Measuring Sparsity in Spatially Interconnected Systems*

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***Abstract*— The goal of this paper is to develop a mathematical framework to measure sparsity of state feedback controllers for spatially interconnected systems. We introduce a new algebra of infinite-dimensional matrices equipped with a matrix quasi-norm which is defined using ℓ^q quasi–norm for $0 < q \leq 1$. When $q = 0$, the value of the matrix quasi-norm is equal to the maximum number of nonzero entries in rows or columns of a matrix. When $0 < q \leq 1$, the proposed matrix algebra forms a mathematical object so called q–Banach algebra, which is not a Banach algebra. We show that this matrix algebra is inverse-closed. Moreover, we prove that the unique solutions of Lyapunov and Riccati equations belong to this matrix algebra. We show that there exists a nonzero q for which the value of the matrix quasi-norm reflects a reasonable estimate for sparsity of a spatially decaying matrix.**

I. INTRODUCTION

In number of important applications, automatic control implementations are still spatially centralized, in the sense that the controller interfaces with the physical system at a fixed and relatively small number of actuators and sensors. However, recent technological advances has opened up new possibilities to change this picture by making the idea of small devices with actuating, sensing, computing, and telecommunications capabilities feasible. Distributing a large array of such devices in a spatial configuration gives unprecedented capabilities for control. This results in distribution of the control variables in space, in addition to the internal states of the underlying system, see [1] and references in there. Important questions that arise are (i) how to design controllers for these systems with regard to global objectives; (ii) how to determine the communication requirements in the controller array; and (iii) how can these control algorithms be implemented in a distributed fashion.

The goal of this paper is to develop some of the fundamental insights and tools that will allow us to exploit architectural properties of the underlying systems to design optimal controllers with sparse information structures. Our primary focus is on an important class of spatially distributed systems, so called *spatially decaying (SD) systems*, for which the corresponding optimal controllers have an inherently *semi-decentralized* architecture, which we refer to as "localized" [1]–[4]. This architecture determines the communication requirements in the controller array. We propose a new methodology that is based on exploiting spatial decay property of the dynamics of the underlying systems. The proposed novel method relates sparsity features of a spatially decaying system to a mathematical object so called *q-Banach algebra*, which is not a Banach algebra.

In this paper, we focus on a special class of q-Banach algebras, the class of spatially decaying matrices. This q-Banach algebra is denoted by $\mathcal{S}_{q,w}(\mathbb{G})$ where $0 < q \leq 1$ and w is a coupling weight function with certain properties, and $\mathbb{G}$ is the underlying spatial domain. The matrix norm on $\mathcal{S}_{q,w}(\mathbb{G})$ is defined using some weighted form of ℓ^q quasi–norm. In order to get a sense on importance of this class of matrix norms, let us assume that $w = 1$. When $q = 0$, the matrix norm on $\mathcal{S}_{q,w}$ gives us exactly the maximum number of nonzero entries in each row or column of a given matrix.

The family $\mathcal{S}_{q,w}(\mathbb{G})$ of infinite dimensional matrices on $\mathbb{G}$ is known as Gröchenig-Schur class. It was introduced by Schur [10] and Gröchenig and Leinert [7] for $q = 1$, by Jaffard [8] for $q = \infty$, and by Sun [11], [12] for $1 \leq q \leq \infty$. We refer the reader to [3], [5], [9], [13] for various families of infinite dimensional matrices and their applications in frame theory, time-frequency analysis, operator algebra, sampling and optimization.

In Section II, it is shown that for the class of sub-exponentially and polynomially decaying matrices, one can compute an estimate for q, where $0 < q \leq 1$, such that the value of the matrix norm on $\mathcal{S}_{q,w}(\mathbb{G})$ provides a reasonable measure to estimate sparsity of a spatially decaying (SD) matrix. In Section III, we introduce $\mathcal{S}_{q,w}(\mathbb{G})$ and verify several algebraic properties for this family of matrices. We show that $\mathcal{S}_{q,w}(\mathbb{G})$ is not a Banach algebra, but it is an inverse-closed subalgebra of $\mathcal{B}(\ell^2)$. This result enables us later in Section IV to introduce spatially distributed systems on $\mathcal{S}_{q,w}(\mathbb{G})$ and show that the unique solutions of Lyapunov equations and algebraic Riccati equations belong to $\mathcal{S}_{q,w}(\mathbb{G})$. These results are novel as our methodologies do not require $\mathcal{S}_{q,w}(\mathbb{G})$ to be a Banach algebra.

II. SPARSITY MEASURE FOR SD MATRICES

In this section, we introduce a class of sparsity measures for spatially decaying matrices. We consider the set of all matrices $A = [a(i,j)]_{i,j\in\mathbb{G}}$ which is equipped with the following matrix quasi-norm

$$\|A\|^q_{\mathcal{S}_{q,1}} := \max\left\{\sup_{i\in\mathbb{G}} \sum_{j\in\mathbb{G}} |a(i,j)|^q,\ \sup_{j\in\mathbb{G}} \sum_{i\in\mathbb{G}} |a(i,j)|^q\right\}$$

for some $0 < q \leq 1$, where $\mathbb{G}$ is the underlying spatial domain. We focus on two families of matrices which appear in most real-world applications:

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(i) Sub-exponentially decaying matrices

$$|a(i,j)| \ \le\ Ce^{-\alpha|i-j|^{\beta}};$$

(ii) Polynomially decaying matrices

$$|a(i,j)| \ \le\ C(1+|i-j|)^{-\alpha},$$

where $\alpha > 0$ and $\beta \in (0,1)$. In order to understand sparsity measures for SD matrices, we consider spatial truncations of a SD matrix. For a given matrix A and truncation length T, we represent the truncated matrix by A_T which can be obtained by setting

$$a(i,j) = 0 \qquad \text{if} \qquad |i-j| \ge T.$$

For the class of SD matrices, one can associate a truncation threshold to T in the following sense: if $|a(i,j)| \ \le\ \epsilon(T)$, then replace $a(i,j)$ with zero.

In theory, the small values of q leads to better estimate of sparsity for SD matrices. This is particularly true for SD matrices with slow decaying rates, e.g., polynomially decaying matrices. However, for matrices with fast decaying rates, such as sub-exponentially decaying matrices, larger values of q could also result in reasonable estimate for sparsity.

Our goal is to show that the value of matrix quasi-norm $\|A\|^q_{\mathcal{S}_{q,1}}$ is a reasonable measure for sparsity of a SD matrix if q is chosen appropriately. In this section, we only focus on sub-exponential decay. Our discussion can be extended to polynomial decay as well. Let us consider the class of random matrices A for which the entries are defined by

$$a(i,j) \ =\ r_{ij}\, e^{-\alpha|i-j|^{\beta}}, \tag{1}$$

where r_{ij} are drawn from the normal distribution $\mathcal{N}(0,\sigma^2)$. We define a quantity to study asymptotic behavior of our proposed sparsity measure. For a given truncation length T for matrix A, we introduce the following quantity

$$\varphi_A(\delta,T) \ =\ \frac{1}{2T-1}\,\mathbf{E}\big[\|A_T\|^q_{\mathcal{S}_{q,1}}\big]$$

where $\mathbf{E}$ is the expectation operation. For a given truncation length T, we compute a truncation threshold through

$$\epsilon(T) = e^{-\alpha T^{\beta}}. \tag{2}$$

In order to reach to the acceptable error range δ, we select q to satisfy

$$\epsilon^q = 1-\delta, \tag{3}$$

and its value is given by

$$q(T) = \frac{\ln(1-\delta)^{-1}}{\alpha}\ T^{-\beta}. \tag{4}$$

Proposition 2.1: Suppose that A is a random matrix defined by (1) and $0<\delta<1$. If we assume that $\delta\times 100$ indicates the acceptable percentage error for estimating sparsity of the band matrix A_T, then

$$\lim_{T\to\infty}\ \varphi_A(\delta,T) \ \ge\ 1-\delta \tag{5}$$

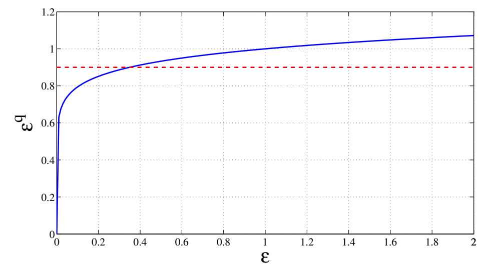

Fig. 1: By a smart selection of parameter $0 < q \le 1$, one can filter out entries of a matrix which are less than a given threshold ϵ. This figure shows that $\epsilon^q \approx 0.9$ for $q = 0.1$ and $\epsilon \approx 0.3$.

where $q := q(T)$ is chosen as in (4).

Proof: In the proof, we simply use q instead of $q(T)$ in (4). In order to calculate the expected value of the matrix norm, we use the expected value of the central absolute moments of each entry which is given by

$$\mathbf{E}[|r_{ij}|^q] \ =\ \sigma^q\ \frac{2^{\frac{q}{2}}\ \Gamma(\frac{q+1}{2})}{\sqrt{\pi}}. \tag{6}$$

From the norm definition, we have

$$\|A_T\|^q_{\mathcal{S}_{q,1}} \ \ge\ \sum_{|i-j|<T} |a(i,j)|^q \ge e^{-\alpha q T^{\beta}} \sum_{|i-j|<T} |r_{ij}|^q,$$

for every $i\in\mathbb{G}$. Let us fix i and take expectations from both sides of the inequality, it follows that

$$\mathbf{E}[\|A_T\|^q_{\mathcal{S}_{q,1}}] \ \ge\ e^{-\alpha q T^{\beta}} \sum_{|i-j|<T} \mathbf{E}[|r_{ij}|^q].$$

From (2), (3), and (6), we have

$$\mathbf{E}[\|A_T\|^q_{\mathcal{S}_{q,1}}] \ \ge\ (1-\delta)\ (2T-1)\ \sigma^q\ \frac{2^{\frac{q}{2}}\ \Gamma(\frac{q+1}{2})}{\sqrt{\pi}}$$

We emphasize that q depends on T and is given by (4). Therefore,

$$\lim_{T\to\infty} q(T) = 0.$$

Moreover,

$$\lim_{T\to\infty}\ \sigma^q\ \frac{2^{\frac{q}{2}}\ \Gamma(\frac{q+1}{2})}{\sqrt{\pi}} = \lim_{q\to 0}\ \sigma^q\ \frac{2^{\frac{q}{2}}\ \Gamma(\frac{q+1}{2})}{\sqrt{\pi}} = 1.$$

From this result, we can conclude that

$$\lim_{T\to\infty}\ \frac{1}{2T-1}\,\mathbf{E}[\|A_T\|^q_{\mathcal{S}_{q,1}}] \ \ge\ (1-\delta).$$

■

III. q-Banach Algebras of Infinite Matrices

In this section, we will briefly introduce an important class of matrices, denoted by $\mathcal{S}_{q,w}$, and their properties which are essential for development of a general theory to study sparsity in spatially distributed systems. This class of matrices forms a mathematical object so called q-Banach Algebra [6], which is not a Banach algebra, but it is an inverse-closed subalgebra of $\mathcal{B}(\ell^2)$. In the following, we

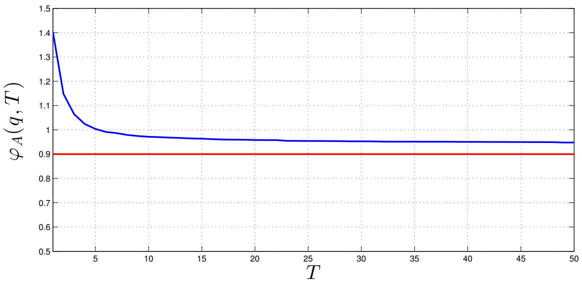

Fig. 2: This curves depicts asymptotic behavior of $\phi_A(T)$ as a function of T for a 100×100 sub-exponentially decaying matrix defined by (1) with parameters $\alpha = 0.6, \beta = 0.8$, and $\sigma = 3$. For $\delta = 0.1$, it shows that matrix norm $\|A_T\|_{\mathcal{S}_{q,1}}^q$ represents sparsity of matrix A_T with at most 10% error for all $2 \le T \le 50$. It turns out that q is a strictly decreasing function of T. For this example, every $0 < q \le 0.08$ results in at most 10% error in measuring sparsity of matrix A_T for all $2 \le T \le 50$.

will define $\mathcal{S}_{q,w}$ and examine its properties. In order to characterize how fast/slow entries of a SD matrix decays to zero, we use weight functions.

Definition 3.1: A weight function w on $\mathbb{G} \times \mathbb{G}$ satisfies

(i) $w(i,j) \ge 1$ for all $i, j \in \mathbb{G}$,
(ii) $w(i,j) = w(j,i)$ for all $i, j \in \mathbb{G}$,
(iii) $\sup_{i \in \mathbb{G}} w(i,i) < \infty$.

We can interpret $w(i,j)$ as communication cost between subsystems i and j. The property (i) implies that there is a minimal cost for such a communication, after normalization, we assume that the minimal cost is one unit. The property (ii) indicates that the communication cost is symmetric and independent of communication direction, i.e., the cost from subsystem i to subsystem j is the same as the communication cost from subsystem j to subsystem i. The property (iii) implies that the operation cost for subsystem i is uniformly bounded, in other words, there are finite self-loop for all subsystems.

Definition 3.2: Let $0 < q \le 1$ and w be a weight function on $\mathbb{G} \times \mathbb{G}$. We define the class of spatially decaying infinite matrices on $\mathbb{G}$ by

$$\mathcal{S}_{q,w} = \Big\{ A = (a(i,j))_{i,j \in \mathbb{G}} : \ \|A\|_{\mathcal{S}_{q,w}} < \infty \Big\}, \qquad (7)$$

where

$$\|A\|_{\mathcal{S}_{q,w}} := \max \Big\{ \sup_{i \in \mathbb{G}} \Big(\sum_{j \in \mathbb{G}} |a(i,j)|^q w(i,j) \Big)^{1/q}, \ \sup_{j \in \mathbb{G}} \Big(\sum_{i \in \mathbb{G}} |a(i,j)|^q w(i,j) \Big)^{1/q} \Big\}. \qquad (8)$$

This class of spatially decaying matrices is not a Banach algebra as its quasi-norm does not satisfy the triangular inequality. In the following, we verify some of the basic properties of this class.

Proposition 3.3: Suppose that $0 < q \le 1$ and w is a weight function on $\mathbb{G} \times \mathbb{G}$. The algebra $(\mathcal{S}_{q,w}, \|\cdot\|_{\mathcal{S}_{q,w}})$ satisfies the following properties:

(i) If $A \in \mathcal{S}_{q,w}$, then $cA \in \mathcal{S}_{q,w}$ for all $c \in \mathbb{R}$. Moreover

$$\|cA\|_{\mathcal{S}_{q,w}} = |c| \|A\|_{\mathcal{S}_{q,w}} \qquad (9)$$

for all $c \in \mathbb{R}$ and $A \in \mathcal{S}_{q,w}$.

(ii) If $A \in \mathcal{S}_{q,w}$, then A is a bounded operator on ℓ^2. Moreover, its operator norm is dominated by its $\mathcal{S}_{q,w}$ norm,

$$\|A\|_{\mathcal{B}(\ell^2)} \le \|A\|_{\mathcal{S}_{q,w}} \qquad (10)$$

for all $A \in \mathcal{S}_{q,w}$.

(iii) If $A, B \in \mathcal{S}_{q,w}$, then $A + B \in \mathcal{S}_{q,w}$. Moreover

$$\|A+B\|_{\mathcal{S}_{q,w}}^q \le \|A\|_{\mathcal{S}_{q,w}}^q + \|B\|_{\mathcal{S}_{q,w}}^q \qquad (11)$$

for all $A, B \in \mathcal{S}_{q,w}$.

(iv) If there exists a positive number K such that

$$w(i,j) \le K w(i,k) w(k,j) \quad \text{for all } i,j,k \in \mathbb{G}, \qquad (12)$$

then

$$\|AB\|_{\mathcal{S}_{q,w}}^q \le K \|A\|_{\mathcal{S}_{q,w}}^q \|B\|_{\mathcal{S}_{q,w}}^q \qquad (13)$$

for all $A, B \in \mathcal{S}_{q,w}$.

Proof: Due to space limitations, we eliminate the proof. ■

We observe that $\mathcal{S}_{q,w}$ is a sub-algebra of $\mathcal{B}(\ell^2)$, but it is not a Banach sub-algebra of $\mathcal{B}(\ell^2)$. Nonetheless, we show that $\mathcal{S}_{q,w}$ is inverse-closed. Before stating this result, we need to make the following assumption on weight functions.

Assumption 3.4: We assume that there exist a companion weight u, a positive exponent $\theta \in (0,1)$, and a positive constant D such that

$$w(i,j) \le w(i,k)u(k,j) + u(i,k)w(k,j) \qquad (14)$$

for all $i, j, k \in \mathbb{G}$, and

$$\sup_{i \in \mathbb{G}} \inf_{\tau \ge 0} \Big(\Big(\sum_{\rho(i,j) < \tau, j \in \mathbb{G}} |u(i,j)|^{2/(2-q)} \Big)^{1-q/2} + t \sup_{\rho(i,j) \ge \tau, j \in \mathbb{G}} \frac{u(i,j)}{w(i,j)} \Big) \le D t^{\theta}, \qquad (15)$$

where $t \ge 1$ and ρ is a nonnegative symmetric function on $\mathbb{G} \times \mathbb{G}$.

A weight w satisfying (12) is known to be *submultiplicative* [7], [12], [13]. One may verify that a weight satisfying (14) and (15) is submultiplicative with $K \le 2D$ [12], [13].

Example 3.5: In this example, we discuss the weight assumptions (14) and (15). First, consider the polynomial weight function

$$w_\alpha(i,j) = (1 + |i-j|)^{\alpha}, \qquad \alpha > 0.$$

For the polynomial weight w_α, we have

$$w_\alpha(i,j) \le 2^{\alpha}(w_\alpha(i,k) + w_\alpha(k,j)). \qquad (16)$$

The reason is that at least one of $|i-k|$ and $|k-j|$ is larger than or equal to $|i-j|/2$, which implies that $(1+|i-j|) \le \max(1+2|i-k|, 1+2|k-j|) \le 2\max(1+|i-k|, 1+|k-j|)$.

Thus, we may select the companion weight $u_\alpha(i,j) = 2^\alpha$. It is straightforward to verify that u_α is a weight function. Moreover, it satisfies (15) with $D = 2^{\alpha+2}$ and $\theta = (1 - q/2)/(\alpha + 1 - q/2)$. This is true because

$$\begin{aligned}
&\inf_{\tau\geq 0}\Big(\sum_{|i-j|<\tau}(u_\alpha(i,j))^{2/(2-q)}\Big)^{1-q/2} + t\sup_{|i-j|\geq\tau}\frac{u(i,j)}{w(i,j)}\\
&\leq\ 2^\alpha\inf_{\tau\geq 0}\Big(\Big(\sum_{|i-j|<\tau}1\Big)^{1-q/2} + 2^\alpha t(1+\tau)^{-\alpha}\Big)\\
&\leq\ 2^\alpha\inf_{\tau\geq 0}\Big((1+2\tau)^{1-q/2} + t(1+\tau)^{-\alpha}\Big)\\
&\leq\ 2^{\alpha+1}\inf_{\tau\geq 0}\Big((1+\tau)^{1-q/2} + t(1+\tau)^{-\alpha}\Big)\\
&\leq\ 2^{\alpha+2}t^{(1-q/2)/(\alpha+1-q/2)}.
\end{aligned}$$

The above argument shows that the polynomial weight function $w_\alpha(i,j) = (1 + |i-j|)^\alpha, \alpha > 0$, satisfies the requirement for the weight w in Assumption 3.4.

Next, let us consider the sub-exponential weight function $e_{D,\delta}(i,j) = e^{D|i-j|^\delta}$ where $D > 0$ and $\delta \in (0,1)$. We recall that $(1+t)^\delta \leq 1 + (2^\delta - 1)t^\delta$ for all $t \in [0,1]$ as both sides are equal when $t = 0$ and $t = 1$ and derivative of the function $(1+t)^\delta - 1 - (2^\delta - 1)t^\delta$ has only one zero $(2^\delta - 1)^{1/(1-\delta)}/\big(1 - (2^\delta - 1)^{1/(1-\delta)}\big)$. Therefore,

$$e_{D,\delta}(i,j) \leq e_{D,\delta}(i,k)e_{D(2^\delta-1),\delta}(k,j) + e_{D,\delta}(i,k)e_{D(2^\delta-1),\delta}(k,j)$$

for all $i,j,k \in \mathbb{Z}$. This holds as either $|i-k| \geq |k-j|$ or $|i-k| \leq |k-j|$. Therefore, we may select $e_{D(2^\delta-1),\delta}$ as the companion weight. The weight $e_{D(2^\delta-1),\delta}$ satisfies properties of Definition 3.1 and inequality (14). Now, let us verify (15) by taking $\theta \in (2^\delta - 1, 1)$,

$$\begin{aligned}
&\inf_{\tau\geq 0}\Big\{\Big(\sum_{|i-j|<\tau}e_{D(2^\delta-1),\delta}(i-j))^{2/(2-q)}\Big)^{1-q/2} + t\sup_{|i-j|\geq 2}e_{D(2^\delta-2),\delta}(i-j)\Big\}\\
&\leq\ \inf_{\tau\geq 0}\Big\{e^{D(2^\delta-1)\tau^\delta}\big(\sum_{|i-j|<\tau}1\big)^{1-q/2} + t\ e^{D(2^\delta-2)\tau^\delta}\Big\}\\
&\leq\ \inf_{\tau\geq 0}\Big\{e^{D(2^\delta-1)\tau^\delta}(1+2\tau)^{1-q/2} + t\ e^{D(2^\delta-2)\tau^\delta}\Big\}\\
&\leq\ t^{2^\delta-1}\big((1+2(\ln t/D)^{1/\delta})^{1-q/2}+1\big).
\end{aligned}$$

This proves that the sub-exponential weight function also satisfies conditions of Assumption 3.4.

The following result shows that if a matrix in $\mathcal{S}_{q,w}$ is invertible with an inverse in $\mathcal{B}(\ell^2)$, then the inverse also belongs to $\mathcal{S}_{q,w}$. This result is particularly important to us as it enables us to prove that the unique solution of the algebraic Riccati equation also belongs to $\mathcal{S}_{q,w}$.

Theorem 3.6: Let $0 < q \leq 1$ and w be a weight function satisfying (14) and (15). For any $A \in \mathcal{S}_{q,w}$ with $A^{-1} \in \mathcal{B}(\ell^2)$, we have that $A^{-1} \in \mathcal{S}_{q,w}$.

Proof: Due to space limitations, we eliminate the proof. ■

IV. Spatially Distributed Systems Defined on q-Banach Algebras

The introduction of algebra $\mathcal{S}_{q,w}$ automatically leads to the introduction of the class of spatially decaying systems over $\mathcal{S}_{q,w}$. Consider a linear system with constant matrices:

$$\dot{x} = Ax + Bu \quad (17)$$

$$y = Cx + Du \quad (18)$$

where $x, u, y \in L^2([0,\infty);\ell^2)$.

Definition 4.1: The linear system (17)-(18) is called spatially decaying if $A, B, C, D \in \mathcal{S}_{q,w}$ for some $0 < q \leq 1$ and weight function w on $\mathbb{G}\times\mathbb{G}$ satisfying (14) and (15).

Definition 4.2: We say that A is *exponentially stable* if

$$\|e^{tA}\|_{\mathcal{B}(\ell^2)} \leq Ee^{-\alpha t} \quad \text{for all } t \geq 0. \quad (19)$$

The following result shows that the unique positive definite solution to the Lyapunov equation is in $\mathcal{S}_{q,w}$.

Theorem 4.3: Suppose that $q \in (0,1]$ and w is a weight satisfying (14) and (15). If $Q \in \mathcal{S}_{q,w}$ is a strictly positive definite operator on ℓ^2, and $A \in \mathcal{S}_{q,w}$ is exponentially stable on ℓ^2, then the unique strictly positive definite solution P of the Lyapunov equation

$$AP + PA^T + Q = 0. \quad (20)$$

belongs to $\mathcal{S}_{q,w}$.

In order to prove Theorem 4.3, we need the following result about exponential stability for infinite matrices in $\mathcal{S}_{q,w}$.

Lemma 4.4: Let $0 < q \leq 1$ and w be a weight satisfying (14) and (15). If $A \in \mathcal{S}_{q,w}$ satisfies

$$\|e^{tA}\|_{\mathcal{B}(\ell^2)} \leq Ee^{-\alpha t} \quad \text{for all } t \geq 0, \quad (21)$$

where $E, \alpha > 0$, then for any $\beta < \alpha$, there exists a positive constant E_β such that

$$\|e^{tA}\|_{\mathcal{S}_{q,w}} \leq E_\beta e^{-\beta t} \quad \text{for all } t \geq 0. \quad (22)$$

Proof: Due to space limitations, we eliminate the proof. ■

Applying this result, we proceed to prove Theorem 4.3.

Proof: [Proof of Theorem 4.3] From our assumptions, $A \in \mathcal{S}_{q,w}$ satisfies the exponential stability condition

$$\|e^{tA}\|_{\mathcal{B}(\ell^2)} \leq Ee^{-\alpha t}, \quad t \geq 0 \quad (23)$$

where $E, \alpha > 0$. By taking $t_0 > 0$ and $\beta \in (0,\alpha)$, we define

$$Q_{t_0} = \sum_{n,m=0}^{\infty}\frac{t_0^{m+n+1}}{(m+n+1)m!n!}(A^T)^m QA^n. \quad (24)$$

Then

$$\begin{aligned}
\|Q_{t_0}\|^q_{\mathcal{S}_{q,w}} &\leq \sum_{n,m=0}^{\infty}\Big(\frac{t_0^{m+n+1}}{(m+n+1)m!n!}\Big)^q\|(A^T)^m QA^n\|^q_{\mathcal{S}_{q,w}}\\
&\leq\ \sum_{n,m=0}^{\infty}\Big(\frac{t_0^{m+n+1}}{(m+n+1)m!n!}\Big)^q(2D)^{m+n}\times\\
&\qquad \|A\|^{(m+n)q}_{\mathcal{S}_{q,w}}\|Q\|^q_{\mathcal{S}_{q,w}}
\end{aligned}$$

$$\leq \|Q\|_{\mathcal{S}_{q,w}}^q t_0^q \Big(\sum_{m=0}^{\infty} \frac{(2D)^m (t_0\|A\|_{\mathcal{S}_{q,w}})^{qm}}{(m!)^q}\Big)^2 < \infty,$$

where the second inequality follows from (13). Therefore Q_{t_0} belongs to $\mathcal{S}_{q,w}$ and also is a bounded operator on ℓ^2. Moreover, Q_{t_0} is positive definite as Q is positive definite and

$$\begin{aligned}\langle Q_{t_0}x, x\rangle &= \sum_{m,n=0}^{\infty} \frac{t_0^{m+n+1}}{(m+n+1)m!n!} x^T (A^T)^m Q A^n x \\ &= \int_0^{t_0} x^T e^{tA^T} Q e^{tA} x dt > 0 \qquad (25)\end{aligned}$$

for any $0 \neq x \in \ell^2$.

Define $P_n, n \geq 0$, iteratively by

$$P_0 = Q_0, \qquad (26)$$

and

$$P_n = e^{t_0 A^T} P_{n-1} e^{t_0 A} + Q_0 \quad \text{for } n \geq 1. \qquad (27)$$

By induction, we can show that $P_n \geq Q_0, n \geq 0$, are positive definite by (25), and

$$P_n = \sum_{k=0}^{n} e^{kt_0 A^T} Q_0 e^{kt_0 A}, \quad n \geq 0. \qquad (28)$$

By Lemma 4.4, there exists a positive constant E_β for any $\beta < \alpha$ such that

$$\|e^{tA}\|_{\mathcal{S}_{q,w}} \leq E_\beta e^{-\beta t} \quad \text{for all } t \geq 0. \qquad (29)$$

This implies that $P_n, n \geq 0$, belong to $\mathcal{S}_{q,w}$, and $P_n, n \geq 1$, converges as

$$\begin{aligned}\sum_{k=1}^{\infty} \|e^{kt_0 A^T} Q_0 e^{kt_0 A}\|_{\mathcal{S}_{q,w}}^q &\leq 4D^2 E_\beta^2 \|Q_0\|_{\mathcal{S}_{q,w}}^q \sum_{k=1}^{\infty} e^{-2\beta q t_0 k} \\ &< \infty. \qquad (30)\end{aligned}$$

Now, we define

$$P_\infty = \lim_{n\to\infty} P_n. \qquad (31)$$

The above limit converges in the $\mathcal{S}_{q,w}$ norm and then also in the operator norm on ℓ^2 by (28) and (30). Then P_∞ is strictly positive definite and belongs in $\mathcal{S}_{q,w}$. Moreover, we have

$$\begin{aligned}& A^T P_\infty + P_\infty A + Q \\ &= \lim_{n\to\infty} A^T \Big(\sum_{k=0}^{n} A^T e^{kt_0 A^T} \Big(\int_0^{t_0} e^{t_0 A^T} Q e^{tA} dt\Big) e^{kt_0 A}\Big) \\ &\quad + \Big(\sum_{k=0}^{n} e^{kt_0 A^T} \Big(\int_0^{t_0} e^{t_0 A^T} Q e^{tA} dt\Big) e^{kt_0 A}\Big) A + Q \\ &= \lim_{n\to\infty} A^T \Big(\int_0^{(n+1)t_0} e^{tA^T} Q e^{tA} dt\Big) \\ &\quad + \Big(\int_0^{(n+1)t_0} e^{t_0 A^T} Q e^{tA} dt\Big) A + Q \\ &= \lim_{n\to\infty} \int_0^{(n+1)t_0} \frac{d}{dt}(e^{tA^T} Q e^{tA}) dt + Q \\ &= \lim_{n\to\infty} e^{(n+1)t_0 A^T} Q e^{(n+1)t_0 A} = 0 \qquad (32)\end{aligned}$$

where the last limit holds as

$$\|e^{(n+1)t_0 A}\|_{\mathcal{B}(\ell^2)}^q \leq E e^{-(n+1)t_0\alpha} \to 0 \quad \text{as } n \to \infty$$

by exponential stability of the matrix A. Therefore, P_∞ is the desired strictly positive definite matrix in $\mathcal{S}_{q,w}$. ■

Theorem 4.5: Suppose that $q \in (0, 1]$ and w is a weight function satisfying (14) and (15), $A, R, Q \in \mathcal{S}_{q,w}$, R strictly positive definite and Q positive definite matrices on ℓ^2. If $(A, Q^{\frac{1}{2}})$ is exponentially detectable, then there exists a unique strictly positive definite solution $X \in \mathcal{B}(\ell^2)$ to the algebraic Riccati equation

$$A^* X + XA - XRX + Q = 0 \qquad (33)$$

and $A_X = A - RX$ is exponentially stable on ℓ^2. Moreover, $X \in \mathcal{S}_{q,w}$.

Proof: The results on ℓ^2 are well-known [15]. Consider the algebra of all 2×2 matrices with entries from $\mathcal{S}_{q,w}$ which is denoted by $\mathcal{M}_2(\mathcal{S}_{q,w})$. the corresponding Hamiltonian matrix

$$H = \begin{bmatrix} A & -R \\ -Q & -A^* \end{bmatrix}$$

Thus, it follows that spectrum $\sigma(A_X)$ is contained in the open left half complex plane. Suppose that Ω is a Cauchy domain contained in the open left half complex plane such that $\sigma(A_X) \subset \Omega$, and Γ be the boundary of the Cauchy domain Ω. Then $\lambda I - A_X$ is invertible and has bounded inverse in $\mathcal{B}(\ell^2)$ for all $\gamma \in \Gamma$. Moreover its inverse $(\lambda I - A_X)^{-1}$ is continuous and bounded on $\lambda \in \Gamma$,

$$\sup_{\lambda\in\Gamma} \|(\lambda I - A_X)^{-1}\|_{\mathcal{B}(\ell^2)} < \infty. \qquad (34)$$

Recall the spectrum of A_X^* is also contained in the open left half complex plane and λ is contained in the left half complex plane for any $\gamma \in \Gamma$. Thus $\lambda I + A_X^*$ has its spectrum contained in the open left half complex plane for any $\gamma \in \Gamma$. Thus $\lambda I + A_X^*$ is invertible and its inverse $(\lambda I + A_X^*)^{-1}$ is continuous and bounded on Γ,

$$\sup_{\lambda\in\Gamma} \|(\lambda I + A_X^*)^{-1}\|_{\mathcal{B}(\ell^2)} < \infty. \qquad (35)$$

By direct computation, we have that

$$\lambda I_2 - H = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} \lambda I - A_X & R \\ 0 & \lambda I + A_X^* \end{bmatrix} \begin{bmatrix} I & 0 \\ X & I \end{bmatrix}^{-1}$$

where $I_2 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ is the unit matrix in $\mathcal{M}_2(\mathcal{S}_{q,w})$. This together with the continuity and boundedness of $(\lambda I - A_X)^{-1}$ and $(\lambda I + A_X^*)^{-1}$ implies that

$$E := \frac{1}{2\pi i} \int_\Gamma (\lambda I_2 - H)^{-1} d\lambda \qquad (36)$$

belongs to $\mathcal{B}(\ell^2)$. A careful verification indicates that

$$(\lambda I_2 - H)^{-1} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \times$$
$$\begin{bmatrix} (\lambda I - A_X)^{-1} & -(\lambda I - A_X)^{-1} R (\lambda I + A_X^*)^{-1} \\ 0 & (\lambda I + A_X^*)^{-1} \end{bmatrix} \times$$
$$\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}^{-1}. \tag{37}$$

Hence, $(\lambda I_2 - H)^{-1}$ is analytic on a neighborhood of the set Γ. Applying functional calculus implies that

$$\begin{aligned} E &= \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} I & Z \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ X & I \end{bmatrix}^{-1} \\ &= \begin{bmatrix} I - ZX & Z \\ X(I - ZX) & XZ \end{bmatrix} \end{aligned} \tag{38}$$

for some operator $Z \in \mathcal{B}(\ell^2)$. As $HE = EH$, we see that Z satisfies the Lyapunov equation

$$A_X Z + Z A_X^* + R = 0. \tag{39}$$

As A_X has its spectrum on the open half complex plane and R is strictly positive definitive, Z is uniquely determined by the above equation. According to Theorem 4.3, we have

$$Z \quad \text{is strictly positive definitive.} \tag{40}$$

By the assumption on R, Q and A, both $\lambda I_2 - H = \begin{bmatrix} \lambda I - A & R \\ Q & \lambda I + A^* \end{bmatrix}$ belong to $\mathcal{M}_2(\mathcal{S}_{q,w})$ for any $\lambda \in \Gamma$. Recall from (34), (36) and (35) that $\lambda I_2 - H$ is invertible on $\mathcal{B}(\ell^2)$. This together with the strict positivity of R gives that $\lambda I_2 - H$ has inverse in the algebra $\mathcal{M}_2(\mathcal{S}_{q,w})$. Also we see that $(\lambda I_2 - H)^{-1}$ is analytic about $\lambda \in \Gamma$, and hence $E \in \mathcal{S}_{q,w}$. This together with (38) implies that both Z and XZ belong to $\mathcal{S}_{q,w}$. On the other hand, as shown in (40), Z is strictly positive definite solution of a Lyapunov equation (39), then $Z^{-1} \in \mathcal{S}_{q,w}$ by the inverse-closedness of $\mathcal{S}_{q,w}$ in $\mathcal{B}(\ell^2)$ obtained in Theorem 3.6. Hence $X = (XZ)Z^{-1} \in \mathcal{S}_{q,w}$. This proves the desired conclusion. ■

Remark 4.6: In the proof of Theorem 4.5, we use some of the ideas of [14]. However, we highlight that $\mathcal{S}_{q,w}$ is not a Banach algebra. Therefore, we have taken some additional steps to show that the unique solution of the Riccati equation belongs to $\mathcal{S}_{q,w}$. Our proof only uses algebraic and inverse-closedness properties of $\mathcal{S}_{q,w}$. A recent result in [16], which considers the special case $q = 1$, also uses properties of Banach algebras. Our results does not require the underlying algebra to be a Banach algebra.

V. Conclusions

We introduce a useful class of matrix algebras so called q-Banach algebras and focus on a special class of such matrix algebras which is equipped with a quasi-norm. It is shown that the quasi-norm plays an important role in measuring sparsity in spatially distributed systems. Our next goal is to find reasonable estimates for parameter $0 < q \le 1$ such that the value of the matrix quasi-norm reflects a reasonable estimate for sparsity of the unique solutions of Lyapunov and Riccati equations for spatially decaying systems.

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