sum_{n=1}^{\infty} \alpha_n \mu(A_n) = 0$.*

Proof. First note that, if $\emptyset \simeq \sum_{n=1}^{\infty} \alpha_n A_n$, then $\sum_{n=1}^{\infty} \alpha_n |\mu|(A_n) = 0$. Let $\varepsilon > 0$ and let $n_0 \in \mathbb{N}$ be such that

$$\sum_{n=n_0+1}^{\infty} |\mu|(A_n) < \varepsilon.$$

Let $C_1, \ldots, C_{n_1}, D_1, \ldots, D_{n_2} \in \mathcal{R}$ be such that $C_j \cap D_k = \emptyset$ for all j and k and such that

$$\alpha_1 A_1 + \cdots + \alpha_{n_0} A_{n_0} = C_1 + \cdots + C_{n_1} - D_1 - \cdots - D_{n_2}. \tag{3.1}$$

Note that

$$\emptyset \simeq C_1 + \dots + C_{n_1} - D_1 - \dots - D_{n_2} + \alpha_{n_0+1}A_{n_0+1} + \alpha_{n_0+2}A_{n_0+2} + \dots$$

If $V = \operatorname{supp}(D_1 \cup \dots \cup D_{n_2})$, then

$$\emptyset \simeq -D_1 - \dots - D_{n_2} + \alpha_{n_0+1}(A_{n_0+1} \cap V) + \alpha_{n_0+2}(A_{n_0+2} \cap V) + \dots$$

and hence

$$\begin{aligned}0 &= -|\mu|(D_1) - \dots - |\mu|(D_{n_2}) + \alpha_{n_0+1}|\mu|(A_{n_0+1} \cap V) + \alpha_{n_0+2}|\mu|(A_{n_0+2} \cap V) + \dots \\ &< -|\mu|(D_1) - \dots - |\mu|(D_{n_2}) + \varepsilon.\end{aligned}$$

Consequently

$$|\mu|(D_1) + \dots + |\mu|(D_{n_2}) < \varepsilon \tag{3.2}$$

and since

$$0 = \sum_{n=1}^{n_1} |\mu|(C_n) - \sum_{n=1}^{n_2} |\mu|(D_n) + \sum_{n=n_0+1}^{\infty} \alpha_n |\mu|(A_n),$$

also

$$|\mu|(C_1) + \dots + |\mu|(C_{n_2}) < 2\varepsilon. \tag{3.3}$$

From (3.1), (3.2), and (3.3) we obtain

$$\begin{aligned}\left\|\sum_{n=1}^{n_0} \alpha_n \mu(A_n)\right\| &= \|\mu(C_1) + \dots + \mu(C_{n_1}) - \mu(D_1) - \dots - \mu(D_{n_2})\| \\ &\le \|\mu(C_1)\| + \dots + \|\mu(C_{n_1})\| + \|\mu(D_1)\| + \dots + \|\mu(D_{n_2})\| \\ &\le |\mu|(C_1) + \dots + |\mu|(C_{n_1}) + |\mu|(D_1) + \dots + |\mu|(D_{n_2}) \\ &< 3\varepsilon\end{aligned}$$

For any $m > n_0$ we have

$$\begin{aligned}\left\|\sum_{n=1}^{m} \alpha_n \mu(A_n)\right\| &\le \left\|\sum_{n=1}^{n_0} \alpha_n \mu(A_n)\right\| + \left\|\sum_{n=n_0+1}^{m} \alpha_n \mu(A_n)\right\| \\ &\le \left\|\sum_{n=1}^{n_0} \alpha_n \mu(A_n)\right\| + \sum_{n=n_0+1}^{m} \|\mu(A_n)\| \\ &\le \sum_{n=1}^{m} |\mu|(A_n) < 4\varepsilon\end{aligned}$$

Since ε is arbitrary, we obtain $\sum_{n=1}^{\infty} \alpha_n \mu(A_n) = 0$. □

Corollary 3.6. *If* $S \simeq \sum_{n=1}^{\infty} \alpha_n A_n$ *and* $S \simeq \sum_{n=1}^{\infty} \beta_n B_n$*, then* $\sum_{n=1}^{\infty} \alpha_n \mu(A_n) = \sum_{n=1}^{\infty} \beta_n \mu(B_n)$*.*

Proof. It suffices to note that

$$\emptyset \simeq \alpha_1 A_1 - \beta_1 B_1 + \alpha_2 A_2 - \beta_2 B_2 + \dots$$

and then use Lemma 3.5. $\square$

This lemma justifies the following definition of the $\mathbb{E}$-valued μ-measure of any μ-measurable subset of X.

Definition 3.7. For $S \simeq \sum_{n=1}^{\infty} \alpha_n A_n$, we define $\mu(S) = \sum_{n=1}^{\infty} \alpha_n \mu(A_n)$.

It remains to show that μ is σ-additive on $\Sigma(\mathcal{R}, |\mu|)$. We start with a lemma that is a vector version of Lemma 2.10.

Lemma 3.8. *If* $U \simeq \sum_{n=1}^{\infty} \alpha_n U_n$ *for some* $U_1, U_2, \ldots \in \Sigma(\mathcal{R}, |\mu|)$*, then* $\mu(U) = \sum_{n=1}^{\infty} \alpha_n \mu(U_n)$.

Proof. By Lemma 2.10, for every $k \in \mathbb{N}$ there exists an expansion $U_k \simeq \sum_{n=1}^{\infty} \beta_{k,n} B_{k,n}$, with $U_{k,n} \in \mathcal{R}$ such that

$$\sum_{n=1}^{\infty} |\mu|(B_{k,n}) < |\mu|(U_k) + 2^{-k}.$$

Let $\sum_{n=1}^{\infty} \gamma_n C_n$ be a series arranged from all series $\sum_{n=1}^{\infty} \alpha_k \beta_{k,n} B_{k,n}$. Then $U \simeq \sum_{n=1}^{\infty} \gamma_n C_n$ and

$$\mu(U) = \sum_{n=1}^{\infty} \gamma_n \mu(C_n) = \sum_{n=1}^{\infty} \alpha_n \mu(U_n).$$

$\square$

Theorem 3.9. μ *is* σ*-additive on* $\Sigma(\mathcal{R}, |\mu|)$.

Proof. If $U_1, U_2, \ldots \in \Sigma(\mathcal{R}, |\mu|)$ are disjoint, then $\bigcup_{n=1}^{\infty} U_n \simeq \sum_{n=1}^{\infty} U_n$. Therefore $\mu(\bigcup_{n=1}^{\infty} U_n) = \sum_{n=1}^{\infty} \mu(U_n)$, by Lemma 3.8. $\square$

4 Integration of Real-valued Functions with respect to Vector Measures

Let X be a nonempty set, Σ a σ-algebra of subsets of X, and ν a positive measure on Σ. In this section we define the integral of a real-valued function $f \in L^1(X, \Sigma, \nu)$ with respect to a vector measure μ such that $|\mu| = \nu$. In order to understand this section one has to be familiar with the construction of $L^1(X, \Sigma, \nu)$ as presented in [1] or [5]. In particular, we will use the fact that $f \in L^1(X, \Sigma, \nu)$ if and only if there exists a sequence of simple functions (f_n) such that the following two conditions are satisfied:

$$\mathbb{I} \quad \sum_{n=1}^{\infty} \int |f_n| d\nu < \infty;$$

$$\mathbb{II} \quad f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \text{for every } x \in X \text{ such that } \sum_{n=1}^{\infty} |f_n(x)| < \infty.$$

A function $f : X \to \mathbb{R}$ is called a *simple function* if $f(x) = \alpha_1 A_1(x) + \cdots + \alpha_n A_n(x)$ for some $A_1, \ldots, A_n \in \Sigma$ and $\alpha_1, \ldots \alpha_n \in \mathbb{R}$. If conditions $\mathbb{I}$ and $\mathbb{II}$ are satisfied, we write $f \simeq \sum_{n=1}^{\infty} f_n$.

We will also need the following property that follows easily from Lemma 3.8 in [1].

Lemma 4.1. *If $f \in L^1(X, \Sigma, \nu)$ and $\varepsilon > 0$, then there exist $A_n \in \Sigma$ and $\lambda_n \in \mathbb{R}$ such that $f \simeq \sum_{n=1}^{\infty} \lambda_n A_n$ and $\sum\limits_{n=1}^{\infty} |\lambda_n| \nu(A_n) \leq \int |f| + \varepsilon$.*

Let $(\mathbb{E}, \|\cdot\|)$ be a Banach space and let μ be a $\mathbb{E}$-valued measure of bounded variation on (X, Σ). For a simple function $f = \alpha_1 A_1 + \cdots + \alpha_n A_n$, where $A_1, \ldots, A_n \in \Sigma$ and $\alpha_1, \ldots \alpha_n \in \mathbb{R}$, we define

$$\int f d\mu = \alpha_1 \mu(A_1) + \cdots + \alpha_n \mu(A_n).$$

Exercise 4.2. Show that this integral is well-defined linear map from the vector space of all simple functions to $\mathbb{E}$.

Lemma 4.3. *If f is a simple function, then $\left\| \int f d\mu \right\| \leq \int |f| d|\mu|$.*

Proof. If f is a simple function, then $f = \alpha_1 A_1 + \cdots + \alpha_n A_n$ for some disjoint $A_1, \ldots, A_n \in \Sigma$. Then

$$\begin{aligned} \left\| \int f d\mu \right\| &= \|\alpha_1 \mu(A_1) + \cdots + \alpha_n \mu(A_n)\| \\ &\leq \|\alpha_1 \mu(A_1)\| + \cdots + \|\alpha_n \mu(A_n)\| \\ &\leq |\alpha_1| |\mu|(A_1) + \cdots + |\alpha_n| |\mu|(A_n) \\ &= \int |f| d|\mu|. \end{aligned}$$

□

Lemma 4.4. *If $f \in L^1(X, \Sigma, |\mu|)$ and $f \simeq \sum_{n=1}^{\infty} f_n$, where f_n's are simple functions, then the series $\sum_{n=1}^{\infty} \int f_n d\mu$ is absolutely convergent in $\mathbb{E}$.*

Proof. From Lemma 4 we obtain

$$\sum_{n=1}^{\infty}\left\|\int f_n d\mu\right\| \le \sum_{n=1}^{\infty}\int |f_n| d|\mu| < \infty.$$

□

Lemma 4.5. *If* $0 \simeq \sum_{n=1}^{\infty} f_n$*, where* f_n*'s are simple functions, then*

$$\sum_{n=1}^{\infty}\int f_n d\mu = 0.$$

Proof. If $0 \simeq \sum_{n=1}^{\infty} f_n$, then $\sum_{n=1}^{\infty}\int |f_n| d|\mu| < \infty$. For an arbitrary $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0+1}^{\infty}\int |f_n|\, d|\mu| < \frac{\varepsilon}{2}.$$

Since

$$-(f_1 + \cdots + f_{n_0}) \simeq f_{n_0+1} + f_{n_0+2} + \ldots,$$

we have

$$\int |f_1 + \cdots + f_{n_0}|\, d|\mu| \le \int |f_{n_0+1}|\, d|\mu| + \int |f_{n_0+2}|\, d|\mu| + \cdots < \frac{\varepsilon}{2}$$

and consequently

$$\begin{aligned}\left\|\sum_{n=1}^{\infty}\int f_n d\mu\right\| &\le \left\|\sum_{n=1}^{n_0}\int f_n d\mu\right\| + \left\|\sum_{n=n_0+1}^{\infty}\int f_n d\mu\right\| \\ &\le \int |f_1 + \cdots + f_{n_0}|\, d|\mu| + \sum_{n=n_0+1}^{\infty}\left\|\int f_n d\mu\right\| \\ &\le \frac{\varepsilon}{2} + \sum_{n=n_0+1}^{\infty}\int |f_n|\, d|\mu| < \varepsilon.\end{aligned}$$

Since ε is arbitrary, we conclude that $\sum_{n=1}^{\infty}\int f_n d\mu = 0$. □

Lemma 4.6. *If* $f \simeq \sum_{n=1}^{\infty} f_n$ *and* $f \simeq \sum_{n=1}^{\infty} g_n$*, where* f_n*'s and* g_n*'s are simple functions, then*

$$\sum_{n=1}^{\infty}\int f_n d\mu = \sum_{n=1}^{\infty}\int g_n d\mu.$$

Proof. If $f \simeq \sum_{n=1}^{\infty} f_n$ and $f \simeq \sum_{n=1}^{\infty} g_n$, then

$$0 \simeq f_1 - g_1 + f_2 - g_2 + \ldots$$

and consequently, by the above lemma,

$$0 = \int f_1 d\mu - \int g_1 d\mu + \int f_2 d\mu - \int g_2 d\mu + \dots$$

Absolute convergence of the series, by Lemma 4.4, implies $\sum_{n=1}^{\infty} \int f_n d\mu = \sum_{n=1}^{\infty} \int g_n d\mu$. □

Definition 4.7. If $f \in L^1(X, \Sigma, |\mu|)$, then we define

$$\int f d\mu = \sum_{n=1}^{\infty} \int f_n d\mu,$$

where $f \simeq \sum_{n=1}^{\infty} f_n$ for some simple functions f_n.

Lemma 4.6 shows that the integral is well-defined.

Theorem 4.8. *For any $f \in L^1(X, \Sigma, |\mu|)$ we have*

$$\left\| \int f d\mu \right\| \leq \int |f| d|\mu|.$$

Proof. Let $f \in L^1(X, \Sigma, |\mu|)$. By Lemma 4.1, for any $\varepsilon > 0$ there exist $A_n \in \Sigma$ and $\lambda_n \in \mathbb{R}$ such that $f \simeq \sum_{n=1}^{\infty} \lambda_n A_n$ and $\sum\limits_{n=1}^{\infty} |\lambda_n||\mu|(A_n) \leq \int |f| d|\mu| + \varepsilon$. Then

$$\begin{aligned}
\left\| \int f d\mu \right\| &= \left\| \sum_{n=1}^{\infty} \int (\lambda_n A_n) d\mu \right\| \\
&= \left\| \sum_{n=1}^{\infty} \lambda_n \mu(A_n) \right\| \\
&\leq \sum_{n=1}^{\infty} \| \lambda_n \mu(A_n) \| \\
&\leq \sum_{n=1}^{\infty} |\lambda_n||\mu|(A_n) \\
&\leq \int |f| d|\mu| + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the desired inequality follows. □

We have shown that $\int : L^1(X, \mathcal{M}, |\mu|) \to E$ is a bounded linear map.

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