

The Riemann integral

without Riemann sums

Piotr Mikusiński

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Contents

Introduction	1
1 Simple functions	2
2 Riemann integrable functions	7
3 Integrability of linear combinations and products	11
4 The Riemann integral on an interval	14
5 Integrability of limits of sequences of integrable functions	19
6 The Fundamental Theorems of Calculus	21

Introduction

In the standard approach a function $f(x)$ defined on an interval $[a, b]$ is called Riemann integrable if the limit

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*)(x_i - x_{i-1}) \quad \text{as } \max\{x_1 - x_0, x_2 - x_1, \dots, x_n - x_{n-1}\} \rightarrow 0$$

exists and is independent of partitions and choices of sample points

$$a = x_0 \leq x_1^* \leq x_1 \leq x_2^* \leq x_2 \leq \dots \leq x_{n-1} \leq x_n^* \leq x_n = b$$

and then the limit is denoted by $\int_a^b f(x)dx$.

This approach emphasizes partitions and sample points. It leads to more complicated arguments where we have to deal with unnecessary technicalities. For example, when forming Riemann sums for the function in Figure 1 we are required to keep adding partition points between 2 and 5 although it is obvious that it does not improve the accuracy of the approximation.

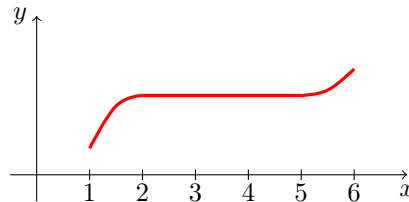


Figure 1: A function with a flat part

The approach presented here is completely equivalent to the standard approach, that is, it defines the same class of Riemann integrable functions and the same integral, but it is simpler and more intuitive. For example, we do not have to deal with the problem described above. It allows us to simplify proofs of properties of Riemann integrable functions and the Riemann integral.

We present here the Riemann integral for functions on the real line $\mathbb{R}$, but this approach has a natural extension to functions in $\mathbb{R}^N$. All functions in our presentation are real valued and defined on $\mathbb{R}$ unless explicitly stated otherwise.

In Sections 1-3 we use the notation $\int f$ instead of the standard $\int_a^b f(x)dx$ because we define Riemann integrability of a function defined on $\mathbb{R}$ rather than an interval $[a, b]$ as is done in the traditional approach. For a Riemann integrable function f we define the integral $\int f$ without any reference to a specific interval. The notation $\int_a^b f(x)dx$ is introduced in Section 4 where Riemann integrability on an interval is introduced and the connection with Riemann integrability on the real line, as presented in Sections 1-3, is carefully explained. In Section 5 we investigate the question whether $f_n \rightarrow f$ implies $\int f_n \rightarrow \int f$. Finally, in Section 6 we present the Fundamental Theorem of Calculus and related properties of the integral.

The presentation is intended for students who have completed an introductory single variable calculus course and are curious about why the concepts they learned work the way they do. Most of the material is accessible at that level, though some proofs draw on ideas and techniques typically covered in a first course in advanced calculus. Readers who are comfortable with the results but find certain proofs technically demanding are encouraged to skip or skim them on a first reading and return to them later.

1 Simple functions

First we need to define the integral for simple functions. We start with the observation that the area under the graph of the function

$$\varphi(x) = \begin{cases} 1, & \text{if } x \in [a, b) \\ 0 & \text{otherwise} \end{cases},$$

where $-\infty < a < b < \infty$, is $b - a$ (see Fig. 2). Note that the area does not change if we replace the interval $[a, b)$ by $[a, b]$, $(a, b]$, or (a, b) .

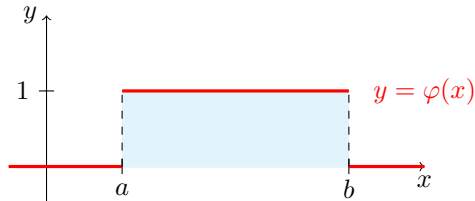


Figure 2: A brick function

We will call the area under the graph of $y = \varphi(x)$ the *integral* of φ and write

$$\int \varphi = b - a.$$

It seems that the left-hand side of this equation does not depend on a and b , but it does, since $\varphi(x)$ depends on a and b . A function like $\varphi(x)$ above, that is 1 on an interval $[a, b)$ and 0 everywhere else, will be called a *brick function on $[a, b)$* . If φ is a brick function on $[a, b)$, then the interval $[a, b)$ will be called the *support* of φ and denoted by $\text{supp } \varphi$. Thus, if φ is a brick function on $[a, b)$, then $\text{supp } \varphi = [a, b)$. Since the brick function on $[a, b)$ is the characteristic function of the interval $[a, b)$ the notation $\chi_{[a, b)}$ can be used when convenient. When we say “the brick function on $[a, b)$ ” we always assume that $-\infty < a < b < \infty$.

We use brick functions to build simple functions. A simple function may look like this.

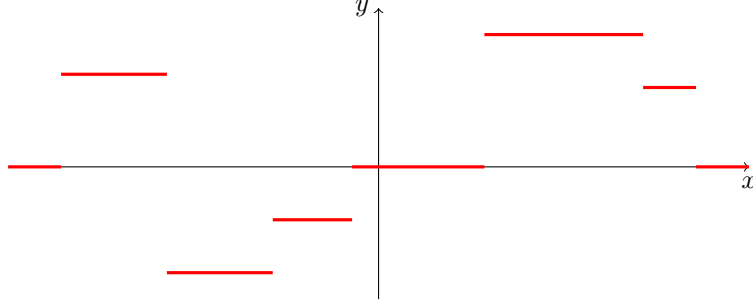


Figure 3: A simple function

Definition 1.1. A function f is called a *simple function* if f is a linear combination of brick functions, that is, if

$$f = \alpha_1 \varphi_1 + \alpha_2 \varphi_2 + \cdots + \alpha_n \varphi_n = \sum_{i=1}^n \alpha_i \varphi_i,$$

where $\varphi_1, \varphi_2, \dots, \varphi_n$ are brick functions and $\alpha_1, \alpha_2, \dots, \alpha_n$ are real numbers.

In the definition above the equality $f = \alpha_1 \varphi_1 + \alpha_2 \varphi_2 + \cdots + \alpha_n \varphi_n$ means that $f(x) = \alpha_1 \varphi_1(x) + \alpha_2 \varphi_2(x) + \cdots + \alpha_n \varphi_n(x)$ for every $x \in \mathbb{R}$. In general, if f and g are any functions on $\mathbb{R}$, the equality $f = g$ means $f(x) = g(x)$ for every $x \in \mathbb{R}$.

The simple function in Figure 3 can be written as a linear combination of 5 brick functions. In general, there are many different ways of representing a simple function as a linear combination of brick functions. For example, the simple function in Figure 4 can be described as the sum of the brick function on $[1, 2)$ plus 2 times the brick function on $[2, 3)$ plus the brick function on $[3, 5)$, but it can also be written as the sum of the brick function on $[1, 5)$ plus the brick function on $[2, 3)$.

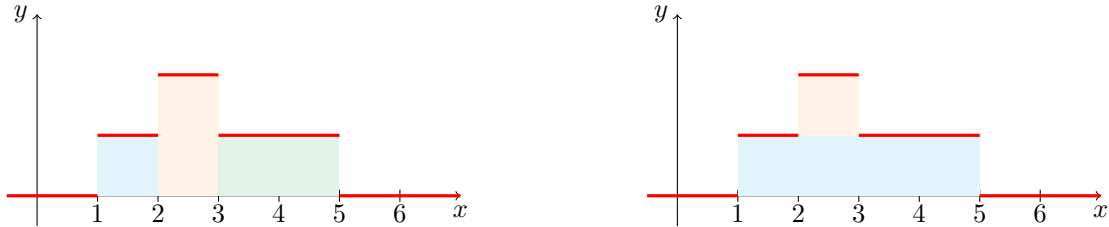


Figure 4: Two representations of a simple function

Since the area under the graph of the function

$$f(x) = \begin{cases} \alpha, & \text{if } x \in [a, b) \\ 0 & \text{otherwise} \end{cases}$$

is $\alpha(b - a)$ (see Fig. 5), it is natural to define $\int f = \alpha(b - a)$. Note that, if φ is a brick function on $[a, b)$, that is $\varphi(x) = 1$ if $x \in [a, b)$ and $\varphi(x) = 0$ otherwise, then $f(x) = \alpha\varphi(x)$ and we have

$$\int \alpha\varphi = \int f = \alpha(b - a) = \alpha \int \varphi.$$

This observation motivates the following definition.

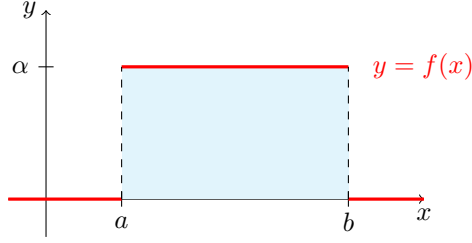


Figure 5: The area of the shaded rectangle is $\alpha(b - a)$.

Definition 1.2. For a simple function

$$f(x) = \alpha_1 \varphi_1(x) + \alpha_2 \varphi_2(x) + \cdots + \alpha_n \varphi_n(x) = \sum_{i=1}^n \alpha_i \varphi_i(x)$$

we define

$$\int f = \alpha_1 \int \varphi_1 + \alpha_2 \int \varphi_2 + \cdots + \alpha_n \int \varphi_n = \sum_{i=1}^n \alpha_i \int \varphi_i.$$

The value of $\int f$ is called the *integral* of f .

Note that while the integral of a simple function is defined using a linear combination of brick functions, this representation is not unique. It is therefore necessary to show that the integral is independent of the representation used.

Theorem 1.3. *The integral of a simple function is independent of the representation.*

Proof. We need to show that, if

$$\sum_{i=1}^n \alpha_i \varphi_i = \sum_{j=1}^m \beta_j \psi_j,$$

then

$$\sum_{i=1}^n \alpha_i \int \varphi_i = \sum_{j=1}^m \beta_j \int \psi_j.$$

First we show that, if $\sum_{i=1}^n \alpha_i \varphi_i = 0$, then $\sum_{i=1}^n \alpha_i \int \varphi_i = 0$. Let $\text{supp } \varphi_i = [a_i, b_i)$ for $i = 1, \dots, n$, and let $c_0 < c_1 < \cdots < c_m$ be all the distinct numbers in the set $\{a_1, \dots, a_n, b_1, \dots, b_n\}$. Then for every $i = 1, \dots, n$ and every $j = 1, \dots, m$ there are numbers $\gamma_{i,j} \in \{0, 1\}$ such that

$$\varphi_i = \sum_{j=1}^m \gamma_{i,j} \psi_j,$$

where ψ_j is the brick function on $[c_{j-1}, c_j)$. Note that for every brick function φ_i on $[a_i, b_i)$ there are indices $k < l$ such that $c_k = a_i$ and $c_l = b_i$. Then $\gamma_{i,j} = 1$ for $k+1 \leq j \leq l$ and $\gamma_{i,j} = 0$ for the remaining j 's. Consequently,

$$\begin{aligned} \int \varphi_i &= c_l - c_k = (c_{k+1} - c_k) + (c_{k+2} - c_{k+1}) + \cdots + (c_l - c_{l-1}) \\ &= \int \psi_{k+1} + \int \psi_{k+2} + \cdots + \int \psi_l = \sum_{j=k+1}^l \int \psi_j = \sum_{j=1}^m \gamma_{i,j} \int \psi_j \end{aligned}$$

and thus

$$\sum_{i=1}^n \alpha_i \int \varphi_i = \sum_{i=1}^n \alpha_i \left(\sum_{j=1}^m \gamma_{i,j} \int \psi_j \right) = \sum_{i=1}^n \sum_{j=1}^m \alpha_i \gamma_{i,j} \int \psi_j = \sum_{j=1}^m \sum_{i=1}^n \alpha_i \gamma_{i,j} \int \psi_j = \sum_{j=1}^m \left(\sum_{i=1}^n \alpha_i \gamma_{i,j} \right) \int \psi_j.$$

Now, if $\sum_{i=1}^n \alpha_i \varphi_i = 0$, we have

$$0 = \sum_{i=1}^n \alpha_i \varphi_i = \sum_{i=1}^n \alpha_i \left(\sum_{j=1}^m \gamma_{i,j} \psi_j \right) = \sum_{i=1}^n \sum_{j=1}^m \alpha_i \gamma_{i,j} \psi_j = \sum_{j=1}^m \sum_{i=1}^n \alpha_i \gamma_{i,j} \psi_j = \sum_{j=1}^m \left(\sum_{i=1}^n \alpha_i \gamma_{i,j} \right) \psi_j.$$

Since $\sum_{j=1}^m \left(\sum_{i=1}^n \alpha_i \gamma_{i,j} \right) \psi_j = 0$ and the supports $\psi_1, \dots, \psi_m$ are disjoint, we must have

$$\sum_{i=1}^n \alpha_i \gamma_{i,j} = 0$$

for every $j = 1, \dots, m$ and consequently

$$\sum_{i=1}^n \alpha_i \int \varphi_i = \sum_{j=1}^m \left(\sum_{i=1}^n \alpha_i \gamma_{i,j} \right) \int \psi_j = 0.$$

This shows that, if $\sum_{i=1}^n \alpha_i \varphi_i = 0$, then $\sum_{i=1}^n \alpha_i \int \varphi_i = 0$.

Now assume

$$\sum_{i=1}^n \alpha_i \varphi_i = \sum_{j=1}^m \beta_j \psi_j$$

for some brick functions $\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_m$ and some numbers $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m$. Then

$$\alpha_1 \varphi_1 + \dots + \alpha_n \varphi_n - \beta_1 \psi_1 - \dots - \beta_m \psi_m = 0$$

and thus, by the first part of the proof, we have

$$\alpha_1 \int \varphi_1 + \dots + \alpha_n \int \varphi_n - \beta_1 \int \psi_1 - \dots - \beta_m \int \psi_m = 0$$

or equivalently

$$\alpha_1 \int \varphi_1 + \dots + \alpha_n \int \varphi_n = \beta_1 \int \psi_1 + \dots + \beta_m \int \psi_m.$$

□

The integral of a simple function has an intuitive interpretation as “signed area”. The areas of the rectangles above the x -axis count as positive and the areas of the rectangles below the x -axis count as negative. For the function in Figure 6, $\int f$ is the sum of the areas of the blue rectangles minus the sum of the areas of the orange rectangles.

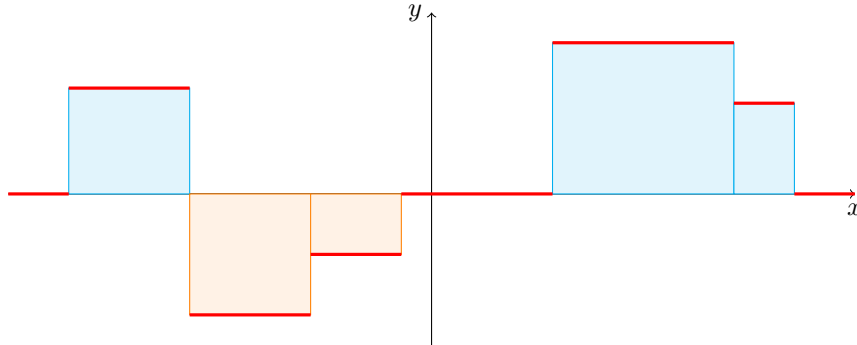


Figure 6: The integral of a simple function

The following theorem captures two fundamental properties of simple functions and the integral of simple functions: The set of all simple functions is a vector space and the integral is a linear functional on that space.

Theorem 1.4. If $f_1, f_2, \dots, f_n$ are simple functions and $a_1, a_2, \dots, a_n$ are real numbers, then the function $f = a_1 f_1 + a_2 f_2 + \dots + a_n f_n = \sum_{i=1}^n a_i f_i$ is also a simple function and we have

$$\int f = a_1 \int f_1 + a_2 \int f_2 + \dots + a_n \int f_n = \sum_{i=1}^n a_i \int f_i.$$

In other words,

$$\int \left(\sum_{i=1}^n a_i f_i \right) = \sum_{i=1}^n a_i \int f_i.$$

The proof of the above theorem is left as an exercise. Consider constructing your proof following these steps:

- Show that, if f is a simple function and α is a real number, then αf is a simple function.
- Show that, if f and g are simple functions, then $f + g$ is a simple function.
- Use (a), (b), and mathematical induction to show that, if $f_1, f_2, \dots, f_n$ are simple functions and $a_1, a_2, \dots, a_n$ are real numbers, then $f = a_1 f_1 + a_2 f_2 + \dots + a_n f_n = \sum_{i=1}^n a_i f_i$ is a simple function.
- Show that, if f is a simple function and α is a real number, then $\int \alpha f = \alpha \int f$.
- Show that, if f and g are simple functions, then $\int (f + g) = \int f + \int g$.
- Use (d) and (e) and the mathematical induction to show that, if $f_1, f_2, \dots, f_n$ are simple functions and $a_1, a_2, \dots, a_n$ are real numbers, then $\int \left(\sum_{i=1}^n a_i f_i \right) = \sum_{i=1}^n a_i \int f_i$.

In the next theorem we list some properties of simple functions that will be used in subsequent sections.

Theorem 1.5. Let f and g be simple functions.

- If $f(x) \leq g(x)$ for all $x \in \mathbb{R}$, then $\int f \leq \int g$.
- The function $|f|(x) = |f(x)|$ is a simple function and $\left| \int f \right| \leq \int |f|$.
- The functions $m(x) = \min\{f(x), g(x)\}$ and $M(x) = \max\{f(x), g(x)\}$ are simple functions.
- The function $h(x) = f(x)g(x)$ is a simple function.

Exercises

1.1. In the proof of Theorem 1.3 we claim that, since $\sum_{j=1}^m \left(\sum_{i=1}^n \alpha_i \gamma_{i,j} \right) \psi_j = 0$ and the supports $\psi_1, \dots, \psi_m$ are disjoint, we must have $\sum_{i=1}^n \alpha_i \gamma_{i,j} = 0$. Justify this claim.

1.2. Prove Theorem 1.4.

1.3. Prove Theorem 1.5. Hints:

- First show that, if $f(x) \geq 0$ for all $x \in \mathbb{R}$, then $\int f \geq 0$. Then use the fact that $f(x) \leq g(x)$ is equivalent to $g(x) - f(x) \geq 0$.
- Represent f as a linear combination of brick functions with disjoint supports.
- $\min\{x, y\} = \frac{1}{2}(x + y - |x - y|)$ and $\max\{x, y\} = \frac{1}{2}(x + y + |x - y|)$.
- Represent both functions f and g as linear combinations of the same brick functions with disjoint supports.

1.4. Prove: Let $\tau_z f$ be the function defined by $\tau_z f(x) = f(x - z)$ for $z \in \mathbb{R}$. If f is a simple function, then $\tau_z f$ is a simple function and $\int \tau_z f = \int f$.

2 Riemann integrable functions

Now we are ready to define Riemann integrable functions on the real line.

Definition 2.1. A function f is called *Riemann integrable* if for every $\varepsilon > 0$ there exist simple functions g and h such that $g(x) \leq f(x) \leq h(x)$ for all $x \in \mathbb{R}$ and $\int(h - g) < \varepsilon$.

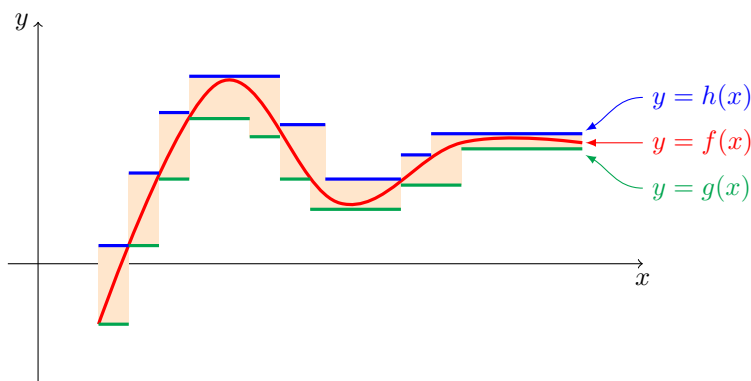


Figure 7: Approximation of an integrable function

Figure 7 illustrates the meaning of the above definition. A function $f(x)$ is Riemann integrable if for an arbitrarily small positive number ε we can find simple functions $g(x)$ and $h(x)$ such that the following three conditions are satisfied

- (a) The graph of $y = g(x)$ is below the graph of $y = f(x)$,
- (b) The graph of $y = h(x)$ is above the graph of $y = f(x)$,
- (c) The area between the graphs of $y = g(x)$ and $y = h(x)$ (the shaded area in Figure 7) is less than ε .

It should be clear that, if we make ε smaller, we most likely will have to find new functions g and h that are closer to f .

Since the Riemann integral is the only type of integration considered in this text, we will simply use the term “integrable” to mean “Riemann integrable”. The following theorem gives an equivalent definition of Riemann integrable functions that some may find more intuitive than the above definition.

Theorem 2.2. A function f is integrable if and only if there exist sequences of simple functions $g_1, g_2, \dots$ and $h_1, h_2, \dots$ such that

$$g_n(x) \leq f(x) \leq h_n(x) \quad \text{for every } x \in \mathbb{R} \text{ and every } n \in \mathbb{N}$$

and

$$\int (h_n - g_n) < \frac{1}{n} \quad \text{for every } n \in \mathbb{N}.$$

Note that the use of $\frac{1}{n}$ is just to make it concrete, but any sequence convergent to 0 will work. In other words, the condition

$$\int (h_n - g_n) < \frac{1}{n} \quad \text{for every } n \in \mathbb{N}$$

is equivalent to the following condition:

$$\int (h_n - g_n) < \varepsilon_n \quad \text{for every } n \in \mathbb{N} \text{ and for some sequence of positive numbers } \varepsilon_1, \varepsilon_2, \dots \text{ such that } \lim_{n \rightarrow \infty} \varepsilon_n = 0.$$

Using this observation we could state the property in a more compact way:

Theorem 2.3. A function f is integrable if and only if there exist sequences of simple functions $g_1, g_2, \dots$ and $h_1, h_2, \dots$ such that

$$g_n(x) \leq f(x) \leq h_n(x) \quad \text{for every } x \in \mathbb{R} \text{ and every } n \in \mathbb{N}$$

and

$$\lim_{n \rightarrow \infty} \int (h_n - g_n) = 0.$$

If convenient, we can always assume that the sequence $g_1, g_2, \dots$ is increasing and that the sequence $h_1, h_2, \dots$ is decreasing. Indeed, if $g_1, g_2, \dots$ and $h_1, h_2, \dots$ are arbitrary sequences of simple functions such that $g_n(x) \leq f(x) \leq h_n(x)$ for every $x \in \mathbb{R}$ and for every $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \int (h_n - g_n) = 0$, then for the sequences

$$g_n^*(x) = \max\{g_1(x), g_2(x), \dots, g_n(x)\} \quad \text{and} \quad h_n^*(x) = \min\{h_1(x), h_2(x), \dots, h_n(x)\}$$

we have

$$g_n(x) \leq g_n^*(x) \leq f(x) \leq h_n^*(x) \leq h_n(x)$$

for every $x \in \mathbb{R}$ and for every $n \in \mathbb{N}$. Then

$$\int g_n \leq \int g_n^* \leq \int h_n^* \leq \int h_n$$

and, since $\lim_{n \rightarrow \infty} \int (h_n - g_n) = 0$, we have

$$\lim_{n \rightarrow \infty} \int (h_n^* - g_n^*) = 0.$$

Note that the sequence $g_1^*, g_2^*, \dots$ is increasing and the sequence $h_1^*, h_2^*, \dots$ is decreasing. Consequently, the sequence $\int g_1^*, \int g_2^*, \dots$ is increasing and the sequence $\int h_1^*, \int h_2^*, \dots$ is decreasing. Since $\int g_n^* \leq \int h_1^*$ for every $n \in \mathbb{N}$, the sequence $\int g_1^*, \int g_2^*, \dots$ is convergent. Similarly, since $\int h_n^* \geq \int g_1^*$ for every $n \in \mathbb{N}$, the sequence $\int h_1^*, \int h_2^*, \dots$ is convergent. Observe that we must have

$$\lim_{n \rightarrow \infty} \int g_n^* = \lim_{n \rightarrow \infty} \int h_n^*,$$

because $\lim_{n \rightarrow \infty} \int (h_n^* - g_n^*) = 0$. Finally, since $\int g_n \leq \int g_n^* \leq \int h_n^* \leq \int h_n$, we have

$$0 \leq \int g_n^* - \int g_n \leq \int h_n^* - \int g_n \leq \int h_n - \int g_n = \int (h_n - g_n) \rightarrow 0$$

as $n \rightarrow \infty$. Consequently, both limits $\lim_{n \rightarrow \infty} \int g_n$ and $\lim_{n \rightarrow \infty} \int h_n$ exist and we have

$$\lim_{n \rightarrow \infty} \int g_n = \lim_{n \rightarrow \infty} \int g_n^* = \lim_{n \rightarrow \infty} \int h_n^* = \lim_{n \rightarrow \infty} \int h_n.$$

The main conclusions of the above argument are captured in the following theorem.

Theorem 2.4. Let $\varepsilon_1, \varepsilon_2, \dots$ be a sequence of positive numbers such that $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. If

$$g_n(x) \leq f(x) \leq h_n(x) \quad \text{for all } x \in \mathbb{R} \text{ and all } n \in \mathbb{N}, \text{ and}$$

$$\int (h_n - g_n) < \varepsilon_n \quad \text{for all } n \in \mathbb{N},$$

then both limits $\lim_{n \rightarrow \infty} \int g_n$ and $\lim_{n \rightarrow \infty} \int h_n$ exist and we have

$$\lim_{n \rightarrow \infty} \int g_n = \lim_{n \rightarrow \infty} \int h_n.$$

Since f is “squeezed” between g_n and h_n , it is natural to define

$$\int f = \lim_{n \rightarrow \infty} \int g_n = \lim_{n \rightarrow \infty} \int h_n.$$

To use this as a definition of the integral of an integrable function we must show that the number $\int f$ is independent of the choice of the approximating sequences of simple functions $g_1, g_2, \dots$ and $h_1, h_2, \dots$. Let $g_1^*, g_2^*, \dots$ and $h_1^*, h_2^*, \dots$ be a different choice of approximating sequences of simple functions. Since the sequences

$$g_1, g_1^*, g_2, g_2^*, \dots \quad \text{and} \quad h_1, h_1^*, h_2, h_2^*, \dots$$

also satisfy the assumptions, both sequences

$$\int g_1, \int g_1^*, \int g_2, \int g_2^*, \dots \quad \text{and} \quad \int h_1, \int h_1^*, \int h_2, \int h_2^*, \dots$$

converge to the same limit. This is possible only if

$$\lim_{n \rightarrow \infty} \int g_n = \lim_{n \rightarrow \infty} \int g_n^* = \lim_{n \rightarrow \infty} \int h_n^* = \lim_{n \rightarrow \infty} \int h_n,$$

which proves the desired independence of the choice of the approximating sequences. Now we can give a formal definition of the integral.

Definition 2.5. For an integrable function f we define

$$\int f = \lim_{n \rightarrow \infty} \int g_n = \lim_{n \rightarrow \infty} \int h_n,$$

where $g_1, g_2, \dots$ and $h_1, h_2, \dots$ are sequences of simple functions such that $g_n(x) \leq f(x) \leq h_n(x)$ for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \int (h_n - g_n) = 0$. The number $\int f$ is called the *Riemann integral* of f .

Since the Riemann integral is the only type of integral considered in this text, we will simply use the term “integral” to mean “Riemann integral”.

The definition of the integral $\int f$ is often given in the following equivalent form: The integral of an integrable function f is

$$\int f = \sup \left\{ \int g : g \text{ is a simple function such that } g(x) \leq f(x) \text{ for all } x \in \mathbb{R} \right\}.$$

Note that we could define the integral of f as the infimum of the integrals of all simple functions that are above f :

$$\int f = \inf \left\{ \int h : h \text{ is a simple function such that } f(x) \leq h(x) \text{ for all } x \in \mathbb{R} \right\}.$$

Theorem 2.6. Let f and g be integrable functions.

- (a) If $f(x) \leq g(x)$ for all $x \in \mathbb{R}$, then $\int f \leq \int g$.
- (b) The function $|f|(x) = |f(x)|$ is integrable and $\left| \int f \right| \leq \int |f|$.
- (c) The functions $m(x) = \min\{f(x), g(x)\}$ and $M(x) = \max\{f(x), g(x)\}$ are integrable.

When constructing functions $g(x)$ and $h(x)$ such that $g(x) \leq f(x) \leq h(x)$ for all $x \in \mathbb{R}$ and $\int (h - g) < \varepsilon$ we often have to deal with some exceptional points that make the construction more cumbersome. It turns out that, as long as there are finitely many points like that, we can simply ignore them. The following theorem makes this statement precise. Note that it implies that we can safely change the values of f at a finite number of points without affecting its integrability.

Theorem 2.7. *If for every $\varepsilon > 0$ there exist simple functions g and h such that $\int (h - g) < \varepsilon$ and $g(x) \leq f(x) \leq h(x)$ for every $x \in \mathbb{R}$ except a finite number of points, then the function f is integrable.*

Proof. Let $\varepsilon > 0$ and let f be a function such that there exist simple functions g and h such that $\int (h - g) < \varepsilon/2$ and $g(x) \leq f(x) \leq h(x)$ for every $x \notin \{c_1, \dots, c_n\}$ for some $c_1, \dots, c_n \in \mathbb{R}$. Note that we must have $g(x) \leq h(x)$ for every $x \in \mathbb{R}$. If we define

$$f^*(x) = \begin{cases} f(x), & \text{if } x \notin \{c_1, \dots, c_n\} \\ \frac{1}{2}(g(x) + h(x)) & \text{if } x \in \{c_1, \dots, c_n\} \end{cases},$$

then

$$g(x) \leq f^*(x) \leq h(x) \tag{1}$$

for every $x \in \mathbb{R}$.

Now let

$$M = \max_{x \in \mathbb{R}} |f(x) - f^*(x)| = \max\{|f(c_1) - f^*(c_1)|, \dots, |f(c_n) - f^*(c_n)|\}.$$

If $M = 0$, then $f = f^*$ and integrability follows directly from (1), so we can assume that $M > 0$. Let $\delta = \frac{\varepsilon}{4Mn}$ and let φ_k be the brick function on $[c_k, c_k + \delta)$ for $k = 1, \dots, n$. Then

$$-M\varphi_1(x) - \dots - M\varphi_n(x) \leq f(x) - f^*(x) \leq M\varphi_1(x) + \dots + M\varphi_n(x) \tag{2}$$

for every $x \in \mathbb{R}$. Note that $f(x) - f^*(x) = 0$ for $x \notin \{c_1, \dots, c_n\}$, so (2) holds trivially at such points, and at $x = c_k$ it holds because $|f(c_k) - f^*(c_k)| \leq M = M\varphi_k(c_k)$.

Adding (1) and (2) gives us

$$g(x) - M\varphi_1(x) - \dots - M\varphi_n(x) \leq f(x) \leq h(x) + M\varphi_1(x) + \dots + M\varphi_n(x)$$

for every $x \in \mathbb{R}$. Define

$$g^*(x) = g(x) - M\varphi_1(x) - \dots - M\varphi_n(x) \quad \text{and} \quad h^*(x) = h(x) + M\varphi_1(x) + \dots + M\varphi_n(x).$$

Then g^* and h^* are simple functions,

$$g^*(x) \leq f(x) \leq h^*(x)$$

for every $x \in \mathbb{R}$, and

$$\begin{aligned} \int (h^* - g^*) &= \int ((h + M\varphi_1 + \dots + M\varphi_n) - (g - M\varphi_1 - \dots - M\varphi_n)) \\ &= \int (h - g) + 2M \int (\varphi_1 + \dots + \varphi_n) \\ &= \int (h - g) + 2Mn\delta \\ &< \frac{\varepsilon}{2} + 2Mn \frac{\varepsilon}{4Mn} = \varepsilon. \end{aligned}$$

Since ε is an arbitrary positive number, this shows that f is an integrable function. □

Exercises

2.1. Prove Theorem 2.2.

2.2. Prove Theorem 2.3.

2.3. Show that simple functions are integrable and that for simple functions the integrals defined in Definitions 1.2 and 2.5 are the same.

2.4. If f is an integrable function, show that

$$\int f = \sup \left\{ \int g : g \text{ is a simple function such that } g(x) \leq f(x) \text{ for all } x \in \mathbb{R} \right\}$$

and

$$\int f = \inf \left\{ \int h : h \text{ is a simple function such that } f(x) \leq h(x) \text{ for all } x \in \mathbb{R} \right\}.$$

2.5. Prove Theorem 2.6. Compare this exercise with Exercise 1.3 and consider the hints given for that exercise.

2.6. Let $a < b$. Show that the following functions are integrable: $\chi_{[a,b]}$, $\chi_{(a,b)}$, $\chi_{[a,b)}$, and $\chi_{\{a\}}$. Then show that $\int \chi_{[a,b]} = \int \chi_{(a,b)} = \int \chi_{[a,b)} = b - a$ and $\int \chi_{\{a\}} = 0$.

2.7. Show that integrable functions are bounded, that is, for every integrable function $f(x)$ there exist numbers m and M such that $m \leq f(x) \leq M$ for all $x \in \mathbb{R}$. Moreover, there is a brick function φ such that $m\varphi(x) \leq f(x) \leq M\varphi(x)$ for all $x \in \mathbb{R}$. (Note that this implies that for every integrable function f we have $\text{supp } f \subset [a, b]$ for some $-\infty < a < b < \infty$.)

2.8. Show that, if f is an integrable function such that $f(x) \geq 0$ for all $x \in \mathbb{R}$, then there are sequences of simple functions $g_1, g_2, \dots$ and $h_1, h_2, \dots$ and a positive number M such that

$$0 \leq g_n(x) \leq f(x) \leq h_n(x) \leq M$$

for every $x \in \mathbb{R}$ and every $n \in \mathbb{N}$, and such that $\lim_{n \rightarrow \infty} \int (h_n - g_n) = 0$.

2.9. Show that for any integrable function f there exists a positive constant C and a brick function φ such that $0 \leq f(x) + C\varphi(x)$ for all $x \in \mathbb{R}$.

2.10. If g and h are simple functions such that $g(x) \leq h(x)$ for every $x \notin \{c_1, \dots, c_n\}$ for some $c_1, \dots, c_n \in \mathbb{R}$, show that that we must have $g(x) \leq h(x)$ for every $x \in \mathbb{R}$.

2.11. Prove: Let $\tau_z f$ be the function defined by $\tau_z f(x) = f(x - z)$ for $z \in \mathbb{R}$. If f is an integrable function, then $\tau_z f$ is an integrable function and $\int \tau_z f = \int f$ for all $z \in \mathbb{R}$.

2.12. Prove: If a function f is integrable and $g(x) = f(x)$ for all but a finite number of values of x , then g is integrable and $\int g = \int f$.

2.13. True or false: If a function f is integrable and $g(x) = f(x)$ for all values of x except an infinite sequence $\{x_1, x_2, \dots\}$, then g is integrable and $\int g = \int f$.

3 Integrability of linear combinations and products

Theorem 3.1. If f is an integrable function and c is a real number, then the function cf is also integrable and we have

$$\int cf = c \int f.$$

Proof. If f is an integrable function, then there exist sequences of simple functions $g_1, g_2, \dots$ and $h_1, h_2, \dots$ such that $g_n(x) \leq f(x) \leq h_n(x)$ for every $x \in \mathbb{R}$ and every $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \int (h_n - g_n) = 0$.

Since the zero function is integrable, the function cf is integrable and $\int cf = c \int f$ when $c = 0$.

Now assume that $c > 0$. Then $cg_1, cg_2, \dots$ and $ch_1, ch_2, \dots$ are simple functions such that that $cg_n(x) \leq cf(x) \leq ch_n(x)$ for every $x \in \mathbb{R}$ and every $n \in \mathbb{N}$ and

$$\lim_{n \rightarrow \infty} \int (ch_n - cg_n) = \lim_{n \rightarrow \infty} \int c(h_n - g_n) = \lim_{n \rightarrow \infty} c \int (h_n - g_n) = c \lim_{n \rightarrow \infty} \int (h_n - g_n) = 0.$$

If $c < 0$, then $cg_1, cg_2, \dots$ and $ch_1, ch_2, \dots$ are simple functions such that that $ch_n(x) \leq cf(x) \leq cg_n(x)$ for every $x \in \mathbb{R}$ and every $n \in \mathbb{N}$ and

$$\lim_{n \rightarrow \infty} \int (cg_n - ch_n) = \lim_{n \rightarrow \infty} \int c(g_n - h_n) = \lim_{n \rightarrow \infty} c \int (g_n - h_n) = c \lim_{n \rightarrow \infty} \int (g_n - h_n) = -c \lim_{n \rightarrow \infty} \int (h_n - g_n) = 0.$$

Consequently, in both cases the function $cf(x)$ is integrable and

$$\int cf = \lim_{n \rightarrow \infty} \int cg_n = c \lim_{n \rightarrow \infty} \int g_n = c \int f.$$

□

Theorem 3.2. If f_1 and f_2 are integrable functions, then the function $f_1 + f_2$ is also integrable and we have

$$\int (f_1 + f_2) = \int f_1 + \int f_2.$$

Proof. Assume f_1 and f_2 are integrable functions. Since f_1 is an integrable function, there exist sequences of simple functions $g_{1,1}, g_{1,2}, \dots$ and $h_{1,1}, h_{1,2}, \dots$ such that

$$g_{1,n}(x) \leq f_1(x) \leq h_{1,n}(x) \quad \text{and} \quad \lim_{n \rightarrow \infty} \int (h_{1,n} - g_{1,n}) = 0.$$

Similarly, since f_2 is an integrable function, there exist sequences of simple functions $g_{2,1}, g_{2,2}, \dots$ and $h_{2,1}, h_{2,2}, \dots$ such that

$$g_{2,n}(x) \leq f_2(x) \leq h_{2,n}(x) \quad \text{and} \quad \lim_{n \rightarrow \infty} \int (h_{2,n} - g_{2,n}) = 0.$$

Consequently,

$$g_{1,n}(x) + g_{2,n}(x) \leq f_1(x) + f_2(x) \leq h_{1,n}(x) + h_{2,n}(x)$$

for every $x \in \mathbb{R}$ and every $n \in \mathbb{N}$, and we have

$$\lim_{n \rightarrow \infty} \int ((h_{1,n} + h_{2,n}) - (g_{1,n} + g_{2,n})) = \lim_{n \rightarrow \infty} \int (h_{1,n} - g_{1,n}) + \lim_{n \rightarrow \infty} \int (h_{2,n} - g_{2,n}) = 0.$$

Since $g_{1,n} + g_{2,n}$ and $h_{1,n} + h_{2,n}$ are simple functions, the above shows that the function $f_1 + f_2$ is integrable and

$$\int (f_1 + f_2) = \lim_{n \rightarrow \infty} \int (g_{1,n} + g_{2,n}) = \lim_{n \rightarrow \infty} \int g_{1,n} + \lim_{n \rightarrow \infty} \int g_{2,n} = \int f_1 + \int f_2.$$

□

By induction, Theorems 3.1 and 3.2 can be generalized to the following important property of integrable functions.

Theorem 3.3. If $f_1, f_2, \dots, f_m$ are integrable functions and $a_1, a_2, \dots, a_m$ are real numbers, then the function $f = a_1 f_1 + a_2 f_2 + \dots + a_m f_m = \sum_{i=1}^m a_i f_i$ is also integrable and we have

$$\int f = a_1 \int f_1 + a_2 \int f_2 + \dots + a_m \int f_m = \sum_{i=1}^m a_i \int f_i.$$

In other words,

$$\int \left(\sum_{i=1}^m a_i f_i \right) = \sum_{i=1}^m a_i \int f_i.$$

Theorem 3.4. If f and g are integrable functions, then the function fg is also integrable.

Proof. We will prove this theorem in three steps.

Step 1 First we assume that f is a function such that there are sequences of simple functions $g_1, g_2, \dots$ and $h_1, h_2, \dots$ such that

$$0 \leq g_n(x) \leq f(x) \leq h_n(x) \leq M$$

for some M , every $x \in \mathbb{R}$ and every $n \in \mathbb{N}$, and such that $\lim_{n \rightarrow \infty} \int (h_n - g_n) = 0$. Since

$$(g_n(x))^2 \leq (f(x))^2 \leq (h_n(x))^2,$$

for every $x \in \mathbb{R}$ and every $n \in \mathbb{N}$, $(g_1)^2, (g_2)^2, \dots$ and $(h_1)^2, (h_2)^2, \dots$ are simple functions, and

$$\int ((h_n)^2 - (g_n)^2) = \int (h_n + g_n)(h_n - g_n) \leq 2M \int (h_n - g_n),$$

it follows that $\lim_{n \rightarrow \infty} \int ((h_n)^2 - (g_n)^2) = 0$. This proves that the function $(f)^2$ is integrable.

Step 2 Now let f be an arbitrary integrable function. Then there is a brick function $\varphi = \chi_{[a,b]}$ such that $f(x) = 0$ for every $x \notin [a, b]$. Note that $f\varphi = f$ and $(\varphi)^2 = \varphi$, which will be used below. Moreover, there exists a positive constant C such that $0 \leq f(x) + C\varphi(x)$ for all $x \in \mathbb{R}$. By Exercise 2.8, there are sequences of simple functions $g_1, g_2, \dots$ and $h_1, h_2, \dots$ and a positive number M such that

$$0 \leq g_n(x) \leq f(x) + C\varphi(x) \leq h_n(x) \leq M$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, and such that $\lim_{n \rightarrow \infty} \int (h_n - g_n) = 0$.

By Step 1, the function $(f + C\varphi)^2$ is integrable. Consequently, the function

$$(f)^2 = (f + C\varphi)^2 - 2fC\varphi - C^2(\varphi)^2 = (f + C\varphi)^2 - 2Cf - C^2\varphi$$

is integrable by Step 1 and Theorem 3.3.

Step 3 Let f and g be arbitrary integrable functions. Since

$$fg = \frac{1}{2} ((f+g)^2 - (f)^2 - (g)^2),$$

the function fg is integrable by Step 2 and Theorem 3.3. □

Exercises

3.1. Prove Theorem 3.3.

3.2. True or false?

- (a) If f and g are integrable, then $\int(fg) = \int f \int g$.
- (b) If f and g are integrable, then $\int(fg) \leq \int f \int g$.
- (c) If f and g are non-negative integrable, then $\int(fg) \leq \int f \int g$.
- (d) If $fg = 0$, then $\int f \int g = 0$.

3.3. True or false?

- (a) If f is an integrable function, then the function $(f)^n$ is integrable for every $n \in \mathbb{N}$.
- (b) If f is a non-negative integrable function, then the function $\sqrt{f}$ is integrable.
- (c) If f a non-negative integrable, then the function $(f)^\alpha$ is integrable for every $\alpha > 0$.

3.4. Is the function $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$ integrable?

3.5. True or false?

- (a) If $f_1, f_2, \dots$ are integrable functions and the series $\sum_{n=1}^{\infty} f_n(x)$ converges for every $x \in \mathbb{R}$, then the function $f(x) = \sum_{n=1}^{\infty} f_n(x)$ is integrable and we have $\int f = \sum_{n=1}^{\infty} \int f_n$.
- (b) If $f_1, f_2, \dots$ are integrable functions and the series $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely for every $x \in \mathbb{R}$, then the function $f(x) = \sum_{n=1}^{\infty} f_n(x)$ is integrable and we have $\int f = \sum_{n=1}^{\infty} \int f_n$.
- (c) Let $f_1, f_2, \dots$ be integrable functions. If the series $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely for every $x \in \mathbb{R}$ and $\sum_{n=1}^{\infty} \int |f_n| < \infty$, then the function $f(x) = \sum_{n=1}^{\infty} f_n(x)$ is integrable and we have $\int f = \sum_{n=1}^{\infty} \int f_n$.

3.6. All proofs in this section use characterization of integrability presented in Theorem 2.3. Prove the theorems in this section directly from Definition 2.1.

4 The Riemann integral on an interval

Definition 4.1. Let $-\infty < a < b < \infty$. A function f is called **integrable on $[a, b]$** if the function

$$f\chi_{[a,b]} = \begin{cases} f(x), & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

is integrable. If a function f is integrable on $[a, b]$, we define

$$\int_a^b f(x)dx = \int f\chi_{[a,b]}.$$

The use of $[a, b)$ instead of $[a, b]$ in the above definition is for technical convenience. The proofs of Theorems 4.2 and 4.3 illustrate it well. In the proof of Theorem 4.2 we use the fact that if g and h are simple functions, so are the functions $g\chi_{[c,d]}$ and $h\chi_{[c,d]}$. In the proof of Theorem 4.3 we use the equality $f\chi_{[a,b]} = f\chi_{[a,c]} + f\chi_{[c,b]}$. Since it is not true that $f\chi_{[a,b]} = f\chi_{[a,c]} + f\chi_{[c,b]}$ because of a single point, we would have to deal with that point which introduces unnecessary technical complications.

A function does not have to be defined outside an interval $[a, b]$ to be integrable on an interval $[a, b]$. Since the general definition of integrability is given for functions defined for all real values of x , there is a formal problem when $f(x)$ is not defined for all real values of x . This problem can be easily fixed by defining $f(x) = 0$ for every $x \notin [a, b]$. We write $f : [a, b] \rightarrow \mathbb{R}$ to indicate that f is defined on $[a, b]$.

Theorem 4.2. If f is integrable on $[a, b]$ and $[c, d] \subseteq [a, b]$, then f is integrable on $[c, d]$.

Proof. Let $\varepsilon > 0$. If f is integrable on $[a, b]$, then there are simple functions g and h such that

$$g(x) \leq f(x)\chi_{[a,b]}(x) \leq h(x)$$

for all $x \in \mathbb{R}$ and $\int (h - g) < \varepsilon$. If $[c, d] \subseteq [a, b]$, then

$$g(x)\chi_{[c,d]}(x) \leq f(x)\chi_{[c,d]}(x) \leq h(x)\chi_{[c,d]}(x)$$

for all $x \in \mathbb{R}$ and $\int (h\chi_{[c,d]} - g\chi_{[c,d]}) < \varepsilon$. Since $g\chi_{[c,d]}$ and $h\chi_{[c,d]}$ are simple functions and ε is an arbitrary positive number, the function $f\chi_{[c,d]}$ is integrable. $\square$

Theorem 4.3. If f is integrable on $[a, b]$ and $c \in (a, b)$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Proof. Note that the functions $f\chi_{[a,c]}$ and $f\chi_{[c,b]}$ are integrable by Theorem 4.2. Since $f\chi_{[a,b]} = f\chi_{[a,c]} + f\chi_{[c,b]}$, we have

$$\int_a^b f(x)dx = \int f\chi_{[a,b]} = \int (f\chi_{[a,c]} + f\chi_{[c,b]}) = \int f\chi_{[a,c]} + \int f\chi_{[c,b]} = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

$\square$

The formal definition of Riemann integrability and the integral is very important from the theoretical point of view. All properties of integrable functions and the integral are directly or indirectly derived from the definition. However, in applications we almost never use the definition to check if a function is integrable, because there are theorems that give simple to use conditions that are sufficient for integrability. The following two theorems are good examples of such conditions.

Theorem 4.4. *If $f : [a, b] \rightarrow \mathbb{R}$ is monotone, then it is integrable.*

Proof. Assume first that $f : [a, b] \rightarrow \mathbb{R}$ is increasing. Let $\varepsilon > 0$. If $f(b) = f(a)$, then f is constant, and integrability is immediate. Otherwise choose $n \in \mathbb{N}$ such that

$$\frac{b-a}{n}(f(b) - f(a)) < \varepsilon.$$

Define

$$a_j = a + j \frac{b-a}{n} \quad \text{and} \quad \varphi_j = \chi_{[a_{j-1}, a_j]}$$

for $j = 1, \dots, n$. Now we define simple functions

$$g = \sum_{j=1}^n f(a_{j-1})\varphi_j \quad \text{and} \quad h = \sum_{j=1}^n f(a_j)\varphi_j.$$

Since f is increasing, for every $x \in [a_{j-1}, a_j]$ we have

$$f(a_{j-1}) \leq f(x) \leq f(a_j),$$

and consequently

$$g(x) \leq f(x) \leq h(x)$$

for every $x \in [a, b]$. Moreover,

$$\begin{aligned} \int (h - g) &= \int \left(\sum_{j=1}^n f(a_j)\varphi_j - \sum_{j=1}^n f(a_{j-1})\varphi_j \right) \\ &= \int \left(\sum_{j=1}^n (f(a_j) - f(a_{j-1}))\varphi_j \right) \\ &= \sum_{j=1}^n (f(a_j) - f(a_{j-1})) \int \varphi_j \\ &= \sum_{j=1}^n (f(a_j) - f(a_{j-1})) \frac{b-a}{n} \\ &= \frac{b-a}{n} \sum_{j=1}^n (f(a_j) - f(a_{j-1})) \\ &= \frac{b-a}{n} (f(b) - f(a)) < \varepsilon. \end{aligned}$$

Thus f is integrable on $[a, b]$.

If f is decreasing, then $-f$ is increasing so integrability of f follows. $\square$

All proofs presented so far, while requiring a certain level of mathematical maturity, do not require any concepts that are not covered in a standard calculus course. The proof of the following important result uses uniform continuity, a standard topic in an advanced calculus course.

A function $f : [a, b] \rightarrow \mathbb{R}$ is continuous if for every $\varepsilon > 0$ and every $x \in [a, b]$ there exists a $\delta_x > 0$ such that $|f(x) - f(y)| < \varepsilon$ for every $y \in [a, b]$ such that $|x - y| < \delta_x$. The subscript x in δ_x emphasizes the fact that δ_x depends on x . If we take a different $x \in [a, b]$ we may have to adjust the δ_x even if we use the same ε . A function is called uniformly continuous if for every ε it is possible to find a $\delta > 0$ that will work for every x in the domain of the function. It turns out that every continuous function on a bounded closed interval is uniformly continuous. This is a nontrivial result – it may seem surprising that a local property (continuity at each individual point) automatically becomes a global one on such intervals.

The assumptions that the interval is closed and bounded are essential. For example the function $f(x) = 1/x$ is continuous on the bounded (but not closed) interval $(0, 1]$, but it is not uniformly continuous on that interval. On the other hand, the function $f(x) = x^2$ is continuous on the closed (but not bounded) interval $[0, \infty)$, but it is not uniformly continuous on that interval.

Theorem 4.5. *If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then it is integrable.*

Proof. Let $\varepsilon > 0$. Since f is a continuous function on a bounded closed interval, it is uniformly continuous and thus there is a $\delta > 0$ such that

$$|f(s) - f(t)| < \frac{\varepsilon}{b-a} \quad \text{whenever } |s - t| < \delta.$$

Let $n \in \mathbb{N}$ be such that $\frac{b-a}{n} < \delta$. For $j = 1, 2, \dots, n$ we define

$$I_j = \left[a + (j-1)\frac{b-a}{n}, a + j\frac{b-a}{n} \right).$$

Since f is continuous, f is bounded and

$$\sup_{x \in I_j} f(x) - \inf_{x \in I_j} f(x) < \frac{\varepsilon}{b-a}$$

for every $j = 1, 2, \dots, n$. Let φ_j be the characteristic function of the interval I_j for $j = 1, 2, \dots, n$. Now we define

$$g = \sum_{j=1}^n \left(\inf_{x \in I_j} f(x) \right) \varphi_j \quad \text{and} \quad h = \sum_{j=1}^n \left(\sup_{x \in I_j} f(x) \right) \varphi_j.$$

Note that

$$g(x) \leq f(x) \leq h(x) \quad \text{for every } x \in [a, b]$$

and

$$\begin{aligned} \int (h - g) &= \int \left(\sum_{j=1}^n \left(\sup_{x \in I_j} f(x) \right) \varphi_j - \sum_{j=1}^n \left(\inf_{x \in I_j} f(x) \right) \varphi_j \right) \\ &= \int \left(\sum_{j=1}^n \left(\sup_{x \in I_j} f(x) - \inf_{x \in I_j} f(x) \right) \varphi_j \right) \\ &= \sum_{j=1}^n \left(\sup_{x \in I_j} f(x) - \inf_{x \in I_j} f(x) \right) \int \varphi_j \\ &= \sum_{j=1}^n \left(\sup_{x \in I_j} f(x) - \inf_{x \in I_j} f(x) \right) \frac{b-a}{n} \\ &< \sum_{j=1}^n \frac{\varepsilon}{b-a} \frac{b-a}{n} = \varepsilon. \end{aligned}$$

Thus f is integrable on the interval $[a, b]$. □

In the following theorem we give another condition that implies integrability of a function on an interval. The proof is based on another advanced tool from real analysis, namely compactness. A set $S \subset \mathbb{R}$ is compact if every open cover of S contains a finite subcover. This means that, if $S \subseteq \bigcup_{i \in I} U_i$, where U_i 's are open subsets of $\mathbb{R}$ indexed by the set I , then $S \subseteq \bigcup_{k=1}^n U_{i_k}$ for some $i_1, \dots, i_n \in I$. For S to be a compact set, this has to be true for every open cover.

In the proof of the next theorem we use the fact that every bounded closed interval $[a, b]$ is compact. In our case the open cover has the form

$$[a, b] \subseteq \bigcup_{c \in [a, b]} (c - \delta_c, c + \delta_c),$$

so the indexing set is the interval $[a, b]$ itself and the open sets are the open intervals $(c - \delta_c, c + \delta_c)$. While it is not obvious that the interval $[a, b]$ is compact, it is relatively easy to see that both the fact that it is closed and that it is bounded are essential here. Indeed, the open cover

$$(0, 1) = \bigcup_{n=3}^{\infty} \left(\frac{1}{n}, 1 - \frac{1}{n} \right)$$

does not have a finite subcover and the open cover

$$[0, \infty) = \bigcup_{n=1}^{\infty} (-1, n)$$

does not have a finite subcover.

Note that the assumption that the limit $\lim_{x \rightarrow c} f(x)$ exists for every $c \in [a, b]$ does not imply continuity of f on $[a, b]$.

Theorem 4.6. *If $f : [a, b] \rightarrow \mathbb{R}$ is a function such that the limit $\lim_{x \rightarrow c} f(x)$ exists for every $c \in [a, b]$, then f is integrable.*

Proof. Assume $f : [a, b] \rightarrow \mathbb{R}$ is a function such that the limit $\lim_{x \rightarrow c} f(x)$ exists for every $c \in [a, b]$. Let ε be an arbitrary positive number. For every $c \in [a, b]$ there exists a $\delta_c > 0$ such that

$$\left| f(x) - \lim_{x \rightarrow c} f(x) \right| < \frac{\varepsilon}{2(b-a)} \quad \text{whenever } 0 < |x - c| < \delta_c.$$

Note that

$$[a, b] \subseteq \bigcup_{c \in [a, b]} (c - \delta_c, c + \delta_c).$$

Since $[a, b]$ is a compact set, there are $c_1, \dots, c_n \in [a, b]$ such that

$$[a, b] \subseteq (c_1 - \delta_{c_1}, c_1 + \delta_{c_1}) \cup \dots \cup (c_n - \delta_{c_n}, c_n + \delta_{c_n}).$$

There is a $b_1 \in \{c_1, \dots, c_n\}$ such that $a \in (b_1 - \delta_{b_1}, b_1 + \delta_{b_1})$. We define $a_0 = a$ and $a_1 = b_1 + \delta_{b_1}$, so that $[a_0, a_1] \subset (b_1 - \delta_{b_1}, b_1 + \delta_{b_1})$.

Next we take a $b_2 \in \{c_1, \dots, c_n\}$ such that $a_1 \in (b_2 - \delta_{b_2}, b_2 + \delta_{b_2})$ and define $a_2 = b_2 + \delta_{b_2}$, so that $[a_1, a_2] \subset (b_2 - \delta_{b_2}, b_2 + \delta_{b_2})$.

In general, if $a_0, a_1, \dots, a_{j-1}$ are defined, we take a $b_j \in \{c_1, \dots, c_n\}$ such that $a_{j-1} \in (b_j - \delta_{b_j}, b_j + \delta_{b_j})$ and define $a_j = b_j + \delta_{b_j}$ so that $[a_{j-1}, a_j] \subset (b_j - \delta_{b_j}, b_j + \delta_{b_j})$.

Since there are only n intervals in the finite subcover, this process terminates after at most n steps and produces numbers

$$a_0 < a_1 < \dots < a_m$$

such that $[a, b] \subset [a_0, a_m]$. Note that for all $j \in \{1, \dots, m\}$ we have

$$\left| f(x) - \lim_{x \rightarrow b_j} f(x) \right| < \frac{\varepsilon}{2(b-a)} \quad \text{for all } x \in [a_{j-1}, a_j].$$

Now we define

$$g(x) = \begin{cases} \lim_{x \rightarrow b_j} f(x) - \frac{\varepsilon}{2(b-a)} & \text{if } x \in [a_{j-1}, a_j] \text{ for } j = 1, \dots, m-1 \\ \lim_{x \rightarrow b_m} f(x) - \frac{\varepsilon}{2(b-a)} & \text{if } x \in [a_{m-1}, b] \\ 0 & \text{otherwise} \end{cases}$$

and

$$h(x) = \begin{cases} \lim_{x \rightarrow b_j} f(x) + \frac{\varepsilon}{2(b-a)} & \text{if } x \in [a_{j-1}, a_j] \text{ for } j = 1, \dots, m-1 \\ \lim_{x \rightarrow b_m} f(x) + \frac{\varepsilon}{2(b-a)} & \text{if } x \in [a_{m-1}, b] \\ 0 & \text{otherwise} \end{cases}$$

Then g and h are simple functions such that $h(x) \geq g(x)$ for every $x \in \mathbb{R}$. Since $h(x) - g(x) < \frac{\varepsilon}{b-a}$ for every $x \in \mathbb{R}$ and $g(x) = h(x) = 0$ if $x \notin [a, b]$, we have

$$\int (h - g) < \frac{\varepsilon}{b-a} (b-a) = \varepsilon.$$

Moreover, since $g(x) \leq f(x) \leq h(x)$ for every $x \notin \{c_1, \dots, c_n, b\}$, integrability of f follows by Theorem 2.7. $\square$

When the notation $\int_a^b f(x)dx$ was introduced we assumed that $-\infty < a < b < \infty$. Sometimes it is convenient to use $\int_a^b f(x)dx$ when $a = b$ or $a > b$, but the meaning has to be defined since these cases are not covered by our original definition.

Definition 4.7. For an integrable function f we define

$$\int_a^a f = 0 \quad \text{and} \quad \int_a^b f = - \int_b^a f.$$

In Theorem 4.3 it was important that the point c is between a and b . With the above convention we can generalize the theorem as follows.

Theorem 4.8. If $f : [a, b] \rightarrow \mathbb{R}$ is integrable, then

$$\int_\alpha^\beta f(x)dx = \int_\alpha^\gamma f(x)dx + \int_\gamma^\beta f(x)dx$$

for any $\alpha, \beta, \gamma \in [a, b]$.

Proof. The case when $\alpha < \gamma < \beta$ is covered by Theorem 4.3. Consider the case when $\alpha < \beta < \gamma$. Then, again by Theorem 4.3, we have

$$\int_\alpha^\gamma f(x)dx = \int_\alpha^\beta f(x)dx + \int_\beta^\gamma f(x)dx$$

and thus

$$\int_\alpha^\beta f(x)dx = \int_\alpha^\gamma f(x)dx - \int_\beta^\gamma f(x)dx.$$

Since $-\int_\beta^\gamma f(x)dx = \int_\gamma^\beta f(x)dx$, we obtained the desired equality

$$\int_\alpha^\beta f(x)dx = \int_\alpha^\gamma f(x)dx + \int_\gamma^\beta f(x)dx.$$

The other cases can be proved in a similar way (see Exercise 4.9). □

Exercises

4.1. Show that an integrable function is integrable on $[a, b]$ for all $-\infty < a < b < \infty$.

4.2. Show that the following definition is equivalent to Definition 4.1: A function f is called integrable on $[a, b]$ if the function $f\chi_{[a,b]}$ is integrable. If f is integrable on $[a, b]$, we define $\int_a^b f(x)dx = \int f\chi_{[a,b]}$.

4.3. True or false?

- (a) If f is an integrable function, then $\int f = \int_a^b f(x)dx$ for all $-\infty < a < b < \infty$.
- (b) If f is an integrable function, then $\int f \geq \int_a^b f(x)dx$ for all $-\infty < a < b < \infty$.
- (c) If f is an integrable function, then $\int f = \int_a^b f(x)dx$ for some $-\infty < a < b < \infty$.

4.4. How would you modify the presented proof of Theorem 4.4 to show that a decreasing function is integrable?

4.5. Does Theorem 4.5 remain true if $[a, b]$ is replaced by $[a, b)$?

4.6. Show that the function $f(x) = 1/x$ is not uniformly continuous on the interval $(0, 1]$. Note that it suffices to show that, for example, there is no $\delta > 0$ such that $\left| \frac{1}{x} - \frac{1}{y} \right| < 1$ for any $x, y \in (0, 1]$ such that $|x - y| < \delta$.

4.7. Show that the function $f(x) = x^2$ is not uniformly continuous on $[0, \infty)$. Note that it suffices to show that, for example, there is no $\delta > 0$ such that $|x^2 - y^2| < 1$ for any $x, y \in [0, \infty)$ such that $|x - y| < \delta$.

4.8. Show that the assumption that the limit $\lim_{x \rightarrow c} f(x)$ exists for every $c \in [a, b]$ does not imply continuity of f on $[a, b]$.

4.9. Complete the proof of Theorem 4.8 by considering the case when γ is to the left of α and β , and the case when two of the numbers α, β, γ are equal. Is it necessary to consider any other cases?

5 Integrability of limits of sequences of integrable functions

In our approach a function is integrable if it can be approximated by simple functions, in a certain specific way, and then the integrals of the approximating simple functions converge to the integral of the function. In this case we can say, somewhat informally, that $f_n \rightarrow f$ implies $\int f_n \rightarrow \int f$. Is this a general property of the integral? More precisely, we are going to address the following two important questions:

- If a sequence of integrable functions (f_n) converges to a function f , is f necessarily an integrable function?
- If a sequence of integrable functions (f_n) converges to an integrable function f , is it necessarily true that the limit $\lim_{n \rightarrow \infty} \int f_n$ exists and equals $\int f$?

Example 5.1. Consider the following sequence of functions

$$f_n(x) = \begin{cases} 1 & \text{if } x \in (-n, n] \\ 0 & \text{otherwise} \end{cases}.$$

Clearly functions f_n are integrable. Since $f(x) = \lim_{n \rightarrow \infty} f_n(x) = 1$ for every $x \in \mathbb{R}$, the limit is the constant function 1, which is not integrable because its support is not bounded (recall that by Exercise 2.7 every integrable function vanishes outside some bounded interval).

We can try to fix the problem by imposing that the approximating functions vanish outside a fixed bounded interval. The next example shows that it does not help.

Example 5.2. Consider the following sequence of functions

$$f_n(x) = \begin{cases} \frac{1}{x} & \text{if } x \in (\frac{1}{n}, 1] \\ 0 & \text{otherwise} \end{cases}.$$

Clearly functions f_n are integrable. Since

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = \begin{cases} \frac{1}{x} & \text{if } x \in (0, 1] \\ 0 & \text{otherwise} \end{cases},$$

the limit function f is not integrable because it is not bounded (recall that by Exercise 2.7 every integrable function is bounded).

Now we look at the second question. We assume that limit function is integrable and ask if in such case the limit of the integrals equals the integral of the limit function. As the following two examples show, both things can go wrong. First, it is possible that a sequence of integrable functions f_n converges to an integrable function, but the sequence of the integrals $\int f_n$ is not convergent.

Example 5.3. Consider the following sequence of functions

$$f_n(x) = \begin{cases} n^3 & \text{if } x \in (\frac{1}{n+1}, \frac{1}{n}] \\ 0 & \text{otherwise} \end{cases}.$$

Clearly functions f_n are integrable and $f(x) = \lim_{n \rightarrow \infty} f_n(x) = 0$ for every $x \in \mathbb{R}$, so the limit function is integrable. Since

$$\int f_n = \frac{n^3}{n(n+1)} = \frac{n^2}{n+1} \rightarrow \infty,$$

the sequence of the integrals $\int f_n$ is not convergent.

In our final example the sequence of the integrals converges, but not to the integral of the limit.

Example 5.4. Consider the following sequence of functions

$$f_n(x) = \begin{cases} n^2 & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right] \\ 0 & \text{otherwise} \end{cases}.$$

Clearly functions f_n are integrable and $f(x) = \lim_{n \rightarrow \infty} f_n(x) = 0$ for every $x \in \mathbb{R}$, so the limit function is integrable. Since

$$\int f_n = \frac{n^2}{n(n+1)} = \frac{n}{n+1} \rightarrow 1,$$

we have $\lim_{n \rightarrow \infty} \int f_n = 1$, but since $\int f = 0$, $\lim_{n \rightarrow \infty} \int f_n \neq \int f$.

In the above examples the condition “sequence of functions (f_n) converges to a function f ” was interpreted as $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for every x in the domain of f . This type of convergence of sequences of functions is called *pointwise convergence*.

We will introduce a stronger type of convergence of sequences of functions for which the statement “If $f_n \rightarrow f$, then $\int f_n \rightarrow \int f$.” will be true. By stronger convergence we mean a convergence with fewer convergent sequences. In particular, the sequences in the above examples should not be convergent any more.

First we examine more carefully the formal definition of pointwise convergence.

$f_n \rightarrow f$ pointwise on the interval $[a, b]$ if for every $x \in [a, b]$ and every $\varepsilon > 0$ there exists an $n_{x,\varepsilon} \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for every $n > n_{x,\varepsilon}$.

We include x and ε in $n_{x,\varepsilon}$ to emphasize the fact that the number $n_{x,\varepsilon} \in \mathbb{N}$ depends on both x and ε . If we use the same ε for different points x , it may be necessary to find different numbers $n_{x,\varepsilon}$ for the inequality $|f_n(x) - f(x)| < \varepsilon$ to hold. If we insist that for every $\varepsilon > 0$ we can find a number n_ε that will work for every $x \in [a, b]$, then we are going to describe a stronger type of convergence. This type of convergence is called *uniform convergence*.

$f_n \rightarrow f$ uniformly on the interval $[a, b]$ if for every $\varepsilon > 0$ there exists an $n_\varepsilon \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for every $n > n_\varepsilon$ and every $x \in [a, b]$.

Note the similarity between the definitions of uniform continuity and uniform convergence. In both cases, a “local” property (convergence at each point, continuity at each point) is strengthened by requiring a single bound to work everywhere simultaneously.

Theorem 5.5. If the functions f_n are integrable on $[a, b]$ and $f_n \rightarrow f$ uniformly on $[a, b]$, then f is integrable on $[a, b]$ and $\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \int_a^b f_n(x)dx$.

Proof. Since the functions f_n are integrable on $[a, b]$, they are bounded on $[a, b]$ and thus f is bounded on $[a, b]$ being the uniform limit of a sequence of bounded functions.

Let $\varepsilon > 0$. Since $f_n \rightarrow f$ uniformly on $[a, b]$, there is an $n_\varepsilon \in \mathbb{N}$ such that

$$|f(x) - f_{n_\varepsilon}(x)| < \frac{\varepsilon}{4(b-a)} \quad \text{for all } x \in [a, b]$$

and thus

$$-\frac{\varepsilon}{4(b-a)} < f(x) - f_{n_\varepsilon}(x) < \frac{\varepsilon}{4(b-a)} \quad \text{for all } x \in [a, b]. \quad (3)$$

Since f_{n_ε} is integrable on $[a, b]$, there are simple functions g and h such that

$$g(x) \leq f_{n_\varepsilon}(x) \leq h(x) \quad \text{for all } x \in [a, b] \quad (4)$$

and

$$\int_a^b (h(x) - g(x))dx < \frac{\varepsilon}{2}.$$

From inequalities (3) and (4) we get

$$g(x) - \frac{\varepsilon}{4(b-a)}\chi_{[a,b]}(x) < f(x) < h(x) + \frac{\varepsilon}{4(b-a)}\chi_{[a,b]}(x) \quad \text{for all } x \in [a, b].$$

Note that $g - \frac{\varepsilon}{4(b-a)}\chi_{[a,b]}$ and $h + \frac{\varepsilon}{4(b-a)}\chi_{[a,b]}$ are simple functions. Moreover, we have

$$\int_a^b \left(\left(h(x) + \frac{\varepsilon}{4(b-a)}\chi_{[a,b]}(x) \right) - \left(g(x) - \frac{\varepsilon}{4(b-a)}\chi_{[a,b]}(x) \right) \right) dx = \int_a^b (h(x) - g(x))dx + \frac{\varepsilon}{2} < \varepsilon.$$

Since ε is an arbitrary positive number, this shows that f is integrable. Moreover, since

$$\left| \int_a^b (f(x) - f_n(x))dx \right| \leq \int_a^b |f(x) - f_n(x)|dx \leq \sup_{[a,b]} |f(x) - f_n(x)|(b-a) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

because $f_n \rightarrow f$ uniformly on $[a, b]$, we have

$$\lim_{n \rightarrow \infty} \int_a^b (f(x) - f_n(x))dx = 0$$

and thus $\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \int_a^b f_n(x)dx.$

□

Exercises

5.1. Verify the claims made in Examples 5.1 and 5.2.

5.2. Verify the claims made in Examples 5.3 and 5.4.

5.3. Show that the sequences of functions f_n in Examples 5.1-5.4 are not uniformly convergent.

5.4. Show that, if $f_n \rightarrow f$ uniformly on $[a, b]$ and each function f_n is bounded, then the limit function f is also bounded.

5.5. Show that $f_n \rightarrow f$ uniformly on $[a, b]$ if and only if $\sup_{[a,b]} |f(x) - f_n(x)| \rightarrow 0$.

5.6. Let $f, f_1, f_2, \dots$ be functions on $[a, b]$. We say that $\sum_{i=1}^{\infty} f_i = f$ uniformly on $[a, b]$ if $\sum_{i=1}^n f_i \rightarrow f$ as $n \rightarrow \infty$

uniformly on $[a, b]$. Show that if $\sum_{i=1}^{\infty} f_i = f$ uniformly on $[a, b]$ and each function f_n is integrable on $[a, b]$, then f is an integrable function and $\sum_{i=1}^{\infty} \int_a^b f_i(x)dx = \int_a^b f(x)dx.$

6 The Fundamental Theorems of Calculus

In this section we discuss some connections between integrability and other properties of functions we are familiar with from the calculus course, including continuity and differentiability. These properties offer additional insight into integration and also provide essential tools for applications of the integral.

Thus far we only dealt with integrals over a fixed interval. Now we want to make the upper limit of the integral a variable. Note that if f is integrable on $[a, b]$, then it is integrable on $[a, x]$ for every $x \in [a, b]$ and thus $F(x) = \int_a^x f(t)dt$ is a well-defined function on $[a, b]$.

Theorem 6.1. *If f is integrable on $[a, b]$, then the function*

$$F(x) = \int_a^x f(t)dt$$

is continuous on $[a, b]$.

Proof. If f is integrable on $[a, b]$, then f is bounded on $[a, b]$, so there is an $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for every $x \in [a, b]$. For every $c \in [a, b]$ we have

$$|F(x) - F(c)| = \left| \int_a^x f(t)dt - \int_a^c f(t)dt \right| = \left| \int_c^x f(t)dt \right| \leq M|x - c|.$$

Therefore

$$\lim_{x \rightarrow c} |F(x) - F(c)| = 0,$$

which means that F is continuous at c . □

The figure below illustrates an intuitive property of continuous functions on an interval, called the Mean Value Theorem for Integrals (Theorem 6.2). The area below the lower green line is clearly less than the shaded area while the area below the upper green line is clearly more than the shaded area. If we imagine that we are moving the lower green line upwards, then there must be an elevation where the area below that line is equal to the shaded area. Since the function f is continuous, the line at that elevation must intersect the graph of f .

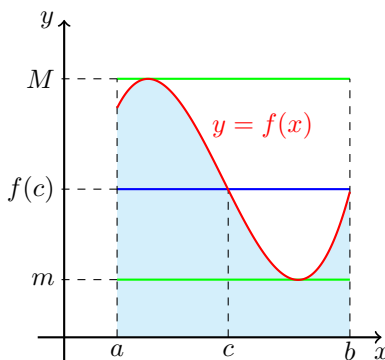


Figure 8: The Mean Value Theorem for Integrals

Theorem 6.2 (Mean Value Theorem for Integrals). *If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then there is a $c \in (a, b)$ such that*

$$\int_a^b f(x) dx = f(c)(b - a)$$

Proof. First we note that for any $c \in (a, b)$ we have $\int_a^b f(x) dx = f(c)(b - a)$ if f is a constant function.

Now assume that f is not constant on $[a, b]$. Since f is a continuous function on $[a, b]$, there are two distinct points $x_1, x_2 \in [a, b]$ such that

$$m = \min_{[a, b]} f(x) = f(x_1) \quad \text{and} \quad M = \max_{[a, b]} f(x) = f(x_2).$$

Then $m \leq f(x) \leq M$ for every $x \in [a, b]$ and consequently

$$m(b - a) = \int_a^b m dx \leq \int_a^b f(x) dx \leq \int_a^b M dx = M(b - a).$$

Now consider the function $g(x) = f(x)(b - a)$. This is a continuous function on the interval $[a, b]$ such that

$$g(x_1) = m(b - a) \leq \int_a^b f(x) dx \leq M(b - a) = g(x_2).$$

Thus, by the Intermediate Value Theorem, there is a point c between x_1 and x_2 such that $g(c) = \int_a^b f(x) dx$ which means that

$$\int_a^b f(x) dx = f(c)(b - a).$$

□

In Theorem 6.1 we prove that the function $F(x) = \int_a^x f(t)dt$ is continuous if f is integrable. If we assume more, namely that f is continuous in $[a, b]$, then F is differentiable. Note that it says that

$$\frac{d}{dx} \left(\int_a^x f(t)dt \right) = f(x),$$

which indicates that, in a sense, differentiation is opposite to integration. It is arguably the most important and surprising property of the integral.

In Theorems 6.3-6.5 we talk about differentiability of a function F on an interval $[a, b]$. By differentiability at the endpoints we mean existence of one-sided derivatives, that is,

$$F'(a) = \lim_{h \rightarrow 0+} \frac{F(a+h) - F(a)}{h} \quad \text{and} \quad F'(b) = \lim_{h \rightarrow 0-} \frac{F(b+h) - F(b)}{h}.$$

Theorem 6.3 (Fundamental Theorem of Calculus I). *If f is continuous on $[a, b]$, then the function*

$$F(x) = \int_a^x f(t)dt$$

is differentiable on $[a, b]$ and $F'(x) = f(x)$.

Proof. For every $x \in (a, b)$ we have

$$\lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \rightarrow 0} \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t)dt = \lim_{h \rightarrow 0} \frac{1}{h} f(c_h)h = \lim_{h \rightarrow 0} f(c_h)$$

where c_h is a number between x and $x+h$ and h is such that $x+h \in [a, b]$. Since $\lim_{h \rightarrow 0} c_h = x$, we have $\lim_{h \rightarrow 0} f(c_h) = f(x)$, by continuity of f . Consequently,

$$\lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} = f(x),$$

proving differentiability of F and the equality $F'(x) = f(x)$.

If $x = a$, then we need to modify the above argument by using $h \rightarrow 0+$ instead of $h \rightarrow 0$. Similarly, for $x = b$ we need to use $h \rightarrow 0-$ instead of $h \rightarrow 0$. $\square$

So far, our focus has been heavily theoretical. While a rigorous foundation is essential for understanding integrability and deriving its core properties, theory alone offers little help when we need to calculate the integral of a specific function. Attempting to build sequences of approximating simple functions and computing their limits is incredibly tedious, if not practically impossible, for everyday calculations. The following theorem gives us a practical way of calculating many integrals.

Theorem 6.4 (Fundamental Theorem of Calculus II - easy version). *If $f : [a, b] \rightarrow \mathbb{R}$ is continuous and $F : [a, b] \rightarrow \mathbb{R}$ is a differentiable function such that $f(x) = F'(x)$ for all $x \in [a, b]$, then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof. Let $G(x) = \int_a^x f(t)dt$. Since $G'(x) = f(x) = F'(x)$ on $[a, b]$, there is a constant C such that $F(x) = G(x) + C$ for all $x \in [a, b]$. Consequently,

$$F(b) - F(a) = G(b) - G(a) = \int_a^b f(t)dt - \int_a^a f(t)dt = \int_a^b f(t)dt.$$

$\square$

For most applications the above easy version is sufficient, but it turns out that the continuity assumption can be replaced by a weaker assumption of integrability. This stronger version is no longer an easy consequence of Theorem 6.3.

Theorem 6.5 (Fundamental Theorem of Calculus II - difficult version). *If $f : [a, b] \rightarrow \mathbb{R}$ is integrable and $F : [a, b] \rightarrow \mathbb{R}$ is a differentiable function such that $f(x) = F'(x)$ for $x \in [a, b]$, then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof. Let $\varepsilon > 0$. There exist simple functions g and h such that $g(x) \leq f(x) \leq h(x)$ for all $x \in [a, b]$ and $\int(h - g) < \varepsilon$. Let the numbers

$$a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$$

contain all discontinuity points of g and h in the interval (a, b) . By the Mean Value Theorem there are numbers $c_1, c_2, \dots, c_n$ such that

$$a_0 < c_1 < a_1 < c_2 < \cdots < a_{n-1} < c_n < a_n$$

and $F(a_j) - F(a_{j-1}) = F'(c_j)(a_j - a_{j-1})$ for $j = 1, \dots, n$. Then

$$F(b) - F(a) = \sum_{j=1}^n (F(a_j) - F(a_{j-1})) = \sum_{j=1}^n F'(c_j)(a_j - a_{j-1}) = \sum_{j=1}^n f(c_j)(a_j - a_{j-1}).$$

Let φ_j be the characteristic function of the interval $[a_{j-1}, a_j)$ for $j = 1, \dots, n$. Then

$$g(x) \leq \sum_{j=1}^n f(c_j)\varphi_j(x) \leq h(x)$$

and thus

$$\int g \leq \int \left(\sum_{j=1}^n f(c_j)\varphi_j \right) \leq \int h.$$

Since

$$\int \left(\sum_{j=1}^n f(c_j)\varphi_j \right) = \sum_{j=1}^n f(c_j) \int \varphi_j = \sum_{j=1}^n f(c_j)(a_j - a_{j-1}) = F(b) - F(a),$$

we have

$$\int g \leq F(b) - F(a) \leq \int h.$$

The above, together with $\int g \leq \int f \leq \int h$ and $\int(h - g) < \varepsilon$, gives us

$$\left| \int f - (F(b) - F(a)) \right| < \varepsilon.$$

Since ε is an arbitrary positive number, we can conclude that

$$\int f = F(b) - F(a).$$

□

In Theorems 6.4 and 6.5 we assume differentiability of a function $F : [a, b] \rightarrow \mathbb{R}$. This should be interpreted as differentiability of F in (a, b) and one-sided differentiability at the endpoints. Since every function satisfying these conditions can be extended to a differentiable function defined on an open interval containing $[a, b]$, we could assume, without loss of generality, that F is defined and differentiable on an open interval containing $[a, b]$. Note that only the values of $F(x)$ and $F'(x)$ for $x \in [a, b]$ are used in the formulation of those theorems and the proofs.

We conclude this section with the proof of the so-called “substitution rule” for integrals.

Theorem 6.6. Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. If $\varphi : [\alpha, \beta] \rightarrow [a, b]$ is a continuously differentiable function, then

$$\int_{\alpha}^{\beta} f(\varphi(x))\varphi'(x)dx = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx.$$

Proof. Let

$$F(x) = \int_a^x f(t)dt.$$

Then $F'(x) = f(x)$ and

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx = F(\varphi(\beta)) - F(\varphi(\alpha))$$

by the Fundamental Theorems of Calculus I and II.

Now consider the function $G(x) = F(\varphi(x))$. Since

$$G'(x) = F'(\varphi(x))\varphi'(x) = f(\varphi(x))\varphi'(x)$$

we have

$$\int_{\alpha}^{\beta} f(\varphi(x))\varphi'(x)dx = G(\beta) - G(\alpha) = F(\varphi(\beta)) - F(\varphi(\alpha)),$$

again by the Fundamental Theorem of Calculus II. Hence

$$\int_{\alpha}^{\beta} f(\varphi(x))\varphi'(x)dx = F(\varphi(\beta)) - F(\varphi(\alpha)) = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx.$$

□

Exercises

6.1. Justify the inequality $|\int_c^x f(t)dt| \leq M|x - c|$ in the proof of Theorem 6.1.

6.2. Show that, if f is integrable on $[a, b]$, then for every $c \in [a, b]$ the function $F(x) = \int_c^x f(t)dt$ is continuous on $[a, b]$.

6.3. Show that, if $f : \mathbb{R} \rightarrow \mathbb{R}$ is integrable, then for every $c \in \mathbb{R}$ the function $F(x) = \int_c^x f(t)dt$ is well-defined and continuous on $\mathbb{R}$.

6.4. Show that, if $f : \mathbb{R} \rightarrow \mathbb{R}$ is integrable on every bounded interval $[a, b]$, then for every $c \in \mathbb{R}$ the function $F(x) = \int_c^x f(t)dt$ is well-defined and continuous on $\mathbb{R}$.

6.5. Show that, if f is integrable on $[a, b]$, then the function $G(x) = \int_a^x F(t)dt$, where $F(x) = \int_a^x f(t)dt$, is differentiable. In other words, but less precisely, if we integrate an integrable function twice, we get a differentiable function.

6.6. Show that, if f is n times continuously differentiable in $[a, b]$, then the function $F(x) = \int_a^x f(t)dt$, is $(n + 1)$ times differentiable in $[a, b]$.

6.7. Show that, if $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, then for every $c \in \mathbb{R}$ the function $F(x) = \int_c^x f(t)dt$ is differentiable on $\mathbb{R}$.

Piotr Mikusiński
Professor Emeritus
University of Central Florida
piotr@ucf.edu