Adaptive nonparametric empirical Bayes estimation via wavelet series: The minimax study

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Abstract

In the present paper, we derive lower bounds for the risk of the nonparametric empirical Bayes estimators. In order to attain the optimal convergence rate, we propose generalization of the linear empirical Bayes estimation method which takes advantage of the flexibility of the wavelet techniques. We present an empirical Bayes estimator as a wavelet series expansion and estimate coefficients by minimizing the prior risk of the estimator. As a result, estimation of wavelet coefficients requires solution of a well-posed low-dimensional sparse system of linear equations. The dimension of the system depends on the size of wavelet support and smoothness of the Bayes estimator. An adaptive choice of the resolution level is carried out using Lepski et al. (1997) method. The method is computationally efficient and provides asymptotically optimal adaptive EB estimators. The theory is supplemented by numerous examples.

1. Introduction

Empirical Bayes (EB) methods are estimation techniques in which the prior distribution, in the standard Bayesian sense, is estimated from the data. They are powerful tools, in particular, when data are generated by repeated execution of the same type of experiment. The EB methods are directly related to the standard Bayes models but there is difference in perspective between the two in the sense that, in the standard Bayesian approach, the prior distribution is assumed to be fixed before any data are observed, whereas, in the EB setting, the prior distribution, in some way or another, is estimated from the observed data.

Consider the following setting. One observes independent two-dimensional random vectors \( (X_1, \theta_1), \ldots, (X_n, \theta_n) \), where each \( \theta_i \) is distributed according to some unknown prior pdf \( g \) and, given \( \theta_i = \theta \), observation \( X_i \) has the known conditional density function \( q(x|\theta) \), so that, each pair \( (X_i, \theta) \) has an absolutely continuous distribution with the density function \( q(x|\theta)g(\theta) \). In each pair, the first component is observable, but the second is not. After the \( (n+1) \)-th observation \( y \equiv X_{n+1} \) is taken, the goal is to estimate \( t(\theta_{n+1}) \).

If the prior density \( g(\theta) \) were known, then the Bayes estimator of \( \theta_{n+1} \) which delivers the minimal mean squared risk would be given by the following equation:

\[
t(y) = \frac{\int_{-\infty}^{\infty} \theta q(y|\theta)g(\theta) \, d\theta}{\int_{-\infty}^{\infty} q(y|\theta)g(\theta) \, d\theta}
\]  

(1.1)

(see, e.g., Carlin and Louis, 2000, or Maritz and Lwin, 1989).

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Since the prior density \(g(\theta)\) is unknown, an EB estimator \(\hat{t}(y; X_1, X_2, \ldots, X_n)\) has to be used. Using notations

\[
p(y) = \int_{-\infty}^{\infty} q(y|\theta)g(\theta) \, d\theta, \quad (1.2)
\]

\[
\Psi(y) = \int_{-\infty}^{\infty} \vartheta q(y|\theta)g(\theta) \, d\theta. \quad (1.3)
\]

\(t(y)\) can be rewritten as

\[
t(y) = \Psi(y)/p(y). \quad (1.4)
\]

There is a variety of methods which allow to estimate \(t(y)\) on the basis of observations \(y; X_1, X_2, \ldots, X_n\). After Robbins (1955, 1964) formulated EB estimation approach, many statisticians have been working on developing EB methods. The comprehensive list of references as well as numerous examples of applications of EB techniques can be found in Carlin and Louis (2000) or Maritz and Lwin (1989).

The EB techniques can be divided into two groups: parametric and nonparametric. Parametric EB methods require that the parametric form of a family, to which the prior distribution belongs, is specified a priori. The past data is then used to estimate the values of the unknown parameters, usually using the maximum likelihood approach (see, e.g., Louis, 1984; Morris, 1983; Casella, 1985; Efron and Morris, 1977 among others).

In nonparametric EB estimation, prior distribution is completely unspecified. One of the approaches to nonparametric EB estimation is based on estimation of the numerator and the denominator in the ratio in (1.4). This approach was introduced by Robbins (1955, 1964) himself and later developed by a number of authors (see, e.g., Brown and Greenshtein, 2009; Datta, 1991, 2000; Ma and Balakrishnan, 2000; Nogami, 1988; Pensky, 1997a,b; Raykar and Zhao, 2011; Singh, 1976, 1979; Walter and Hamedani, 1991 among others). The method provides estimators with good convergence rates, however, it requires relatively tedious three-step procedure: estimation of the top and the bottom of the fraction and then the fraction itself. In particular, one of the approaches is to take advantage of the fact that, in the case of a one-parameter exponential family, the numerator can be expressed as the derivative of the denominator. Wavelets provide an opportunity to construct adaptive wavelet-based EB estimators with better computational properties in this framework (see, e.g., Huang, 1997; Pensky, 1998, 2000, 2002) but the necessity of estimation of the ratio in (1.4) remains. Another nonparametric approach developed in Jiang and Zhang (2009) is based on application of nonparametric MLE technique which is computationally extremely demanding.

In 1983, Robbins introduced a much more simple, local nonparametric EB method, linear EB estimation. Robbins (1983) suggested to approximate Bayes estimator \(t(y)\) locally by a linear function of \(y\) and to determine the coefficients of \(t(y)\) by minimizing the expected squared difference between \(t(y)\) and \(\theta\), with subsequent estimation of the coefficients on the basis of observations \(X_1, \ldots, X_n\). The technique is extremely efficient computationally and was immediately put to practical use, for instance, for prediction of the finite population mean (see, e.g., Ghosh and Meeden, 1986; Ghosh and Lahiri, 1987; Karunamuni and Zhang, 2003).

However, a linear EB estimator has a large bias since, due to its very simple form, it has a limited ability to approximate the Bayes estimator \(t(y)\). For this reason, linear EB estimators are optimal only in the class of estimators linear in \(y\). To overcome this defect, Pensky and Ni (2000) extended approach of Robbins (1983) to approximation of \(t(y)\) by algebraic polynomials. However, although the polynomial-based EB estimation provides significant improvement in the convergence rates in comparison with the linear EB estimator, the system of linear equations resulting from the method is badly conditioned which leads to computational difficulties and loss of precision.

To overcome those difficulties, Pensky and Alotaibi (2005) proposed to replace polynomial approximation of the Bayes estimator \(t(y)\) by its approximation via wavelets, in particular, by expansion over scaling functions at the resolution level \(m\). The method exploits de-correlating property of wavelets and leads to a low-dimensional well-posed sparse system of linear equations. The paper also treated the issue of locally optimal choice of resolution level as \(n \to \infty\): if the resolution level is selected correctly, in accordance with the smoothness of the Bayes estimator, then the suggested EB estimator attains the best convergence rates which can be obtained by application of wavelet-based EB estimator.

However, smoothness of the Bayes estimator \(t(y)\) is hard to assess. For this reason, the EB estimator of Pensky and Alotaibi (2005) is non-adaptive. One of the possible ways of achieving adaptivity would be to replace the linear scaling function based approximation by a traditional wavelet expansion with subsequent thresholding of wavelet coefficients. The deficiency of this approach, however, is that it yields the system of equations which is much less sparse and is growing in size with the number of observations \(n\).

The present paper has two main objectives. The first one is to derive lower bounds for the posterior risk of a nonparametric empirical Bayes estimator. In spite of a 50 years long history of empirical Bayes estimation methods, a general lower bound for the risk of an empirical Bayes estimators has not been derived so far. Only some particular cases of the problem were investigated. Specifically, Penskaya (1995) obtain lower bounds for the posterior risk of nonparametric empirical Bayes estimators of a location parameter. Li et al. (2005) obtained lower bounds for the risk of empirical Bayes estimators in the exponential families. However, since their lower bound was of the form \(Cn^{-1}\), practically no estimator could attain it. Construction of the lower bounds for the risk was attempted also in the empirical Bayes two-action problem in the case of a continuous one-parameter exponential family. Karunamuni (1996) published the paper on the subject but
his results were proved to be inaccurate, at least in the case of the normal distribution, when Liang (2000) constructed an estimator with the convergence rates below the lower bound for the risk. Pensky (2003) derived lower bounds for the loss in the empirical Bayes two-action problem involving normal means. Nevertheless, no general theory for construction of the lower bounds for the prior or posterior risk of an empirical Bayes estimator has ever been developed so far. In what follows, we construct lower bounds for the posterior risk of an empirical Bayes estimator under a general assumption that the marginal density \( p(x) \) given by formula (1.2) is continuously differentiable in the neighborhood of \( y \).

The second purpose of this paper is to provide an adaptive version of the wavelet EB estimator developed in Pensky and Alotaibi (2005). In particular, we preserve the linear structure of the estimator. However, since expansion over scaling functions at the resolution level \( m \) tends to excessive variance when resolution level \( m \) is too high and disproportionately large bias when \( m \) is too small, we choose the resolution level using methodology introduced by Lepski (1991) and further developed by Lepski et al. (1997). The resulting estimator is adaptive and attains optimal convergence rates (within a logarithmic factor of \( n \)). In addition, it has an advantage of computational efficiency since it is based on the solution of low-dimensional sparse system of linear equations the matrix of which tends to a scalar multiple of an identity matrix as the scale \( m \) grows. The theory is supplemented by numerous examples that demonstrate how the estimator can be implemented for various types of distribution families.

The rest of the paper is organized as follows. Section 2 introduces EB estimation algorithm. Section 3 assesses estimation error and describes the choice of the resolution level which delivers the best possible convergence rates when the degree of smoothness of Bayes estimator \( \hat{f}(y) \) is known. Section 4 derives minimax lower bounds for the posterior risk, so that we can verify that the EB estimators constructed in the paper are indeed asymptotically optimal as \( n \to \infty \). Section 5 discusses adaptive choice of the resolution level using Lepski method and proves asymptotic optimality of the resulting EB estimator which is based on this choice. Section 6 provides examples of construction of EB estimators for a variety of distribution families. Section 7 concludes the paper with discussion. Finally, Section 8 contains the proofs of the statements in the paper.

## 2. EB estimation algorithm

In order to construct an estimator of \( \hat{f}(y) \) defined in (1.4), choose a twice continuously differentiable scaling function \( \varphi \) with bounded support and \( s \) vanishing moments, so that

\[
\text{supp } \varphi \in [M_1, M_2],
\]

\[
\int_{-\infty}^{\infty} x^s \sum_{k \in \mathbb{Z}} \varphi(x-k) \varphi(z-k) \, dx = z^s, \quad 0 \leq s < 1
\]

(see, e.g., Walter and Shen, 2001).

Approximate \( \hat{f}(y) \) by a wavelet series, for some fixed \( m \geq 0 \),

\[
t_m(y) = \sum_{k \in \mathbb{Z}} a_{m,k} \varphi_{m,k}(y)
\]

where \( \varphi_{m,k}(y) = 2^{m/2} \varphi(2^m y - k) \), and estimate coefficients of \( t_m(y) \) by minimizing the integrated mean squared error:

\[
\min_{a_{m,k}} \int_{-\infty}^{\infty} \left\{ \sum_{k \in \mathbb{Z}} a_{m,k} \varphi_{m,k}(y) - z \right\}^2 q(y|z) \, dz \, dy.
\]

Taking derivatives of the last expression with respect to \( a_{m,j} \) and equating them to zero, we obtain the system of linear equations

\[
B_m a_m = c_m
\]

with

\[
(B_m)_{ij} = B_{i,j} = \int_{-\infty}^{\infty} \varphi_{m,k}(x) \varphi_{m,j}(x) \, dx = \mathbb{E}[\varphi_{m,k}(X) \varphi_{m,j}(X)],
\]

\[
(c_m)_i = c_i = \int_{-\infty}^{\infty} \varphi_{m,k}(x) \psi(x) \, dx
\]

where we use the symbol \( \mathbb{E} \) for expectation over the joint distribution of \( X_1, X_2, \ldots, X_n \). The expectations over all other distributions are represented in the integral forms. Also, in what follows, we suppress index \( m \) in notations of matrix \( B_m = B \) and vector \( c_m = c \) unless this leads to a confusion.

System (2.5) is an infinite system of equations. However, since we are interested in estimating \( f(x) \) locally at \( x = y \), we shall keep only indices \( k, j \in K_{m,y} \) such that

\[
K_{m,y} = \{ k \in \mathbb{Z} : 2^m y - M_2 - s(M_2 - M_1) \leq k \leq 2^m y - M_1 + s(M_2 - M_1) \},
\]

where \( s \) is the number of vanishing moments of scaling function \( \varphi \) (see (2.2)). Observe that expansion (2.3) actually contains only coefficients \( a_{m,k} \) with \( 2^m y - M_2 \leq k \leq 2^m y - M_1 \), however, in order to ensure fast convergence of the bias of the estimator
to zero, we need to keep more terms in the system of equations (2.5) (see Lemma A.3 in Pensky and Alotaibi, 2005 for more details).

The entries (2.6) of the matrix $B$ are unknown and can be estimated by sample means

$$
\hat{B}_{jk} = n^{-1} \sum_{i=1}^{n} [\phi_{m,k}(X_i)\phi_{m,j}(X_i)].
$$

(2.9)

In order to estimate $c_j$, one needs to find functions $u_{mj}(x)$ such that for any $\theta$

$$
\int_{-\infty}^{\infty} q(x;\theta)u_{mj}(x) \, dx = \int_{-\infty}^{\infty} \partial q(x;\theta)\phi_{m,j}(x) \, dx.
$$

(2.10)

Then, multiplying both sides of (2.10) by $g(\theta)$ and integrating over $\theta$, we obtain

$$
\mathbb{E}u_{mj}(x) = \int_{-\infty}^{\infty} u_{mj}(x)p(x) \, dx = \int_{-\infty}^{\infty} \phi_{m,j}(x)\Psi(x) \, dx = c_j.
$$

Note that functions $u_{mj}(x)$ are the same functions that appeared in the wavelet estimator of the numerator $\Psi(y)$ of the EB estimator (1.4) in Pensky (1997a, 1998) where it was demonstrated that construction of $u_{mj}(x)$ is possible in many particular cases. In Section 6 we show that solutions of equation (2.10) can be easily obtained for a variety of distribution families.

Once functions $u_{mj}(x)$ are derived, coefficients $c_j$ can be estimated by

$$
\hat{c}_j = n^{-1} \sum_{i=1}^{n} u_{mj}(X_i)
$$

(2.11)

and system (2.5) is replaced by $\hat{B}\hat{a} = \hat{c}$. However, though estimators $\hat{B}$ and $\hat{c}$ converge in mean squared sense to $B$ and $c$, respectively, the estimator $\hat{a} = \hat{B}^{-1}\hat{c}$ may not even have finite expectation. To understand this fact, note that both $\hat{B}$ and $\hat{c}$ are asymptotically normal. In one dimensional case, the ratio of two normal random variables has Cauchy distribution and, hence, does not have finite mean. In multivariate case the difficulty remains. To ensure that the estimator of $a$ has finite expectation, we choose $\delta = \delta_n > 0$ and construct an estimator of $a$ of the form

$$
\hat{a}_\delta = (\hat{B} + \delta I)^{-1}\hat{c}
$$

(2.12)

where $I$ is the identity matrix. Observe that matrix $\hat{B}$ is nonnegative definite, so that $(\hat{B} + \delta I)$ is a positive definite matrix and, hence, is nonsingular. Solution $\hat{a}_\delta$ is used for construction of the EB estimator

$$
\hat{t}_m(y) = \sum_{k \in K_{m,\delta}} (\hat{a}_\delta)_{m,k} \phi_{m,k}(y).
$$

(2.13)

### 3. Estimation error and convergence rates

#### 3.1. The posterior and the prior risks

An EB estimator $\hat{t}(y)$ can be characterized by its posterior risk

$$
R(y; \hat{t}) = (p(y))^{-1} \int_{-\infty}^{\infty} (\hat{t}(y) - \theta)^2 q(y;\theta)g(\theta) \, d\theta
$$

which can be partitioned into two components. The first component of this sum is

$$
R(y; \hat{t}(y)) = \inf \int_{-\infty}^{\infty} (t(y) - \theta)^2 q(y;\theta)g(\theta) \, d\theta,
$$

which is independent of $\hat{t}(y)$ and represents the posterior risk of the Bayes estimator (1.1). Thus, we shall measure precision of an EB estimator (2.13) by the second component

$$
R_0(y) = \mathbb{E}(\hat{t}_m(y) - \hat{t}(y))^2
$$

(3.1)

which represents the local error of the EB estimator at the point of observation $y$.

It must be noted that often the precision of an EB estimator is described by

$$
\mathbb{E}R_n(y) = \int_{-\infty}^{\infty} R(y; \hat{t}(y))p(y) \, dy,
$$

which is the difference between the prior risk

$$
\mathbb{E} \int_{-\infty}^{\infty} R(y; \hat{t}(y))p(y) \, dy
$$
of the EB estimator \(\hat{r}(y)\) and the prior risk
\[
\int_{-\infty}^{\infty} R(y; t(y)) p(y) \, dy = \inf \int_{-\infty}^{\infty} R(y; f(y)) p(y) \, dy
\]
of the corresponding Bayes estimator \(t(y)\). However, the risk function (3.1) has several advantages compared with \(\hat{R}_n(y)\). First, \(R_n(y)\) enables one to calculate the mean squared error for the given observation \(y\) which is the quantity of interest. Note that the wavelet series (2.13) is local in a sense that coefficients \((\hat{a}_k)_{m,k}\) change whenever \(y\) changes, hence, working with a local measure of the risk makes much more sense. Using the prior risk for the estimator which is local in nature prevents one from seeing advantages of this estimator. Second, by using the risk function (3.1) we eliminate the influence on the risk function of the observations having very low probabilities. So, the use of \(R_n(y)\) provides a way of getting EB estimators with better convergence rates. Third, posterior risk allows one to assess optimality of EB estimators for majority of familiar distribution families via comparison of the convergence rate of the estimator with the lower bounds for the risk derived in Pensky (1997a,b). Finally, one can pursue evaluation of the prior risk for the estimator (2.13). The derivation will require assumptions similar to the ones in Pensky (1998) and can be accomplished by standard methods.

The error (3.1) is dominated by the sum of two components
\[
R_n(y) \leq 2(R_{1n}(y) + R_{2n}(y))
\]
where the first component \(R_{1n}(y)\) is due to replacement of the Bayes estimator \(t(y)\) by its wavelet representation (2.3), while \(R_{2n} = R_{2n}(y)\) is due to replacement of vector \(a = B^{-1}c\) by \(\hat{a}_s\) given by (2.12)
\[
R_{1n}(y) = (t_m(y) - t(y))^2,
\]
\[
R_{2n}(y) = E \left[ \sum_{k \in \mathbb{K}} |(\hat{a}_s)_{m,k} - a_{m,k}| \psi_{m,k}(y) \right]^2.
\]
We shall refer to \(R_{1n}(y)\) and \(R_{2n}(y)\) as the systematic and the random error components, respectively. Since in this paper we are using the posterior risk as a measure of precision of an EB estimator, from now on we treat \(y\) as a fixed quantity.

### 3.2. The systematic error component

For evaluation of the systematic error component \(R_{1n}\), let us introduce matrix \(U_h\) and vector \(D_h\) with components
\[
(U_{h})_{k,l} = \int_{-\infty}^{\infty} \phi(z + 2^m y - k) \phi(z + 2^m y - l) \, dz,
\]
\[
(D_{h})_{k,l} = \int_{-\infty}^{\infty} \psi(z + 2^m y - k) \, dz.
\]
Observe that \(U_h\) and \(D_h\) are independent of unknown functions \(p(x)\) and \(\psi(x)\), and that \(U_0 = I\) where \(I\) is the identity matrix. Denote
\[
\Omega_{m,y} = \{x : |x-y| \leq 2^{-m}(M_2 - M_4)\}.
\]
Then, the following statement is valid.

**Lemma 1.** Let functions \(p(x)\) and \(\psi(x)\) be \(r \leq s - 1\) times continuously differentiable in the neighborhood \(\Omega_y\) of \(y\) and let \(\Omega_{m,y} \subseteq \Omega_y\), with \(\Omega_{m,y}\) defined in (3.7). Then, for \(R_{1n}\) defined in (3.3), as \(m \to \infty\),
\[
R_{1n}(y) = (t_m(y) - t(y))^2 = o(2^{-2mr}).
\]

Let us now give some insight into the proof of the lemma. Note that
\[
B_{j,k} = 2^m \int_{-\infty}^{\infty} \phi(2^m x - k) \phi(2^m x - j) p(x) \, dx
\]
\[
= \int_{-\infty}^{\infty} \phi(z + 2^m y - k) \phi(z + 2^m y - j) p(y + 2^m z) \, dz
\]
\[
= \sum_{h=0}^{r} 2^{-mh}(h!)^{-1} p^{(h)}(y) U_h + o(2^{-mr}),
\]
where \(U_0 = I\) is the identity matrix. Deriving a similar representation for \(c_h\), we obtain asymptotic expansions of matrix \(B\) and vector \(c\) via matrices \(U_h\) and vectors \(D_h\), respectively, as \(m \to \infty\)
\[
B = p(y) I + \sum_{h=1}^{r} 2^{-mh}(h!)^{-1} p^{(h)}(y) U_h + o(2^{-mr}),
\]

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\[ c = 2^{-m/2} \sum_{l=0}^{m} 2^{-ml} t^{l-1} \psi_t(y) D_l + o(2^{-mr}). \]  
(3.10)

Formula (3.9) establishes that, for large values of \( m \), matrix \( B \) is close to \( p(y)I \), so the system of equations (2.5) is well-conditioned. Therefore, the inverse matrix is close to \( p^{-1}(y)I \), or, more precisely (see Pensky and Alotaibi, 2005),

\[ B^{-1} = p^{-1}(y)I - 2^{-m} p'(y) y p^{-2}(y) U_1 + o(2^{-m}). \]  
(3.11)

Furthermore, if \( m \to \infty \), vector \( a \) in (2.5) tends to \( 2^{-m/2} [\psi(y)/p(y)] D_0 \) where

\[ 2^{-m/2} \sum_k (D_0)_k \psi_{mk}(y) = 1 \]

for any \( y \). The latter implies that the systematic error \( R_m \) tends to zero, as \( m \to \infty \), at a rate \( o(2^{-m}) \).

The exact proof of Lemma 1 can be obtained by a slight modification of the proof of Lemma 3.1 in Pensky and Alotaibi (2005).

3.3. The random error component

In order to calculate the random error component \( R_{2m}(y) \) given by (3.4), introduce vectors \( \psi^{(m)}(y) \), \( \phi = 1, 2, 3, 4 \), with components

\[ \psi^{(m)}_k = \left( \int_{-\infty}^{\infty} y_{mk}(x)^2 dx \right)^{1/2}, \quad k \in K_{my}, \quad \phi = 1, 2, 3, 4 \]  
(3.12)

where \( y_{mk}(x) \) are defined in (2.10), and denote

\[ \gamma_m = \| \psi^{(m)} \| \]  
(3.13)

where \( \|z\| \) is the Euclidean norm of the vector \( z \). The following lemma provides an asymptotic expression for the random error component as \( m, n \to \infty \).

**Lemma 2.** Let \( \delta \sim n^{-1/2} \). Then, under the assumptions of Lemma 1, as \( m, n \to \infty \), the random error component \( R_{2m} \) defined in (3.4) is such that

\[ R_{2m} = O(2^{m-n}(1 + \gamma_m^2)), \quad m, n \to \infty, \]  
(3.14)

provided \( 2^{m-n} \to 0 \) and \( \| \psi^{(2)}(m) \| 2^{m} = o(n^{3}) \) as \( n \to \infty \).

Proofs of this and other statements in the paper are given in Section 8.

Observe that the values of \( \gamma_m^{(m)}(m) \) are independent of the unknown density \( g(\theta) \) and can be calculated explicitly. In Section 6, we bring examples of construction of functions \( y_{mk}(x) \) as well as the asymptotic expressions for \( \gamma_m^{(2)}(m) \), \( \phi = 1, 2 \), for some common special cases (location parameter family, scale parameter family, one-parameter exponential family). In vast majority of situations, including the ones studied in Section 6, \( \gamma_m^{(2)} \) is bounded above by the following expression:

\[ \gamma_m^2 \leq C_2 2^{m}, \quad C_2 > 0, \quad a \in \mathbb{R}, \]  
(3.15)

where \( a \) and \( C_1 \) are the absolute constants independent of \( m \).

It follows from Lemmas 1 and 2 that \( R_n \) given in (3.2) is minimized at the optimal resolution level \( m = m_0 \) where both errors are balanced

\[ m_0 = \arg \min m \left( n^{-1/2} \gamma_m^2 (y_m^2 + 1) + 2^{-2m} \right). \]  
(3.16)

In particular, under assumption (3.15), as \( n \to \infty \), \( m_0 \) is such that

\[ 2^{m_0} \approx 1/(2r + 1 + \max(a, 0)). \]  
(3.17)

Here, we denote \( a = b_0 \) for two sequences, \( (a_n) \) and \( (b_n) \), \( n = 1, 2, \ldots \), of positive real numbers if there exist \( C_1 \) and \( C_2 \) independent of \( n \) such that \( 0 < C_1 < C_2 < \infty \) and \( C_1 \leq o_n/b_n \leq C_2 \).

Then the following statement is true.

**Theorem 1.** Let twice continuously differentiable scaling function \( \psi \) satisfy (2.1) and (2.2). Let functions \( p(x) \) and \( \psi(x) \) be \( r \) times continuously differentiable in the neighborhood \( \Omega_y \) of \( y \) such that \( \Omega_{m,y} \subseteq \Omega_y \) where \( \Omega_{m,y} \) is defined in (3.7) and \( 1/2 \leq r \leq (s-1) \). Choose \( m_0 \) according to (3.16) and let \( \delta \) in (2.12) be such that \( \delta \sim n^{-2} 2^{m_0} \). Then, for any \( y \) such that \( p(y) > 0 \), as \( n \to \infty \), \( R_n(y) \) defined in (3.1) satisfies the following asymptotic relation:

\[ R_n(y) = \varepsilon (\hat{y}_m(y) - t(y))^2 = O(2^{-2m_0}), \quad m \to \infty, \]  
(3.18)

provided \( 2^{m-n} \to 0 \) and \( \| \psi^{(2)}(m_0) \| 2^{m} = o(n^{3}) \) as \( n \to \infty \). In particular, if assumption (3.15) holds, then, as \( n \to \infty \), one has

\[ R_n(y) = O(n^{2r/(2r+1+\max(a,0))}). \]  
(3.19)
Remark 1. In general, it is possible that instead of inequality (3.15) one has
\[ y_m \leq C e^{2^{m_p} \exp(b 2^{m_p})}, \quad b, \beta, C, \gamma \geq 0. \]
In this case, \( 2^{m_0} = (2b)^{-1} \log n \) and
\[ R_n(y) = O(\log n)^{-2r/\beta}. \]
Note that, unlike for the case of \( b = 0 \), the value of \( m_0 \) is independent of the unknown smoothness parameter \( r \), so the estimator is adaptive. However, the rate of convergence is very slow (logarithmic in \( n \)). Fortunately, this case is not common in practice. It does not take place for any of the examples considered in this paper. Moreover, we failed to find an example when this situation takes place.

4. Minimax lower bounds for the posterior risk

Let \( y \) be a fixed point. Consider an \( r \)-times continuously differentiable pdf \( p_0(x) \), and an \( r \)-times continuously differentiable kernel \( k(\cdot) \) with supp \( k = (-1, 1) \) and such that \( \int k(z) \, dz = 0 \). Let \( p_0(\cdot) \) and \( k(\cdot) \) satisfy the following assumptions:

Assumption A1. There exists \( g_0(\theta) \) such that, for any \( x \),
\[ p_0(x) = \int_{-\infty}^{\infty} q(x; \theta) g_0(\theta) \, d\theta. \]

Assumption A2. There exists a function \( \psi_{h,y}(\theta) \) such that, for any \( x \) and \( h > 0 \),
\[ k\left(\frac{x-y}{h}\right) = \int_{-\infty}^{\infty} q(x; \theta) \psi_{h,y}(\theta) \, d\theta. \]  
(4.1)

Assumption A3. Density \( p_0(x) \) is such that, for any \( x \) such that \( |x-y| \leq h \) and some \( C_p > 0 \)
\[ p_0(x) > 2C_p \|k\|_{\infty}. \]

Denote
\[ \psi_0(x) = \int_{-\infty}^{\infty} \partial q(x; \theta) g_0(\theta) \, d\theta, \quad w_{h,y}(x) = \int_{-\infty}^{\infty} \partial q(x; \theta) \psi_{h,y}(\theta) \, d\theta, \]
\[ \rho_r(h) = \left[ \max_{1\leq s\leq r} \left| \frac{d^s}{d\theta^s} [w_{h,y}(x)] \right|_{x}\right]^{-1}. \]  
(4.2)
(4.3)

Let \( \mathcal{G} \) be a class of functions \( g(\theta) \) such that \( p(y) \) defined by (1.2) is \( r \) times continuously differentiable in the neighborhood \( \Omega_y \) of \( y \). The following theorem provides lower bounds of the posterior risk (3.1) at the point \( y \).

Theorem 2. Let \( r \geq 1/2 \). Assumptions A1–A3 hold and \( \rho_r(h) \) and \( w_{h,y}(y) \) be such that, for some nonnegative \( r_1 \) and \( r_2 \),
\[ \rho_r(h) \leq C h^{r_2}, \quad \|w_{h,y}(y)\| \leq C_0 \]  
(4.4)

Then, for any \( y \) such that \( p(y) > 0 \), as \( n \to \infty \),
\[ \Delta_n(y) = \inf_{\mathcal{G}} \sup_{y \in \Omega_y} \mathbb{E}(\hat{y} - y)^2 \geq C n^{-2r_1} \max(r_1, r_2) + (2 \max(r_1, r_2) + 1), \]  
(4.5)

where \( C \) is an absolute constant independent of \( n \). In particular, if \( r_1 = r + 2r_2 \) and \( r_2 = \max(\alpha/2, 0) \) where \( \alpha \) is defined in (3.15), then
\[ \Delta_n(y) \geq C n^{-2r/3 + 1 + \max(\alpha, 0)} \]  
(4.6)

Remark 2. Note that if \( r_1 = r + 2r_2 \) and \( r_2 = \max(\alpha/2, 0) \), then EB estimator (2.13) with \( m = m_0 \) given by formula (3.17) is asymptotically optimal. If \( r_1 \leq r \) and \( r_2 = 0 \), then convergence rates are defined by behavior of \( p(x) \) in the neighborhood of \( y \), otherwise, the rates are defined by behavior of \( \psi(x) \) in the neighborhood of \( y \).

5. Adaptive choice of the resolution level using Lepski method

Note that the value of \( m_0 \) depends on the unknown smoothness \( r \geq 0 \) of functions \( p(x) \) and \( \psi(x) \) which is unknown, i.e., the estimator is nonadaptive. In order to construct an adaptive estimator, we shall apply Lepski method for the optimal selection of the resolution level described in Lepski (1991) and Lepski et al. (1997).
Let \( \gamma_m \) satisfy condition (3.15). Denote
\[
\rho_m^2 = 2m(1 + \gamma_m^2)n\frac{1}{\log n},
\] (5.1)
and let \( m_1 \) and \( m_n \) be such that
\[
2^{m_1} = \log n, \quad 2^{m_n}(\gamma_m^2 + 1)>(\log n)^{-2}n.
\] (5.2)
Denote \( \mathcal{M} = \{m : m_1 \leq m \leq m_n\} \) and observe that under assumption (3.15) with \( b=0, m_n \) is such that
\[
2^{m_n} = \mathcal{C}\left( \frac{n}{\log^2 n} \right)^{(1/\max(0, \alpha))},
\]
so that, for \( m_0 \) given by (3.17), one has \( m_n/m_0 \rightarrow \infty \) as \( n \rightarrow \infty \).

In this situation, the Lepski method suggests to choose resolution level as \( m = \hat{m} \) with
\[
\hat{m} = \min\{m \in \mathcal{M} : \|\hat{f}_m(y) - \hat{f}_m(y)\|^2 \leq \lambda^2 \|\hat{B}_{m,\infty}\|^2 + \|\hat{B}_{\infty}\|^2 \rho_m^2 \text{ for any } j \geq m\},
\] (5.3)
where \( \lambda \) is a positive constant independent of \( m \) and \( n \) and, for any \( k, \hat{B}_{mk} = \hat{B}_k + \delta kI \) where \( \delta_k = 2^{k/2}n^{-1/2} \), and the matrix norm used in (5.3) and throughout the paper is the spectral norm.

Let \( \nu_1 \) and \( \nu_2 \) be small positive values, \( \nu_1 + \nu_2 < 1 \), and \( M \) be the size of vector \( c \) and matrix \( B \). Denote \( C_{\nu} = \sum_k|\nu_k(\nu_k - k)| \) and
\[
D = \|\|P\|\|_{\infty} + \|\|P\|\|_{\infty}^2\|P\|\|_{\infty}M\sqrt{M}[1 - \nu_1]^{-1/2} + 16\|\|P\|\|_{\infty}\|\nu\|_{\infty}^2\|P\|\|_{\infty}^2M^3\sqrt{M}[1 - \nu_2]^{-1}
\]
and let \( \lambda \) be given by equation
\[
\lambda = 16C_{\nu}\|P\|\|_{\infty}^2\sqrt{MD} + 1
\] (5.4)
Then, the following statement is true.

**Theorem 3.** Let twice continuously differentiable scaling function \( \varphi \) satisfy (2.1) and (2.2). Let functions \( p(x) \) and \( \psi(x) \) be \( r \geq 1/2 \) times continuously differentiable in the neighborhood \( \Delta_x \) of \( y \) and let \( \Omega_{m,\nu} \subseteq \Omega_y \) where \( \Omega_{m,\nu} \) is defined in (3.7). Let \( \gamma_m \) satisfy inequality (3.15), Construct EB estimator of the form (2.13) and choose \( \hat{m} \) according to (5.3) with \( \lambda \) defined in (5.5) where \( D \) given by (5.4). If, for any \( k \in \mathbb{K}_{m,y} \),
\[
\|U_{m,k}\|_{\infty} \leq C_2 2^{m/2}/\gamma_m,
\]
then
\[
\|\hat{f}_m(y) - f(y)\|^2 = O(n^{-2r/(2r + 1) + \max(0, \alpha)} \log n).
\] (5.7)

In order to see how the method works, note that the mean squared error can be decomposed as
\[
\Delta = \|\hat{f}_m(y) - f(y)\|^2 = \Delta_1 + \Delta_2
\] (5.8)
where \( m_0 \) is the optimal resolution level defined in formula (3.16) and
\[
\Delta_1 = \|\hat{f}_m(y) - f(y)\|^2 \|\hat{m} \leq m_0\|, \\
\Delta_2 = \|\hat{f}_m(y) - f(y)\|^2 \|\hat{m} > m_0\|.
\]
If \( \hat{m} \leq m_0 \), then, by definition of \( \hat{m} \), one has
\[
\|\hat{f}_m(y) - f(y)\|^2 \leq \lambda^2 \rho_{m_0}^2 \|\hat{B}_{m,\infty}\|^2 + \|\hat{B}_{\infty}\|^2 \rho_{m_0}^2 = O(\rho_{m_0}^2),
\] (5.9)
so that
\[
\Delta_1 \leq 2\|\hat{f}_m(y) - f(y)\|^2 + \|\hat{f}_m(y) - f(y)\|^2 = O(\rho_{m_0}^2)
\] (5.10)
Now, in the case \( \hat{m} > m_0 \), by definition (5.3) of \( \hat{m} \), there exists \( j > m_0 \), such that
\[
\|\hat{f}_m(y) - f(y)\|^2 > \lambda^2 \rho_{m_0}^2 \|\hat{B}_{m,\infty}\|^2 + \|\hat{B}_{\infty}\|^2 \rho_{m_0}^2.
\]
It turns out that probability of such an event is very low. In particular, the following lemma holds.

**Lemma 3.** Let conditions of Theorem 3 hold. If resolution level \( m \) is such that \( m_0 < m \leq m_0 \), then, as \( n \rightarrow \infty \),
\[
\mathbb{P}(\|\hat{f}_m(y) - f(y)\|^2 > \lambda^2 \|\hat{B}_{m,\infty}\|^2 + \|\hat{B}_{\infty}\|^2 \rho_{m_0}^2) = O(n^{-2}).
\] (5.11)
where \( \rho_{m_0}^2 = 2^m(1 + \gamma_m^2)\log n \).

This lemma implies that \( \Delta_2 = O(n^{-1}) = O(\rho_{m_0}^2) \) as \( n \rightarrow \infty \) and, hence, **Theorem 3** is valid. The complete proof of **Theorem 3** is provided in **Section 8**.
6. Examples of construction of EB estimators, lower and upper bounds for the risk

6.1. Location parameter family

In the case when \( \theta \) is a location parameter, one has \( q(x|\theta) = q(x-\theta) \), EB estimator \( t(y) \) in (1.1) is of the form

\[
t(y) = y - \int_{-\infty}^{\infty} (x-\theta) q(x-\theta) g(\theta) \, d\theta / \int_{-\infty}^{\infty} q(x-\theta) g(\theta) \, d\theta.
\]

Hence, in this case, \( u_{m,k}(x) \) is the solution to the following equation:

\[
\int_{-\infty}^{\infty} q(x-\theta) u_{m,k}(x) \, dx = \int_{-\infty}^{\infty} (x-\theta) q(x-\theta) \psi_{m,k}(x) \, dx.
\]

Here we provide a brief construction of \( u_{m,k} \) and evaluation of its variance, for a more detailed derivation see Pensky (2000, 2002). Taking Fourier transforms of both sides, we obtain that

\[
u_{m,k}(x) = 2^{m/2} U_m(2^m x - k),
\]

where \( U_m(\cdot) \) is the inverse Fourier transform of

\[
\hat{U}_m(\omega) = i^k \hat{q}(-2^m \omega)^{-1} \hat{q}(-2^m \omega) \hat{\psi}(\omega).
\]

Here, and in what follows, \( \hat{\psi} \) denotes the Fourier transform of function \( \psi \). Using Parseval identity and taking into account that \( K_m \) has finite number of terms, we derive

\[
y_m^2 \approx \int_{-\infty}^{\infty} |\hat{q}(-2^m \omega)|^{-1} \hat{q}(-2^m \omega)^2 |\hat{\psi}(\omega)|^2 \, d\omega.
\]

Also, the following relation allows to check validity of condition (5.6):

\[
\|u_{m,k}(x)\|_{\infty} \leq 2^{m/2} \|U_m(\cdot)\|_{\infty}.
\]

Now, in order to calculate the lower bounds for the risk, we need to find \( \psi_{h,y}(\theta) \) and \( w_{h,y}(x) \). Let \( \psi_{h,y}(\theta) \) be solution of Eq. (4.1). It is easy to show that \( \psi_{h,y}(\theta) \) is of the form \( \psi_{h,y}(\theta) = \psi_h(\theta-y)/h \), where the Fourier transform \( \hat{\psi}_h(\omega) \) of \( \psi_h(\cdot) \) is of the form

\[
\hat{\psi}_h(\omega) = \hat{k}(\omega)/\hat{q}(\omega/h).
\]

In order to obtain an expression for \( w_{h,y}(x) \), recall that Eq. (4.1) can be rewritten as

\[
w_{h,y}(x) = x - \int_{-\infty}^{\infty} (x-\theta) q(x-\theta) \psi_{h,y}(\theta) \, d\theta = x - w_h\left(\frac{x-y}{h}\right).
\]

where \( w_h(\cdot) \) is the inverse Fourier transform of

\[
\hat{w}_h(\omega) = i^{-1} \hat{k}(\omega) \hat{q}(\omega/h)/\hat{q}(\omega/h).
\]

In this situation, \( \rho_y(h) \) defined in (4.3) and \( w_{h,y}(y) \) are, respectively, given by

\[
w_{h,y}(y) = y - w_h(0), \quad \rho_y(h) = \left[ \frac{\max(h^{-2} |\hat{\psi}_h(0)|)}{1+\alpha} \right]^{-1}.
\]

Below, we consider some special cases.

**Example 1 (Normal distribution).** Let \( q(x|\theta) \) be the pdf of the normal distribution

\[
q(x|\theta) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\theta)^2}{2\sigma^2}},
\]

where \( \sigma > 0 \) is known. Then \( \hat{q}(\omega) = \sqrt{\pi} e^{-\omega^2 \sigma^2/2} \) and, using properties of Fourier transform, we derive

\[
U_m(x) = -2^m \sigma^2 \hat{q}(x).
\]

Since \( \hat{q}(x) \) is square integrable, it follows from (6.4) and (6.5) that \( y_m^2 \approx 2^{2m} \) and \( \|U_m(\cdot)\|_{\infty} \approx 2^m \). Hence, condition (5.6) of Theorem 3 holds. Then, \( \alpha = 2, 2^{m-n(1/2+3)} \), and

\[
\xi^2 \left( t_m(y) - t(y) \right)^2 = O(n^{-2r/(2r+3)} \log n)
\]

by Theorem 3.

In order to verify optimality of the estimator, we derive the lower bounds for the risk following Theorem 2. Using (6.7), by direct calculations, we obtain, \( \hat{w}_h(\omega) = i h^{-1} \omega \sigma^2 k(\omega) \). Consequently, for a fixed value of \( y \), one has \( w_{h,y}(x) = x + \sigma^2 h^{-1} k(h^{-1}(x-y)) \), \( \rho_y(h) = Ch^{-1} \) and \( |w_{h,y}(y)| \leq Ch^{-1} \). Hence, \( r_1 = r + r_2, r_2 = \alpha/2 = 1 \) in (4.4), and application of Theorem 2 yields \( \Delta_e(y) \geq C n^{-2r/(2r+3)} \), so that EB estimator is optimal up to a logarithmic factor.
Remark 3. Note that in the case of the normal distribution,

\[ p^{(1)}(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)^2/2\sigma} \]

\[ p^{(2)}(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)^2/2\sigma} \]

where \( P_r(z) \) is the polynomial in \( z \) of degree \( r \). Since \( P_r(z)e^{-z^2/2} \) is an infinitely smooth uniformly bounded function, \( r \) can take limitlessly large values, as long as \( |\theta|g(\theta) \) is integrable. Then, convergence rate of the EB estimator is bounded only by the number of vanishing moments of the scaling function \( \varphi \).

Example 2 (Double-exponential distribution). Let \( q(\theta) \) be the pdf of the double-exponential distribution

\[ q(\theta) = \frac{1}{2\sigma} e^{-|x-\theta|/\sigma}, \]

where \( \sigma > 0 \) is known. Then \( \hat{q}(\omega) = 1/(1 + \omega^2\sigma^2) \), and, using properties of Fourier transform, we obtain

\[ u_{mk}(x) = \int_{-\infty}^{\infty} \varphi_{mk}(t) \text{sign}(x-t) \exp(-|x-t|/\sigma) \, dt. \]

Note that for any \( \omega \), one has \( \|\hat{q}(\omega)\|^{-1} \hat{q}(\omega) \leq \sigma \). Therefore,

\[ r_m^2 = \int_{-\infty}^{\infty} |\hat{q}(\omega)|^2 \, d\omega = 1, \quad |U_m(x)| \leq \int_{-\infty}^{\infty} |\hat{\varphi}(\omega)| \, d\omega = 1, \]

so that condition (5.6) holds. Consequently, \( \alpha = 0, 2m_n = n^{1/2r+1} \), and

\[ \mathbb{E}[\hat{r}_m(y) - r(y)]^2 = O(n^{-2r/2r+1}) \log n \]

by Theorem 3.

In order to derive lower bounds for the risk, we apply (6.7) to obtain

\[ \hat{\omega}_m(\omega) = -l^{-1} \hat{k}(\omega) - \frac{2\alpha\omega^2\sigma^{-1}}{\sigma^2\omega^2\sigma^2 + 1}. \]

It can be shown that

\[ w_{mk}(x) = x - 2 \int_{-1}^{1} k(t) \text{sign}(x-y-ht) \exp(-\sigma^{-1}|x-y-ht|) \, dt \]

\[ \rho_1(h) = Ch^{-1} \]

and

\[ |w_{h,y}(y)| = y + 2h \left| \int_{-1}^{1} k(t)e^{-ht/\sigma} \, dt \right|. \]

Therefore, \( r_1 = r \) and \( r_2 = 0 \) in (4.4), and application of Theorem 2 yields \( \Delta_n(y) \geq C n^{-2r/2r+1} \), so that EB estimator is optimal up to a logarithmic factor.

6.2. One-parameter exponential family

Let conditional distribution belong to a one-parameter exponential family, i.e.

\[ q(x|\theta) = h(\theta)f(x)e^{-\theta x}, \quad x \in \mathcal{X}, \quad \theta \in \Theta, \]

where \( h(\theta) > 0 \) and \( f(x) > 0 \). Then, Eq. (2.10) is of the form

\[ \int_{\mathcal{X}} f(x)e^{-\theta x} u_{mk}(x) \, dx = \int_{\mathcal{X}} \theta f(x)e^{-\theta x} \varphi_{mk}(x) \, dx. \]

Integrating the right-hand side by parts and solving for \( u_{mk} \) we derive

\[ u_{mk}(x) = \frac{1}{bf(x)} \frac{d}{dx} \left\{ \int_{\mathcal{X}} f(x)e^{-\theta x} \varphi_{mk}(x) \, dx \right\} = \frac{2m/\varphi(2m^2x-k)}{bx^{-b-1}} + \frac{(f(x)x^{-b})^{2m/\varphi(2m^2x-k)}}{bf(x)}. \] (6.8)

Let the value of \( y \) be such that \( c_1 \leq y \leq c_2 \) for some \( 0 < c_1 < c_2 < \infty \). Then, it is easy to show that, if \( k \in K_{m,y} \), then \( k = 2m \). In this case, for any \( k \neq 0 \), one has

\[ \left[ \frac{1}{k} \right](m) \leq 2b^{-2}2^{bm} \int_{M_2} |z + k|^{-2b-2} |\varphi(z)|^2 \, dz + 2b^{-1} \max_{M_1 \leq x-k \leq M_2} |(f(x))^{-1}(f(x)x^{-b})| \leq C2^{bm}, \] (6.9)

so that \( \alpha = 2 \).
In order to evaluate lower bounds for the risk, we need to find \( \psi_{h,y}(\theta) \) and \( w_{h,y}(x) \). Let \( \psi_{h,y}(\theta) \) be solution of Eq. (4.1) and \( w_{h,y}(x) \) be defined by (4.2). It is straightforward to verify that

\[
\psi_{h,y}(\theta) = f(x) \frac{d}{dx} \left[ \frac{1}{f(x)} \frac{d}{d(x)} \frac{f(x)}{b(x)} \right] = f(x) \frac{d}{dx} \left[ \frac{1}{b(x)} \frac{f(x)}{h(x)} \right] - \frac{1}{b(x)} \frac{f(x)}{h(x)}.
\]

(6.10)

For a fixed value of \( y > 0 \), one has \( r_1 = r + 1 \) and \( r_2 = 1 \) in (4.4). Hence, application of Theorem 2 yields

\[
\Delta_n(y) \geq C n^{-2r/(2r+3)}. \quad (6.11)
\]

**Example 3** *(Weibull distribution).* Let \( q(x|\theta) \) be the pdf of the Weibull distribution

\[
q(x|\theta) = b x^{b-1} e^{-\theta x^b}, \quad x > 0, \quad \theta > 0, \quad b \geq 1.
\]

In this case, \( f(x) = x^{b-1} \), \( h(\theta) = b \theta \) and \( u_{m,k}(x) \) is of the form (6.8) with the second term being identical zero

\[
u_{m,k}(x) = \frac{2m/2 \phi(2^m x - k)}{bx^{b-1}}. \quad (6.12)
\]

Hence, it follows from formula (6.9) that

\[
\left[ \frac{1}{b} \right] a^2 2^{m} b^{-2} (M_2-M_1) M_1 + k^{-(2b-2)}.
\]

(6.13)

Since \( k=2^m \) and the set \( K_n \) has a finite number of terms, (6.13) yields that \( r^2 \approx 2^m \).

Finally, it remains to verify whether the condition (5.6) of Theorem 3 holds. Indeed, it follows from (6.12) that for \( k \in K_n \), one has

\[
\sup_x |u_{m,k}(x)| \leq \sup_x \frac{2m/2 \phi(2^m x - k)}{b[M_1+k]} = C 2^m r_m.
\]

(6.14)

so Theorem 3 can be applied. Hence, under assumptions of Lemma 1, one has \( \sigma = 2 \), \( 2^m \sim n^{1/(2r+3)} \), and, by Theorem 3, \( \mathbb{E}[(f(x)-f(y))^2] = O(n^{-2r/(2r+3) \log n}) \). Therefore, the EB estimator is optimal within a log-factor of \( n \) due to (6.11).

**Example 4** *(Gamma distribution).* Let the pdf of the Gamma distribution be given by

\[
q(x|\theta) = \frac{1}{\Gamma(r)} \theta^r x^{r-1} e^{-\theta x^r}, \quad x > 0, \quad \theta > 0, \quad \rho > 1.
\]

(6.15)

In this case, \( b = 1 \), \( f(x) = x^{b-1} \) and \( h(\theta) = \theta^r / \Gamma(r) \). Note that the family of densities (6.15) is also a scale parameter family. By formula (6.8), \( u_{m,k}(x) \) is of the form

\[
u_{m,k}(x) = (\theta-1) x^{1-2m/2} \phi(2^m x - k) + 2^{m/2} \phi(2^m x - k).
\]

If \( y \) is such that \( c_1 \leq y \leq c_2 \) for some \( 0 < c_1 < c_2 < \infty \), then, by calculations similar to Example 3, obtain that \( 2^m \sim n^{1/(2r+3)} \) and Theorem 3 holds, so that, \( \mathbb{E}[(f(x)-f(y))^2] = O(n^{-2r/(2r+3) \log n}) \). Therefore, the EB estimator is optimal within a log-factor of \( n \) due to (6.11).

6.3. **Scale parameter family**

If \( q(x|\theta) \) is a scale parameter family, \( q(x|\theta) = \theta^{-1} q(\theta^{-1} x) \), it is difficult to pinpoint a general rule for finding \( u_{m,k}(x) \), however, as it follows from Example 4, many particular cases can be treated. Below, we consider one more example.

**Example 5** *(Uniform distribution).* Let \( q(x|\theta) \) be given by

\[
q(x|\theta) = \theta^{-1} 1(0 < x < \theta), \quad \theta > 0 .
\]

Then, Eq. (2.10) is of the form

\[
\int_{0}^{\theta} \theta^{-1} u_{m,k}(x) \, dx = \int_{0}^{b} \psi_{m,k}(x) \, dx \quad (6.16)
\]

Taking derivatives of both sides of (6.16) with respect to \( \theta \) and replacing \( \theta \) by \( x \), we derive

\[
u_{m,k}(x) = 2^{-m/2} \int_{M_1}^{2^m x - k} \phi(z) \, dz + x 2^{m/2} \phi(2^m x - k).
\]

(6.17)

Since \( a \leq \theta \leq b \), then also \( a \leq x \leq b \), and it is easy to check that

\[
\int_{a}^{b} x^2 2^m \phi(2^m x - k) \, dx \approx 1, \quad \int_{a}^{b} \left( 2^{-m/2} \int_{M_1}^{2^m x - k} \phi(z) \, dz \right)^2 \, dx = O(2^{-m}).
\]
as $m \to \infty$. Then, $\gamma_m \approx 1$, $\alpha = 0$ and condition (5.6) holds. Therefore, $2^{m_0} \sim n^{1/(2r+1)}$ and, by Theorem 3, one has $\mathbb{E}[\hat{t}_n^2(y) - t(y)]^2 = O(n^{-3/(2r+1)} \log n)$.

Now, in order to calculate the lower bound for the risk, we need to find $\psi_{h,n}(\theta)$ and $w_{h,n}(x)$. According to (4.1) and (4.2), functions $\psi_{h,n}(\theta)$ and $w_{h,n}(x)$ satisfy equations

$$
\int_{x}^{b} \theta^{-1} \psi_{h,n}(\theta) \, d\theta = k(h^{-1}(x-y)), \quad \int_{x}^{b} \psi_{h,n}(\theta) \, d\theta = w_{h,n}(x).
$$

Now, taking derivatives of both sides of the first equation with respect to $x$ and solving for $\psi_{h,n}(\theta)$, we obtain

$$
\psi_{h,n}(\theta) = -h^{-1} \theta k'(h^{-1}(x-y)).
$$

It can be shown that

$$
w_{h,n}(x) = xh(h^{-1}(x-y)) + hK(h^{-1}(x-y)),
$$

where $K'(z) = k(z)$. Note that $r_1 = r$ and $r_2 = 0$, hence, applying Theorem 2, we obtain the following lower bounds for the risk $\Delta_n(y) \geq Cn^{-3/(2r+1)}$, so that the EB estimator is optimal, up to a logarithmic factor.

7. Discussion

The present paper achieves two main objectives. The first one is to derive lower bounds for the posterior risk of a nonparametric empirical Bayes estimator under general assumptions. The present paper is the first one to accomplish this task. The second purpose of this paper is to provide an adaptive wavelet-based method of EB estimation. The method is nonparametric empirical Bayes estimator under general assumptions. The present paper is the first one to accomplish this task. The technique works for a variety of families of conditional distributions. Computationally, it leads to solution of a finite system of linear equations which, due to decorrelation property of wavelets, is sparse and well-conditioned. The size of the system depends on a size and regularity of the wavelet which is used for representation of the EB estimator $t(y)$.

A non-adaptive version of the method was introduced in Pensky and Alotaibi (2005). However, since no mechanism for choosing the resolution level $m$ of the expansion was suggested, the Pensky and Alotaibi (2005) paper remained of a theoretical interest only. In the present paper, we use Lepski method for choosing an optimal resolution level $m$ and show that the resulting EB estimator remains nearly asymptotically optimal (within a logarithmic factor of the number of observations $n$). We also show that the EB estimators constructed in the paper are asymptotically optimal (up to a logarithmic factor) as $n \to \infty$.

Finally, we should comment that, although the choice of a wavelet basis for representation of $t(y)$ is convenient, it is not unique. Indeed, one can use a local polynomial or a kernel estimator for representation of $t(y)$. In this case, the challenge of finding support of the estimator for the local polynomials or bandwidth for a kernel estimator can be addressed by Lepski method in a similar manner. However, the disadvantage of abandoning wavelets will be that the system of equations will cease to be sparse and well-posed.

8. Proofs

In what follows, we suppress index $m$ in notations of matrix $B_m = B$, $B_{m,s} = B_s$, $\hat{B}_m = \hat{B}$ and $\hat{B}_{m,s} = \hat{B}_s$, and vector $c_m = c$ unless this leads to a confusion. Proofs of Lemma 2 are based on the following lemmas.

**Lemma 4.** Let $B$, $c$, $\hat{B}$ and $\hat{c}$ be defined in (2.6), (2.9), (2.7) and (2.11), respectively, and let $M$ be the size of the vector $c$. If $n \to \infty$, $2^m/n \to 0$, one has

$$
\mathbb{E}[\hat{B} - B]^2 = O(n^{-2m}), \quad \sigma = 1, 2, 4,
$$

$$
\mathbb{E}[\hat{c} - c]^2 = O(n^{-2m}),
$$

$$
\mathbb{E}[\hat{c} - c]^4 = O(n^{-3} \| \hat{\psi}^{(2)}(m) \|^2 + n^{-2} \| \hat{\psi}^{(4)}(m) \|^2),
$$

and

$$
\mathbb{E}[\hat{c} - c]^8 = O(n^{-3} \| \hat{\psi}^{(4)}(m) \|^2 + n^{-2} \| \hat{\psi}^{(2)}(m) \|^4 + n^{-2} \| \hat{\psi}^{(4)}(m) \|^2)
$$

$$
+ O(n^{-3} \| \hat{\psi}^{(4)}(m) \|^2 \| \hat{\psi}^{(2)}(m) \|^2 + n^{-4} \| \hat{\psi}^{(4)}(m) \|^8).
$$

If, in addition, (5.6) holds, then, as $n \to \infty$,

$$
\mathbb{E}[\hat{c} - c]^2 = O(n^{-2} \mathbb{E}[\hat{\psi}^{(2,m)}(m)]), \quad \sigma = 1, 2, 4.
$$


Proof of Lemma 4. Recall that \( \hat{B}_{j,k} - B_{j,k} = n^{-1} \sum_{t=1}^{n} \eta_{t} \), where \( \eta_{t} = \varphi_{m,k}(X_{t}) \varphi_{m,j}(X_{t}) \), \( t = 1, \ldots, n \). Hence, taking the second moment we obtain

\[
E[\|\hat{B} - B\|^2] \leq M^2 n^{-1} \sum_{t=1}^{n} \|\eta_{t}\|^2 \leq n^{-1} M^2 [2\|p\|_{\infty}\|\varphi\|_{\infty}^2 2^m] = O(n^{-1} 2^m).
\]

For \( l = 2 \), we apply Jensen's inequality

\[
E \left[ n^{-1} \sum_{t=1}^{n} \varphi_{m,k}(X_{t}) \right]^{4} = O\left(n^{-2} \mathbb{E} [\|\eta_{t}\|^4] \right).
\]

Since \( \mathbb{E} [\|\eta_{t}\|^4] \leq 2^{2m} [\|p\|_{\infty}\|\varphi\|_{\infty}]^4 \) and matrix \( B \) is of finite dimension \( M \), (8.1) is valid for \( l = 2 \). In a similar manner we can show that (8.1) holds for \( l = 4 \).

In order to prove (8.2)–(8.4), recall that \( \hat{c}_{k} - c_{k} = n^{-1} \sum_{t=1}^{n} \xi_{t} \), where \( \xi_{t} = u_{m}(X_{t}) \), \( t = 1, \ldots, n \). Thus,

\[
E[\|\hat{c} - c\|^2] \leq Mn^{-1} \mathbb{E} [\|\xi_{1}\|^2] \leq Mn^{-1} \|p\|_{\infty}^2 = O(n^{-1} 2^m)
\]

which proves (8.2). Now, to prove (8.3), observe that

\[
E \left[ n^{-1} \sum_{t=1}^{n} \eta_{t} \right]^{4} = O(n^{-2} \mathbb{E} [\|\varphi\|_{\infty}^2] \|\varphi\|_{\infty}^2 n^{-2} \mathbb{E} [\|\varphi\|_{\infty}^2])
\]

and note that vector \( c \) is of finite dimension \( M \). The proof of (8.4) can be carried out in a similar manner.

Now, let us check validity of (8.5) when condition (5.6) holds. Observe that, under condition (5.6), for \( k = 1, 2, \ldots \) one has

\[
\|\varphi_{m}(X_{t})\|^2 \leq C_{m} 2^k - 2^m - 1, \quad \eta_{t} \leq \mathbb{E} \|\varphi_{m}(X_{t})\|^2 + C_{m} 2^k - 2^m - 1
\]

(8.6)

Plugging (8.6) into (8.2)–(8.4), obtain (8.5).

\[ \square \]

Lemma 5. Let \( B, c, \hat{B} \) and \( \hat{c} \) be defined in (2.6), (2.9), (2.7) and (2.11), respectively, and let \( M \) be the size of the vector \( c \). If \( 2^m \leq (\log n)^{2} \), then, for any \( r > 0 \),

\[
P(\|\hat{B} - B\|^2 \geq M^2 \|B\|^2 2^{m-1} \log n) \leq 2M^2 n^{-r/8}\|p\|_{\infty}^2.
\]

If, in addition, (5.6) holds, then, for any \( r > 0 \),

\[
P(\|\hat{c} - c\|^2 \geq M^2 \|c\|^2 \log n) \leq 2Mn^{-r/8}\|p\|_{\infty}^2.
\]

Proof of Lemma 5. The proof is based on application of Bernstein inequality

\[
P\left( n^{-1} \sum_{t=1}^{n} Y_{t} > z \right) \leq 2 \exp\left( -\frac{n z^2}{2(\sigma^2 + \|Y\|_{\infty} z)} \right).
\]

where \( Y_{t}, \ t = 1, \ldots, n \), are i.i.d. with \( \mathbb{E} Y_{t} = 0, \mathbb{E} Y_{t}^2 = \sigma^2 \) and \( \|Y\|_{\infty} < \infty \)

Recall that \( \hat{B}_{j,k} \) defined by (2.9) are the unbiased estimators of \( B_{j,k} \) defined in (2.6). Denote \( \eta_{t} = \varphi_{m,k}(X_{t}) \varphi_{m,j}(X_{t}) - B_{j,k} \), so that \( \hat{B}_{j,k} - B_{j,k} = n^{-1} \sum_{t=1}^{n} \eta_{t} \), where \( \eta_{t} \) are i.i.d. with \( \mathbb{E} [\eta_{t}] = 0 \) and \( \sigma^2 = \mathbb{E} [\|\eta_{t}\|^2] \leq \|p\|_{\infty}^2 \|\varphi\|_{\infty}^2 \). Also, \( \|\eta_{t}\|_{\infty} \leq 2^m \|\varphi\|_{\infty}^2 \).

Applying Bernstein inequality with \( z = r^{2m/2} \sqrt{\log n/n} \) for every \( k \in K_{m,v} \) and recalling that \( 2^m \log n/n \rightarrow 0 \) as \( n \rightarrow \infty \), we obtain

\[
P(\|\hat{B}_{j,k} - B_{j,k}\| > r^{2m/2} \sqrt{\log n/n}) \leq 2 \exp\left( \frac{-r^2 \log n}{4\|\varphi\|_{\infty}^2 (\|p\|_{\infty}^2 + 2^{m} r^{2m/2} \sqrt{\log n/n})} \right).
\]

Since matrix \( B \) has \( M^2 \) components, (8.7) is valid.

Now, in order to prove (8.8), recall that \( \hat{c}_{k} \) given by (2.11) is an unbiased estimator of \( c_{k} \), so that, for any \( k \), variables \( \xi_{t} = u_{m}(X_{t}) - c_{k} \), \( t = 1, \ldots, n \), are i.i.d. with \( \mathbb{E} \xi_{t} = 0 \) and

\[
\mathbb{E} \xi_{t}^2 \leq \int_{-\infty}^{\infty} u_{m}(x) u_{m}(x) dx \leq \|p\|_{\infty}^2 \|\varphi\|_{\infty}^2.
\]

In addition, \( \|\xi_{t}\|_{\infty} \leq 2 \|u_{m}\|_{\infty} \leq 2 \gamma_{m} 2^{m/2} \). Thus, applying Bernstein inequality with \( z = n^{-1/2} \gamma_{m} \sqrt{\log n} \), we obtain

\[
P(\|\hat{c}_{k} - c_{k}\| > n^{-1/2} \gamma_{m} \sqrt{\log n}) \leq 2 \exp\left( \frac{-r^2 \log n}{2 (\|p\|_{\infty}^2 + 2^{m} r^{2m/2} \sqrt{\log n/n})} \right).
\]

To complete the proof, note that vector \( c \) has \( M \) components. \[ \square \]
Lemma 6. Let \( \Omega_B = \left\{ \omega : \| \hat{B} - B \| > 0.5 \| B^{-1} \|^{-1} \right\} \). Then,
\[
\| \tilde{a}_s - a \| \leq \| \tilde{c} - c \| \| B^{-1} \| + \| c \| | \delta | B^{-1} \|^2 + 2 \| B^{-1} \|^2 \| \hat{B} - B \| + 2 \delta^{-1}(\Omega_B) + \| \tilde{c} - c \| | \delta | B^{-1} \|^2 + 2 \| B^{-1} \|^2 \| \hat{B} - B \| + 2 \delta^{-1}(\Omega_B).
\]
(8.9)

Proof of Lemma 6. Since \( a = B^{-1}c \) and \( \tilde{a}_s = (\hat{B} + \delta \hat{l})^{-1} \hat{c} \), by the properties of the norm, we obtain
\[
\| \tilde{a}_s - a \| \leq \| B^{-1} \| \| \tilde{c} - c \| + \| B^{-1} \| \| \hat{B} - B \| + \| B^{-1} \| \| \hat{c} - c \|. \tag{8.10}
\]
\[
\| B^{-1} \| \| \hat{B} - B \| + \| B^{-1} \| \| \hat{c} - c \| \tag{8.11}
\]
Now, since \( \hat{B}^{-1} - B^{-1} = (\hat{B} - B)B^{-1}, \| B^{-1} \| \leq \delta^{-1}, \| \hat{B}^{-1} \| \leq \delta^{-1} \) and \( \| B^{-1} \| \leq \delta^{-1} \), for the first part of the right-hand side in (8.11), we obtain
\[
\| B^{-1} \| \| \hat{B} - B \| \leq 2 \| B^{-1} \|^2 \| \hat{B} - B \| \ll 2 \| B^{-1} \|^2 \| \hat{B} - B \| + 2 \delta^{-1}(\Omega_B). \tag{8.12}
\]
For the second part of (8.11) we derive \( \| B^{-1} \| \| \hat{B} - B \| \leq \delta^{-1} \| B^{-1} \|^2 \). Finally, combining (8.10)–(8.12), we derive (8.9). \( \square \)

Proof of Lemma 2. Recall that, since the index set \( K_{my} \) is finite, by (3.4), one has
\[
R_{2n} = \mathbb{E} \left( \sum_{k \in K_{my}} (\| \tilde{a}_s - m_k \| \| m_k(y) \|) \right)^2 = O(2^m \mathbb{E}(\| \tilde{a}_s - a \|^2) \tag{8.13}
\]
In order to find an asymptotic upper bound for \( \mathbb{E}(\| \tilde{a}_s - a \|^2) \), use the observed feature in (8.9) and find expectation in a view of Lemmas 5 and 4. Taking into account that it follows from (3.9) and (3.10) that, as \( m \to \infty \), one has \( \| c \| = O(2^{-m/2}) \) and \( B^{-1} = (p(y))^{-1} + O(2^{-m}) \), where \( p(y) \) is a non-asymptotic value, so that, \( \| B^{-1} \| = O(1) \) and \( \| B^{-1} \|^{-1} = O(1) \) as \( m \to \infty \), obtain
\[
\mathbb{E}(\| \tilde{a}_s - a \|^2) = O(\mathbb{E}(\| \hat{c} - c \|^2) + 2^m \| n \|^2 + \mathbb{E}(\| B - \hat{B} \|^2 + 2^m n p(\Omega_B)))
\]
Applying (8.7) with \( \tau = (2\| B^{-1} \|^{-1} M \sqrt{\log n})^{-1} 2^{-m/2} \sqrt{n} \to \infty \), obtain that, as \( n \to \infty \),
\[
\mathbb{P}(\Omega_B) = O(n^{-h}) \quad \text{for any } h > 0,
\]
i.e., \( \mathbb{P}(\Omega_B) \) tends to zero faster than any negative power of \( n \). Then, result of Lemma 2 follows directly from Lemma 4 and the fact that \( 2^m n^{-1} \to 0 \) and \( \| p(\Omega_B) \|^2 2^m = O(n^2) \) as \( n \to \infty \). \( \square \)

Proof of Theorem 2. Our goal is to construct a lower bound for the minimax risk when \( g \in G_r \). In order to construct lower bound \( \Delta_n(y) \) for (3.1), we use Theorem 2.7 of Tsytakov (2008) which we reformulate here for the case of squared risk.

Lemma 7 (Tsytakov, 2008, Theorem 2.7). Assume that \( Z \) contains elements \( \xi_0, \xi_1, \ldots, \xi_n_0, n_0 \geq 1 \), such that

(i) \( d(\xi_s, \xi_0) \geq 2 \xi > 0 \), for \( 0 \leq s < n_0 \leq n_0 \); 
(ii) \( P_t \in P_0, \text{ for } i = 1, \ldots, n_0, \) and 
\[
K(P_t, P_0) \leq C_{n_0},
\]
where \( P_t = P_t, i = 0, 1, \ldots, n_0, \) and \( C_{n_0} \) is a positive constant. Then, for some absolute positive constant \( C_C \), one has
\[
\inf_{\mathcal{Z}} \sup_{\xi} E[\mathcal{d}(\xi, \xi)] \geq C_C. \tag{8.14}
\]

Now, consider \( \Xi = G_r \), \( \mathcal{d}(g, g) = \| f(y) - g(y) \| \), \( n_0 = 1, p_0(x) = p_0(x) + \zeta(h^{-1}(x - y)) \), and \( \Psi_1(x) = \Psi_0(x) + \zeta(h^{-1}) \). Choose \( \zeta = \zeta_0 \min(h, p_0(h)) = \zeta_0 \max(h, p_0(h)) \), where \( \zeta_0 \leq C_p \) is such that \( p_1(\xi_0) \geq 0 \) and both \( p_1(\xi_0) \) and \( \Psi_1(\xi_0) \) are in \( \hat{G}_r \). Let \( t(x) = \Psi_1(\xi_0)/p_0(y), i = 0, 1 \). Calculating the distance \( d(t_1, t_0) \) at the fixed point \( y \), due to \( p_0(y) \geq p_0(y)/2 \), we obtain
\[
d(t_1, t_0) = \frac{\Psi_0(x) + \zeta h^{-1}(x - y)}{p_0(y) + \zeta h^{-1}(x - y)} - \frac{\Psi_0(x)}{p_0(y)} = \zeta \frac{\zeta h^{-1}(x - y)}{p_0(y) + \zeta h^{-1}(x - y)} \geq \frac{\zeta}{2} \frac{\zeta h^{-1}(x - y)}{p_0(y)}.
\]
Hence,
\[
d(t_1, t_0) \geq \begin{cases} C_C & \text{if } |w_{h^{-1}}(y)| \leq C_0, \\ C_C |w_{h^{-1}}(y)| & \text{if } \lim_{h \to 0} |w_{h^{-1}}(y)| = \infty. \end{cases}
\]
so that \( \chi = \Ch^{\max(r_1) - r_2} \). In order to apply Lemma 7, one needs to verify condition (ii). Observe that 
\[
p_0(x_1, \ldots, x_n) = \prod_{i=1}^{n} p_0(x_i), \quad p_1(x_1, \ldots, x_n) = \prod_{i=1}^{n} \left[ p_0(x_i) + \zeta \left( \frac{X_i - \bar{Y}}{h} \right) \right].
\]

Then, due to the fact that \( \log(1 + x) \leq x \), the Kullback divergence between \( p_1 \) and \( p_0 \) is bounded
\[
K(p_1, p_0) = \int \cdots \int \log \left\{ \prod_{i=1}^{n} \frac{p_1(x_i)}{p_0(x_i)} \right\} \prod_{i=1}^{n} p_1(x_i) \, dx_i
\]
\[
= \sum_{i=1}^{n} \int \log \left( \frac{p_0(x) + \zeta k \left( \frac{X_i - \bar{Y}}{h} \right)}{p_0(x)} \right) \left( p_0(x) + \zeta k \left( \frac{X_i - \bar{Y}}{h} \right) \right) \, dx
\]
\[
\leq \sum_{i=1}^{n} \int h p_0(x)^{-1} k \left( \frac{X_i - \bar{Y}}{h} \right) \left( p_0(x) + \zeta k \left( \frac{X_i - \bar{Y}}{h} \right) \right) \, dx
\]
\[
= n \zeta^2 \int p_0(x)^{-1} k \left( \frac{X_i - \bar{Y}}{h} \right) \, dx.
\]

Now, due to (4.4), one has \( n \zeta^2 h = nh^{2 \max(r_1) + 1} \). Therefore, \( h = n^{1/(2 \max(r_1) + 1)} \) and
\[
J^2 = Cn^{-2 \max(r_1) - 2r_2} \frac{1}{(2 \max(r_1) + 1)},
\]
which completes the proof. \( \Box \)

**Proof of Theorem 1.** Validity of Theorem 1 follows directly from Lemmas 1, 2. \( \Box \)

**Lemma 8.** Let \( \sigma^2 \sim n^{-1}2m \) and assumptions (3.15) and (5.6) hold. Then, under assumptions of Lemma 1, as \( n \to \infty \),
\[
E[\tilde{t}_m(y) - t(y)]^4 = O(n^{-2/2m}r_{m} + 1)^2 + 2^{-4m}r_{m}.
\]

**Proof of Lemma 8.**
\[
E[\tilde{t}_m(y) - t(y)]^4 \leq 8 \cdot E[\tilde{t}_m(y) - t_m(y)]^4 + 8|t_m(y) - t(y)|^4.
\]

By Lemma 1, as \( m, n \to \infty \), one has \( |t_m(y) - t(y)|^4 = o(2^{-4m}) \). For the first term in (8.15), note that
\[
E[\tilde{t}_m(y) - t_m(y)]^4 = O(2^mE[\hat{\hat{\sigma}}_m - \hat{\sigma}_m]^4).
\]

Upper bounds for \( E[\hat{\hat{\sigma}}_m - \hat{\sigma}_m]^4 \) can be derived using Lemma 6. Since, as \( m, n \to \infty \), \( \|c\| = O(2^{-m/2}) \) and \( B^{-1} \|a\| = O(1) \), it follows from (8.1) and (8.5) that
\[
E[\hat{\hat{\sigma}}_m - \hat{\sigma}_m]^4 = O((1 + \|c\|)^4 + O(2^{-2m}[\sigma^2 + E\|\hat{\hat{B}} - B\| + \sigma^4 P(\Omega_8)))
\]
\[
+ o(2^{-m}[\sigma^2 + E\|\hat{\hat{B}} - B\| + \sigma^4 E\|\hat{\hat{B}} - B\|^2 + \sigma^4 E\|\hat{\hat{B}} - B\|^2 P(\Omega_8))
\]
\[
= O(n^{-2}(r^2_{m} + 1)) = O(n^{-2}(r^2_{m} + 1)^2),
\]
which completes the proof. \( \Box \)

**Proof of Lemma 3.** Denote \( R^2_{mn} = E[\hat{\hat{\sigma}}_m - \hat{\sigma}_m]^2 \|\hat{\hat{B}} - B\|^2 r^2_{mn} \) and observe that
\[
P(\tilde{t}_m(y) - t_m(y) \geq R_{mn}) \leq P(\tilde{t}_m(y) - t_m(y)) + |t_m(y) - t(y)| \geq 0.5 R_{mn}
\]
\[
+ P(\tilde{t}_m(y) - t_m(y)) + |t_m(y) - t(y)| \geq 0.5 R_{mn}.
\]

Since \( m > m_0 \) and \( R_{mn} \) is an increasing function of \( m \), one has \( |t_m(y) - t(y)| = o(2^{-m/2}) \) as \( m \to \infty \) and \( R_{mn} > R_{m,n} \). Therefore, it is sufficient to show that
\[
P(\tilde{t}_m(y) - t_m(y) \geq 0.5 R_{mn}) = O(n^{-2} \sigma^2 \|\hat{\hat{\sigma}}_m - \hat{\sigma}_m\|)^2 \|\hat{\hat{B}} - B\|^2 / R_{mn} \to 0 \quad \text{as} \quad m, n \to \infty \),
\]
and, consequently,
\[
P(\tilde{t}_m(y) - t_m(y) \leq 2^{-m/2}(\lambda - 1) R_{mn} / (2C_\sigma)) = O(n^{-2}), \quad n \to \infty.
\]

Recall that \( \hat{\hat{B}}_1 - B^{-1} = \hat{\hat{B}}_1 (B^{-1} \hat{\hat{B}}_1) B^{-1} \), so that, for any \( \delta > 0 \), one has
\[
\|\hat{\hat{B}}_1 - B^{-1}\| \leq \|\hat{\hat{B}}_1\|^2 \|\hat{\hat{B}} - B\| + \delta_m + 2\|\hat{\hat{B}}_1\|^2 \|\hat{\hat{B}} - B\| + \delta_m^2
\]
and, also,
\[
\|\hat{\hat{\sigma}}_m - \hat{\sigma}_m\| \leq \|\hat{\hat{B}}_1 - B^{-1}\| \|\hat{\hat{c}} - c\| + \|\hat{\hat{B}}_1 - B^{-1}\| |c|.
\]

Consequently, probability in (8.16) can be partitioned into three terms:
\[
P(\|\hat{\hat{\sigma}}_m - \hat{\sigma}_m\| \leq 2^{-m/2}(\lambda - 1) R_{mn} / (2C_\sigma)) \leq P_1 + P_2 + P_3 \]
(8.17)
Proof of Theorem 3. where
\[ P_1 = \mathbb{P}
\left( \|\hat{\beta}^{-1} \| \geq \frac{\alpha_1 R_{m}(\lambda - 1) 2^{m/2}}{2^{m/2} C_p} \right) \]
\[ P_2 = \mathbb{P}
\left( \|c\| \|\hat{\beta} - B\| + \delta_m \geq \frac{\alpha_2 \sqrt{1 + \frac{\delta_m^2}{n}} \sqrt{\log n} (\lambda - 1)}{2 \sqrt{nC_p}} \right) \]
\[ P_3 = \mathbb{P}
\left( \|\hat{\beta} - B\| \|c\|^2 + \delta_m^2 \geq \frac{\alpha_3 \sqrt{1 + \frac{\delta_m^2}{n}} \sqrt{\log n} (\lambda - 1)}{2 \sqrt{nC_p}} \right) \]
and \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) are positive constants such that \( \alpha_1 + \alpha_2 + \alpha_3 = 1 \).

Applying (8.8) and taking into account that \( \|\hat{\beta}\| \leq 2\|p\|_\infty \), obtain
\[ P_1 \leq \mathbb{P}
\left( \|\hat{\beta}^{-1} - c\|^2 \geq \frac{1 + \frac{\delta_m^2}{n}}{4\|p\|_\infty \sqrt{nC_p}} \right) \leq 2Mn^{-n} \tag{8.18} \]
where \( \alpha_1 = (128MC_p^2\|p\|_\infty^3 M)^{-1} \alpha_2^2 (\lambda - 1)^2 \).

Recalling that \( \|c\| \leq 2\|p\|_\infty^2 \), using formula (8.7) and taking into account that \( 1 - 4M\|p\|_\infty C_p/(\alpha_2(\lambda - 1) > 1 - \epsilon_1 \), for any small positive constant \( \epsilon_1 \) as \( n \to \infty \), we derive
\[ P_2 \leq \mathbb{P}
\left( \|\hat{\beta} - B\| \geq \frac{(\alpha_2 - 1)2^{m/2}}{4M\|p\|_\infty C_p \sqrt{n}} \right) \leq \mathbb{P}
\left( \|\hat{\beta} - B\| \geq \frac{M2^{m/2}}{\sqrt{n}} \right) \leq 2M^2n^{-\epsilon_2} \tag{8.19} \]
where \( \alpha_2 = (128MC_p^4\|p\|_\infty^2\|p\|_\infty^2 C_p^2\|p\|_\infty^2)^{-1} \alpha_2(1 - \alpha_1)(1 - \epsilon_2)^2 \).

In order to find an upper bound for \( P_3 \), recall that \( \|\hat{\beta} - B\| \leq 2\|p\|_\infty \) and \( \|c\| \leq 2\|p\|_\infty^2 \).

Also, note that \( p(y) \approx (\log n)^{-1/2} \), for any fixed \( y \), as \( n \to \infty \).

Therefore, applying (8.7) and taking into account that, due to (5.2) and \( m \leq m_n \), one has \( (2^{m/2} \sqrt{n})^{-1/2} \) for any small positive constant \( \epsilon_2 \), as \( n \to \infty \), derive
\[ P_3 \leq \mathbb{P}
\left( \|\hat{\beta} - B\| \geq \frac{\alpha_3 (\lambda - 1) \sqrt{1 + \frac{\delta_m^2}{n}} \sqrt{\log n}}{2 \sqrt{nC_p}} \right) \leq \mathbb{P}
\left( \|\hat{\beta} - B\| \leq \frac{M2^{m/2}}{\sqrt{n}} \right) \leq 2M^2n^{-\epsilon_1} \tag{8.20} \]
where \( \alpha_3 = (128MC_p^4\|p\|_\infty^2\|p\|_\infty^2 C_p^2\|p\|_\infty^2)^{-1} \alpha_2(1 - \epsilon_2) \).

Now, in order to complete the proof, combine (8.17)–(8.20) and choose \( \alpha_i \), \( i = 1, 2, 3 \), such that \( \alpha_i \geq 2 \) for \( i = 1, 2, 3 \), and \( P_1 + P_2 + P_3 \) takes minimal value. \( \square \)

Proof of Theorem 3. First, let us show that \( E[\|\hat{\beta}_{\text{est}}^{-1}\|^2 + \|\hat{B}_{\text{est}}^{-1}\|^2] = O(1) \) as \( m, n \to \infty \), so that asymptotic relation (5.9) holds. Indeed, for \( m_1 \leq m \leq m_0 \) and any fixed \( y \), one has
\[ \|\hat{\beta}_{\text{est}}^{-1}\| \leq \|\hat{B}_{\text{est}}^{-1} - B_{\text{est}}^{-1}\| + \|B_{\text{est}}^{-1}\| \leq 2\|B_{\text{est}}^{-1}\| \|\hat{B}_{\text{est}} - B_{\text{est}}\| + 2\|B_{\text{est}}^{-1}\| \|\Omega_m\| + \|B_{\text{est}}^{-1}\| \]
where \( \Omega_m \) is defined in Lemma 6. Then,
\[ E[\|\hat{B}_{\text{est}}^{-1}\|^4] = O(\|\hat{B}_{\text{est}} - B_{\text{est}}\|^4 + \|B_{\text{est}}^{-1}\|^4) = O(1), \]
so that both (5.9) and (5.10) are valid.

In order to find an upper bound for \( \Delta_2 \), note that by Lemmas 8 and 3, one has
\[ \Delta_2 = \mathbb{E}[\|\hat{m}(y) - m(y)\|^2 | m > m_0 \] \[ \leq \sum_{l = m_0 + 1}^{m} \sqrt{\mathbb{E}[\|\hat{m}(y) - m(y)\|^2 \|\hat{B}_{\hat{g}}^{-1}\|] + \|\hat{B}_{\text{est}}^{-1}\|^2} \]
so that \( \Delta_2 = O(n^{-1}) = o(2^{m/n_0}) \) as \( n \to \infty \). \( \square \)

References


