Affine Frame Decompositions and Shift-Invariant Spaces

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Applied and Computational Harmonic Analysis, 20(2006), 74–107.

In this paper, we show that the property of tight affine frame decomposition of functions in L^2 can be extended in a stable way to functions in Sobolev spaces when the generators of the tight affine frames satisfy certain mild regularity and vanishing moment conditions. Applying the affine frame operators Q_j on j-th levels to any function f in a Sobolev space reveals the detailed information $Q_j f$ of f in such tight affine decompositions. We also study certain basic properties of the range of the affine frame operators Q_j such as the topological property of closedness and the notion of angles between the ranges for different levels, and thus establishing some interesting connection to (tight) frames of shift-invariant spaces.

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1. INTRODUCTION

The Sobolev spaces $H^s := H^s(\mathbf{R}), s \in \mathbf{R}$, are often used for representing functions f in many applications. Since these are not sequence spaces, to transmit (store or analyze) $f \in H^s$ by using some 'finite' device, we may have to rely on a normalized tight frame $\{e_\lambda, \lambda \in \Lambda\}$ of the Hilbert space $L^2 := H^0$; that is,

$$f = \sum_{\lambda \in \Lambda} \langle f, e_\lambda \rangle e_\lambda,$$

where the coefficient sequence $\{\langle f, e_\lambda \rangle\}$ constitutes the tight frame decomposition of f. Hence, the transmission (storage or analysis) of the function f reduces to that of this sequence of coefficients. Furthermore, we may want to consider a finite representation of f, if we choose an appropriate finite set $\Lambda' \subset \Lambda$ and quantizations a_λ of $\langle f, e_\lambda \rangle$ specified by certain allowable bit depths, so that $\tilde{f} := \sum_{\lambda \in \Lambda'} a_\lambda e_\lambda$ is a good approximation of f.

[1] Supported by NSF Grant #CCR-0098331 and ARO Grant #W911NF-04-1-0298. This author is also with the Department of Statistics, Stanford University, Stanford, CA 94305. Email: ckchui@standford.edu

To be more specific, let us use a fixed integer $M \geq 2$ as the dilation factor, and consider a wavelet system $\mathcal{F} := \{\psi_{j,k}\}_{\psi\in\Psi, j,k\in\mathbf{Z}}$ that is an orthonormal basis of L^2 generated by some wavelet family Ψ, where, as usual, $\psi_{j,k} := M^{j/2}\psi(M^j\cdot -k)$. Then the orthonormal wavelet system can be used to decompose functions in L^2. Moreover, the sequence of coefficients $\{\langle f, \psi_{j,k}\rangle\}_{\psi\in\Psi,j,k\in\mathbf{Z}}$ in the wavelet decomposition $f = \sum_{\psi\in\Psi}\sum_{j,k\in\mathbf{Z}}\langle f,\psi_{j,k}\rangle\psi_{j,k}$ of an L^2 function gives the time-scale detailed information of f. In particular, under certain very mild assumption on the regularity, order of vanishing moment, and decay at infinity of the wavelets in Ψ, the wavelet system $\mathcal{F}$ can also be used for stable decomposition of functions in Sobolev spaces [20, 33]. As an example, for $M = 2$, the Haar wavelet function H,

$$H(x) = \begin{cases} 1 & \text{for } x \in [0, 1/2), \\ -1 & \text{for } x \in [1/2, 1), \\ 0 & \text{otherwise}, \end{cases}$$

belongs to the Sobolev space $H^\beta, \beta < 1/2$ but not $H^{1/2}$, and has compact support and vanishing moment of order one, while any function f in the Sobolev space $H^\alpha, \alpha \in (-1/2, 1/2)$, has a stable wavelet decomposition $f = \sum_{j,k\in\mathbf{Z}}\langle f, H_{j,k}\rangle H_{j,k}$, namely,

$$A\|f\|_{2,\alpha} \leq \Big(\sum_{j,k\in\mathbf{Z}}(1+2^{2j})^\alpha|\langle f, H_{j,k}\rangle|^2\Big)^{1/2} \leq B\|f\|_{2,\alpha},$$

for some positive constants A, B, where $\|\cdot\|_{2,\alpha}$ is the usual Sobolev norm. Compactly supported orthonormal wavelets with dilation M, and arbitrarily high regularity and order of vanishing moments have been constructed in the literature, with the pioneer work of Daubechies [14] (see the other literature [6, 15, 32, 33, 37]), but all of the known examples with the exception of the above Haar wavelet, do not have explicit analytic formulation expression. Unfortunately, in many applications, it is highly desirable to use wavelets within a certain class of analytically representable functions.

Polynomial splines on a uniform mesh are piecewise polynomials, have explicit analytical formulations, and hence, are the most natural candidates. But if the property of compact support is required, shifts and dilations of such spline generators, other than the Haar example as discussed above, do not form an orthonormal basis of L^2. When allowing redundancy (such as relaxing from an orthonormal basis to a tight frame), compactly supported tight frames generated by splines on uniform meshes can be explicitly constructed by using more than one generators (see [7, 8, 9, 16, 34, 36]). A natural question then is to ask if, analogous to orthogonal wavelet decomposition, the affine frame system associated with splines can be used to decompose functions in a Sobolev space in a stable way. We will give an affirmative answer to this question in this paper (see Theorem 3.1 and Corollary 3.3).

Recall that a finite collection Ψ of L^2-functions is said to generate a *tight affine frame* of L^2, (or, for convenience, Ψ is said to be a tight frame of L^2), if $\mathcal{F} := \{\psi_{j,k}\}_{\psi\in\Psi,j,k\in\mathbf{Z}}$ is a tight frame of L^2, which we will assume, without loss of generality, to be normalized with frame bound constant equal to 1. The *affine frame operator Q_j on j-th level*, $j \in \mathbf{Z}$, of such a tight affine frame is defined by

$$Q_j f = \sum_{\psi\in\Psi}\sum_{k\in\mathbf{Z}}\langle f, \psi_{j,k}\rangle\psi_{j,k}, \ f \in L^2. \tag{1}$$

Hence, it follows from the tight frame representation

$$f = \sum_{\psi\in\Psi} \sum_{j,k\in\mathbf{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k}, \ f \in L^2, \tag{2}$$

that the identity operator I on L^2 can be written as the sum of affine frame operators Q_j, namely:

$$I = \sum_{j\in\mathbf{Z}} Q_j = \cdots + Q_{-1} + Q_0 + Q_1 + \cdots. \tag{3}$$

In this paper, we show that the sum in the above operator decomposition converges strongly in Sobolev spaces, an analytic property of the affine frame operators Q_j, when the tight affine frame generators in Ψ satisfy some mild regularity and vanishing moment conditions (see Theorem 3.1).

By the operator decomposition (3) of the identity operator on L^2, we have the following decomposition of the space L^2,

$$L^2 = \sum_{j\in\mathbf{Z}} W_j = \cdots + W_{-1} + W_0 + W_1 + \cdots,$$

where $W_j = Q_j L^2, j \in \mathbf{Z}$. Clearly, if the system $\mathcal{F} := \{\psi_{j,k}\}_{\psi\in\Psi,j,k\in\mathbf{Z}}$ generated by dilation and shifts of functions in Ψ is an orthonormal system of L^2, then $W_j, j \in \mathbf{Z}$, are the wavelet spaces, and hence are closed in L^2 and mutually orthogonal. These properties of space decompositions are no longer valid in general, when the wavelet decomposition is replaced by the affine frame decomposition. In this paper, we characterize the closedness of the space $Q_j H^\alpha$, a topological property for the affine frame operators Q_j, and study the angle between different $Q_j H^\alpha$, a geometrical property for the affine frame operators Q_j. Loosely speaking, we show that there are three possible geometrical structures associated with the affine frame operators Q_j: (i) The angles between different $Q_j H^\alpha, j \in \mathbf{Z}$, are always zero (or equivalently $Q_0 H^\alpha$ is not closed in L^2, or equivalently $\{\psi(\cdot - k) : \psi \in \Psi, k \in \mathbf{Z}\}$ is not a frame), see Theorems 4.4 and 5.1; (ii) The angles between different $Q_j H^\alpha, j \in \mathbf{Z}$, are always $\pi/2$ (or equivalently both $Q_0 H^\alpha$ and $\tilde{P}_0 H^\alpha$ are closed in L^2, or equivalently $\{\psi(\cdot - k) : \psi \in \Psi, k \in \mathbf{Z}\}$ is a tight frame), see Theorems 4.1 and 5.2; (iii) The angles between different $Q_j H^\alpha, j \in \mathbf{Z}$, are always in the open interval $(0, \pi/2)$ (or equivalently $Q_0 H^\alpha$ is closed in L^2 but $\tilde{P}_0 H^\alpha$ is not closed in L^2, or equivalently $\{\psi(\cdot - k) : \psi \in \Psi, k \in \mathbf{Z}\}$ is a frame but not a tight frame), see Theorems 4.1 and 5.1. For the second case, the frame decomposition $f = \sum_{j\in\mathbf{Z}} Q_j f$ is equivalent to an orthonormal wavelet decomposition, in the sense that Q_j is a projection operator from L^2 to the wavelet spaces W_j, the orthogonal complement of V_j in V_{j+1}, see Theorem 4.1. For the third case, the asymptotic behaviour of the angles between spaces $Q_0 H^\alpha$ and $Q_j H^\alpha$ is related to the Sobolev exponent of the scaling function ϕ, see Theorems 5.3 and 5.4.

The paper is organized as follows. In Section 2, we recall some preliminary results on multiresolution analysis (or MRA) of L^2, tight affine frames associated with an MRA, and frames of a finitely generated shift-invariant space. In Section 3, we establish the property of stable homogeneous, nonhomogeneous and finite decomposition of functions in a Sobolev space (see Theorems 3.1, 3.5 and 3.6). From Theorem 3.1, we conclude that for a finite family Ψ of L^2-functions, if it generates a tight affine frame of L^2, and if, in addition, it satisfies certain mild regularity and vanishing moment conditions, then the corresponding affine frame decomposition is stable in the Sobolev spaces. In Section 4, we study closedness of the shift-invariant spaces $\tilde{P}_0 H^\alpha$ and $Q_j H^\alpha$ in L^2, a topologoical property for the affine frame operators Q_j (see Theorem 4.1), and discuss some interesting connections to other shift-invariant spaces generated by Ψ and the (tight) frame properties of Ψ (see Theorem 4.4). In Section 5, we study the angles θ_j between $\tilde{P}_0 H^\alpha$ and $Q_j H^\alpha, j \ge 0$, a geometric property for the affine frame operators Q_j (see Theorems 5.1, 5.2, 5.3 and 5.4 for details).

2. PRELIMINARIES

Let us first recall the definition of Sobolev spaces and some basic theory of multiresolution analyses (MRA) with dilation M, tight affine frames associated with an MRA, and frames of a finitely generated shift-invariant space.

2.1. Sobolev spaces

For $\alpha \in \mathbf{R}$, let $\mathcal{J}_\alpha$ denote the Bessel potential operator, defined by $\widehat{\mathcal{J}_\alpha f} = (1+|\cdot|^2)^{\alpha/2}\widehat{f}$. Then the Sobolev space H^α, with norm $\|\cdot\|_{2,\alpha}$, is defined by

$$H^\alpha = \left\{ f: \ \|f\|_{2,\alpha} := \left\|\mathcal{J}_\alpha f\|_2 < \infty \right\}.$$

2.2. Multiresolution analyses and scaling functions

A *multiresolution analysis (MRA) with dilation* M is a sequence of closed subspaces $\{V_j\}_{j\in\mathbf{Z}}$ of L^2 such that the following conditions hold: (i) $V_j \subset V_{j+1}$; (ii) $\cup_{j\in\mathbf{Z}} V_j$ is dense in L^2; (iii) $\cap_{j\in\mathbf{Z}} V_j = \{0\}$; (iv) $f \in V_j$ if and only if $f(M\cdot) \in V_{j+1}$; and (v) there exists a compactly supported L^2-function ϕ such that $\{\phi(\cdot - k) : \ k \in \mathbf{Z}\}$ is a Riesz basis of V_0 (see for example [6, 15, 32, 33, 37]). The function ϕ in (v) is called a *scaling function* of the MRA $\{V_j\}_{j\in\mathbf{Z}}$. For an MRA with a compactly supported scaling function, there always exists another compactly supported scaling function f with linear independent shifts (see for instance [25, 37]), meaning that the semi-convolution $f*' : \{d(j)\}_{j\in\mathbf{Z}} \longmapsto \sum_{j\in\mathbf{Z}} d(j) f(\cdot - j)$ is one-to-one on the space of all sequences on $\mathbf{Z}$. Hence, in this paper, the scaling function of an MRA is always assumed to have compact support and linear independent shifts instead of global support and stable shifts (Riesz basis property), as considered in the classical wavelet literatures [6, 15, 32, 33].

Let ϕ be a compactly supported scaling function with linear independent shifts. Since $V_0 \subset V_1$, and ϕ has compact support and linear independent shifts, it follows that

$$\phi = \sum_{j\in\mathbf{Z}} c_0(j)\phi(M\cdot - j), \tag{4}$$

for some finitely supported sequence $c_0 := \{c_0(j)\}_{j\in\mathbf{Z}}$ on $\mathbf{Z}$. Throughout this paper, the Fourier transform $\widehat{f}$ of an integrable function f is given by $\widehat{f}(\xi) = \int_{\mathbf{R}} f(x)e^{-ix\xi}dx$. Taking the Fourier transform of both sides of the refinement equation (4) yields

$$\widehat{\phi}(M\xi) = H_0(\xi)\widehat{\phi}(\xi), \tag{5}$$

where the function H_0, known as the *(two-scale) symbol* of the scaling function ϕ, is defined by

$$H_0(\xi) = \frac{1}{M} \sum_{j \in \mathbf{Z}} c_0(j) e^{-ij\xi}. \tag{6}$$

2.3. Tight affine frames associated with an MRA

We say that a finite collection Ψ of compactly supported L^2-functions generates a *tight affine frame associated with an MRA* $\{V_j\}_{j\in\mathbf{Z}}$ if $\Psi \subset V_1$ and it generates a tight affine frame. Let ϕ be a compactly supported scaling function of the MRA $\{V_j\}_{j\in\mathbf{Z}}$ that has linear independent shifts. Then any function $\psi \in \Psi$ is in the algebraic span of $\phi(M \cdot -k), k \in \mathbf{Z}$, which yields

$$\widehat{\psi}(M\xi) = H_\psi(\xi)\widehat{\phi}(\xi), \tag{7}$$

in the Fourier domain, where $H_\psi(\xi), \psi \in \Psi$, are trigonometric polynomials. The tight frame property of Ψ is characterized via the symbol H_0 of the scaling function ϕ in (6) and the functions $H_\psi, \psi \in \Psi$, in (7) (see [8, 9, 16, 36]).

PROPOSITION 2.1. *Let $\{V_j\}_{j\in\mathbf{Z}}$ be an MRA with compactly supported scaling function ϕ that has linear independent shifts. Let Ψ be a finite collection of compactly supported L^2-functions given by (7). Then Ψ is a tight affine frame if and only if there exists a trigonometric polynomial $S(\xi)$ which satisfies (i) $S(0) \neq 0$; (ii) $S(\xi) \geq 0$ for all $\xi \in \mathbf{R}$; (iii) $S(\xi) = S(-\xi)$ for all $\xi \in \mathbf{R}$; and (iv) for all $m = 0, \ldots, M-1$,*

$$S(M\xi)H_0(\xi)\overline{H_0\Big(\xi + \frac{2m\pi}{M}\Big)} + \sum_{\psi\in\Psi} H_\psi(\xi)\overline{H_\psi\Big(\xi + \frac{2m\pi}{M}\Big)} = \delta_{m0}S(\xi), \tag{8}$$

where H_0 is the symbol of the scaling function ϕ, and $H_\psi, \psi \in \Psi$, are given in (7).

By (8), we have

$$S(\xi) = S(M\xi)|H_0(\xi)|^2 + \sum_{\psi\in\Psi} |H_\psi(\xi)|^2. \tag{9}$$

By applying this formula iteratively, we have

$$S(\xi) = S(M^n\xi) \prod_{j=0}^{n} |H_0(M^j\xi)|^2 + \sum_{j=0}^{n} \Big(\prod_{i=0}^{j-1} |H_0(M^i\xi)|^2\Big)\Big(\sum_{\psi\in\Psi} |H_\psi(M^j\xi)|^2\Big).$$

Hence, taking the limit and using the fact that $\prod_{j=0}^{n} |H(M^j\xi)|^2 \to 0$ as $n \to \infty$ for all $\xi \notin 2\pi\mathbf{Z}$, (which follows from the assumptions that ϕ is compactly supported, has linear independent shifts, and belongs to L^2), we obtain

$$S(\xi) = \sum_{j=0}^{\infty} \Big(\prod_{i=0}^{j-1} |H_0(M^i\xi)|^2\Big) \times \Big(\sum_{\psi\in\Psi} |H_\psi(M^j\xi)|^2\Big). \tag{10}$$

So the function $S(\xi)$, called *vanishing moment recovery (VMR) function* in [8, 9], in Proposition 2.1 is the same as the *fundamental function* Θ *of resolution* of the tight affine frame Ψ in [16, 36].

Multiplying $\hat f(\xi+2k\pi)\overline{\hat\phi(\xi+2k\pi)}\hat\phi(\xi)$ to both sides of (10), and applying (5) and (7), yields

$$\begin{aligned}&\hat f(\xi+2k\pi)\overline{\widehat{\phi}(\xi+2k\pi)}S(\xi)\widehat{\phi}(\xi)\\ =&\sum_{j=1}^{\infty}\sum_{\psi\in\Psi}\hat f(\xi+2k\pi)\overline{\widehat{\psi}(M^j(\xi+2k\pi))}\widehat{\psi}(M^j\xi),\quad k\in\mathbf{Z}.\end{aligned}$$

Then summing over $k\in\mathbf{Z}$ and taking the inverse Fourier transform, we may conclude that

$$\tilde P_0 f=\sum_{j<0}Q_j f,$$

where the operators $\tilde P_j, j\in\mathbf{Z}$, are defined by

$$\tilde P_j f=\sum_{k\in\mathbf{Z}}\langle f,\phi_{j,k}\rangle\tilde\phi_{j,k}=\sum_{k\in\mathbf{Z}}\langle f,\tilde\phi_{j,k}\rangle\phi_{j,k}, \tag{11}$$

and the function $\tilde\phi$ in V_0 is given by $\widehat{\tilde\phi}(\xi)=S(\xi)\widehat{\phi}(\xi)$. By (11) and the dilation invariance of frame operators at different levels, we have

$$\tilde P_j=\sum_{k<j}Q_k,\quad j\in\mathbf{Z}, \tag{12}$$

and

$$Q_j=\tilde P_{j+1}-\tilde P_j,\quad j\in\mathbf{Z}. \tag{13}$$

2.4. Frames of a finitely generated shift-invariant space

For a finite collection Ψ of compactly supported L^2 functions, we define the shift-invariant space $V^2(\Psi)$ by

$$V^2(\Psi)=\Big\{\sum_{\psi\in\Psi}\sum_{k\in\mathbf{Z}}c_\psi(k)\psi(\cdot-k):\ (c_\psi(k))_{k\in\mathbf{Z}}\in\ell^2\quad\text{for any }\psi\in\Psi\Big\}. \tag{14}$$

Here, ℓ^2 denotes, as usual, the space of all square summable sequences on $\mathbf{Z}$. We also use $V^2(\psi_1,\ldots,\psi_N)$ to denote $V^2(\Psi)$ when $\Psi=\{\psi_1,\ldots,\psi_N\}$, and say that Ψ is a *frame of the shift-invariant space* $V^2(\Psi)$ if there exist two positive constants A and B such that

$$A\|f\|_2^2\le\sum_{\psi\in\Psi}\sum_{k\in\mathbf{Z}}|\langle f,\psi(\cdot-k)\rangle|^2\le B\|f\|_2^2,\quad f\in V^2(\Psi).$$

If $A=B$, then we say that Ψ is a *tight frame of the shift-invariant space* $V^2(\Psi)$. Furthermore, if $A=B=1$, the tight frame is said to be normalized.

The (tight) frame for a finitely generated shift-invariant space is characterized in the Fourier domain in [2, 3].

PROPOSITION 2.2. *Let $\psi_1, \ldots, \psi_N$ be compactly supported L^2-functions, and set $\Psi = \{\psi_1, \ldots, \psi_N\}$. Then*

(i) Ψ is a frame of the shift-invariant space $V^2(\Psi)$ if and only if $V^2(\Psi)$ is a closed linear subspace of L^2, which, in turn, is equivalent to the property that the rank of the $N \times \mathbf{Z}$ matrix $(\widehat{\psi}_n(\xi + 2k\pi))_{1\le n\le N, k\in\mathbf{Z}}$ is independent of $\xi \in \mathbf{R}$;

(ii) Ψ is a tight frame of the shift-invariant space $V^2(\Psi)$ if and only if the matrix

$$B(\xi) := \Big(\sum_{k\in\mathbf{Z}} \widehat{\psi}_n(\xi + 2k\pi)\overline{\widehat{\psi}_{n'}(\xi + 2k\pi)}\Big)_{1\le n,n'\le N}$$

satisfies

$$B(\xi)^2 = C_0 B(\xi),\ \xi \in \mathbf{R}, \tag{15}$$

for some positive constant C_0.

3. STABLE AFFINE FRAME DECOMPOSITION IN SOBOLEV SPACES

For the tight affine frame generated by a finite collection Ψ of L^2-functions, the following stable frame decomposition property holds for any $f \in L^2$:

$$f = \sum_{\psi\in\Psi} \sum_{j,k\in\mathbf{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k},$$

while the convergence is unconditional in L^2. The above frame decomposition can be extended to functions in a Sobolev space when the tight affine frame Ψ satisfies some mild regularity and vanishing moment conditions.

THEOREM 3.1. *Let $\beta > 0, \alpha \in (-\beta, \beta)$, and let Ψ be a finite collection of L^2- functions that generate a tight affine frame of L^2, such that any function $\psi \in \Psi$ satisfies the regularity condition:*

$$\sum_{k\in\mathbf{Z}} |\widehat{\psi}(\xi + 2k\pi)|^2 (1 + |\xi + 2k\pi|^2)^\beta \le C_\beta,\ \xi \in \mathbf{R}, \tag{16}$$

as well as the vanishing moment condition:

$$|\widehat{\psi}(\xi)| \le C_\beta |\xi|^\beta \quad \text{as} \quad \xi \to 0. \tag{17}$$

Then the affine frame decomposition

$$f = \sum_{\psi\in\Psi} \sum_{j,k\in\mathbf{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k} = \sum_{j\in\mathbf{Z}} Q_j f, \quad f \in H^\alpha, \tag{18}$$

holds, where the convergence is unconditional in H^α. Furthermore, there exists a positive constant C, independent of $f \in H^\alpha$, such that

$$C^{-1}\|f\|_{2,\alpha}^2 \ \le\ \sum_{j\in\mathbf{Z}} M^{2\alpha j_+} \langle Q_j f, f\rangle \le C\|f\|_{2,\alpha}^2, \tag{19}$$

$$C^{-1}\|f\|_{2,\alpha}^2 \le \sum_{j\in\mathbf{Z}} M^{2\alpha j_+}\|Q_j f\|_2^2 \le C\|f\|_{2,\alpha}^2, \tag{20}$$

and

$$C^{-1}\|f\|_{2,\alpha}^2 \le \sum_{j\in\mathbf{Z}} \|Q_j f\|_{2,\alpha}^2 \le C\|f\|_{2,\alpha}^2, \tag{21}$$

where j_+ *stands for* $\max(j,0)$.

REMARK 3.1. For a finite family Ψ of L^2-functions, we say that Ψ has *stable shifts* if

$$A\sum_{\psi\in\Psi}\sum_{k\in\mathbf{Z}}|c_\psi(k)|^2 \le \Big\|\sum_{\psi\in\Psi}\sum_{k\in\mathbf{Z}} c_\psi(k)\psi(\cdot-k)\Big\|_2^2 \le B\sum_{\psi\in\Psi}\sum_{k\in\mathbf{Z}}|c_\psi(k)|^2$$

holds for all sequences $\{c_\psi(k)\}_{k\in\mathbf{Z}}\in\ell^2, \psi\in\Psi$. Observe that if Ψ has stable shifts, then

$$A\langle Q_j f, f\rangle \le \|Q_j f\|_2^2 \le B\langle Q_j f, f\rangle, \quad f\in H^\alpha$$

for the same positive constants A, B independent of $j\in\mathbf{Z}$. Thus, the middle terms in the estimates in (19) and (20) are equivalent to each other. On the other hand, as we will discuss later, tight affine frames Ψ do not have stable shifts in general (see Theorem 4.4 for details). To the best of our knowledge, the estimate in (21) is new even for $\alpha=0$, when the stable shift assumption of Ψ is dropped.

For $\beta\ge 0$, we say that $\psi\in \mathrm{Lip}\,\beta$ if $D^\gamma\psi, 0\le\gamma\le\beta_0$, are continuous, and

$$|D^{\beta_0}\psi(x) - D^{\beta_0}\psi(y)| \le C|x-y|^{\beta-\beta_0},\ x,y\in\mathbf{R},$$

where β_0 is the largest integer strictly less than β, and C is a positive constant. We denote the class of all compactly supported functions in $\mathrm{Lip}\,\beta$ by $\mathrm{Lip}_0\,\beta$. The Sobolev exponent $s_2(f)$ of an L^2-function f is defined by

$$s_2(f) := \sup\{\beta : f \text{ satisfies } (16)\}$$

and the Hölder exponent $\alpha_\infty(f)$ of a continuous function f by

$$\alpha_\infty(f) := \sup\{\beta : f\in \mathrm{Lip}\,\beta\}.$$

For example, for the m^{th} order cardinal B-spline N_m, we have $s_2(N_m)=m+1/2$ and $\alpha_\infty(N_m)=m$, and hence

$$s_2(N_m) < s_2(N_m).$$

In general, we have the following result on the Hölder exponent and Sobolev exponent of a compactly supported continuous function.

PROPOSITION 3.2. *Let ψ be a compactly supported continuous function. Then its Hölder exponent $\alpha_\infty(\psi)$ and Sobolev exponent $s_2(\psi)$ satisfy*

$$\alpha_\infty(\psi) \le s_2(\psi). \tag{22}$$

REMARK 3.2. The estimates in (19) and (20) are known when the regularity condition (16) for Ψ is replaced by $\Psi \subset \mathrm{Lip}_0\, \beta$ (see [18, 19, 20] and the references therein). In that particular case, $\{\psi_{j,k}\}_{\psi\in\Psi, j,k\in\mathbf{Z}}$ constitutes the so-called atoms of the Sobolev space H^α, as well as atoms of some Triebel-Lizorkin spaces and Besov spaces. In view of Proposition 3.2, the assertion in Theorem 3.1 generalizes this result of the frame decomposition of functions in Sobolev spaces.

REMARK 3.3. For a scaling function ϕ, it is easier to verify $\phi \in H^\beta$ than $\phi \in \mathrm{Lip}_0\, \beta$. In particular, the question of whether or not a scaling function ϕ belongs to H^β reduces to finding the maximum norms of all eigenvalues of a finite matrix generated explicitly by the symbol of the scaling function ϕ (see for instance, [17, 27, 38]). So the regularity condition (16) for the tight frame Ψ is easier to be justified than $\Psi \in \mathrm{Lip}_0\, \beta$, when Ψ is compactly supported and is associated with some MRA, while most of known tight frames satisfy those two conditions. For any compactly supported function ψ, the Sobolev exponent $s_2(\psi)$ is usually larger than the Hölder exponent $\alpha_\infty(\psi)$. So functions in a Sobolev space H^α, where $\min_{\psi\in\Psi} \alpha_\infty(\psi) \le \alpha < \min_{\psi\in\Psi} s_2(\psi)$, have stable affine frame decomposition by Theorem 3.1. In particular, for spline frames, an application of Theorem 3.1 gives the following optimal result.

COROLLARY 3.3. *Let N_m be the m^{th} order cardinal B-spline, and Ψ be a finite family of compactly supported functions defined by*

$$\widehat{\psi}(M\xi) = H_\psi(\xi)\widehat{N}_m(\xi)$$

for some trigonometric polynomials H_ψ that satisfy

$$|H_\psi(\xi)| \le C|\xi|^{m+1} \quad \text{as} \quad \xi \to 0.$$

Let $\alpha \in (-m-1/2, m+1/2)$. Then if Ψ is a tight frame of L^2, any function $f \in H^\alpha$ has a stable frame decomposition of the form (18) and the coefficients in the frame decomposition satisfies the estimates in (19), (20) and (21).

REMARK 3.4. For a tight frame Ψ of L^2, the frame decomposition has minimal energy in the sense that the energy $E := \sum_{\psi\in\Psi}\sum_{j,k\in\mathbf{Z}} |a_{\psi;j}(k)|^2$ of a decomposition $f = \sum_{\psi\in\Psi}\sum_{j,k\in\mathbf{Z}} a_{\psi;j}(k)\psi_{j,k}$ is minimum for the frame decomposition, that is,

$$\sum_{\psi\in\Psi}\sum_{j,k\in\mathbf{Z}} |\langle f, \psi_{j,k}\rangle|^2 \le \sum_{\psi\in\Psi}\sum_{j,k\in\mathbf{Z}} |a_{\psi;j}(k)|^2,$$

(see [7]). A similar but weaker result can be established for frame decomposition of functions in Sobolev spaces, as follows.

COROLLARY 3.4. *Let β, α, Ψ be as in Theorem 3.1. Then the frame decomposition has quasi-minimal energy in Sobolev space H^α in the sense that there exists a positive constant C, which depends only on α, β and Ψ, such that if $f \in H^\alpha$ has a decomposition $f = \sum_{\psi\in\Psi}\sum_{j,k\in\mathbf{Z}} a_{\psi;j}(k)\psi_{j,k}$ with finite energy $\sum_{\psi\in\Psi}\sum_{j,k\in\mathbf{Z}} M^{2j+\alpha}|a_{\psi;j}(k)|^2$ in the Sobolev space H^α, then*

$$\sum_{\psi\in\Psi}\sum_{j,k\in\mathbf{Z}} M^{2j+\alpha}|\langle f, \psi_{j,k}\rangle|^2 \le C\sum_{\psi\in\Psi}\sum_{j,k\in\mathbf{Z}} M^{2j+\alpha}|a_{\psi;j}(k)|^2.$$

The assumptions in Theorem 3.1 that the tight affine frame Ψ is compactly supported and is associated with an MRA, can be removed. However, under these assumptions, in addition to the property of homogeneous frame decomposition (18), functions in a Sobolev space have nonhomogeneous frame decomposition (23) and finite frame decomposition (28) as well.

THEOREM 3.5. *Let $\beta > 0, \alpha \in (-\beta, \beta)$, and let $\phi \in H^\beta$ be a compactly supported scaling function of an MRA $\{V_j\}_{j\in\mathbf{Z}}$ that has linear independent shifts. Assume that $\Psi \subset V_1$ is a finite collection of compactly supported L^2 functions, which generate a tight affine frame of L^2, and that any function $\psi \in \Psi$ satisfies the vanishing moment condition (17). Let $\tilde{P}_0$ and $\tilde{\phi}$ be defined as in (11). Then the nonhomogeneous frame decomposition*

$$\begin{aligned} f &= \tilde{P}_0 f + \sum_{j=0}^{\infty} Q_j f \qquad (23)\\ &:= \sum_{k\in\mathbf{Z}}\langle f, \tilde{\phi}_{0,k}\rangle\phi_{0,k} + \sum_{j=0}^{\infty}\sum_{\psi\in\Psi}\sum_{k\in\mathbf{Z}}\langle f, \psi_{j,k}\rangle\psi_{j,k}, \quad f\in H^\alpha, \end{aligned}$$

holds, where the convergence is unconditional in H^α. Furthermore, there exists a positive constant C such that

$$\begin{aligned} C^{-1}\|f\|_{2,\alpha} &\le \Big(\sum_{k\in\mathbf{Z}}|\langle f, \tilde{\phi}_{0,k}\rangle|^2\Big)^{1/2} + \Big(\sum_{j=0}^{\infty}\sum_{\psi\in\Psi}\sum_{k\in\mathbf{Z}} M^{2j\alpha}|\langle f, \psi_{j,k}\rangle|^2\Big)^{1/2}\\ &= \Big(\sum_{k\in\mathbf{Z}}|\langle f, \tilde{\phi}_{0,k}\rangle|^2\Big)^{1/2} + \Big(\sum_{j=0}^{\infty} M^{2j\alpha}\langle Q_j f, f\rangle\Big)^{1/2}\\ &\le C\|f\|_{2,\alpha}, \qquad (24) \end{aligned}$$

$$C^{-1}\|f\|_{2,\alpha} \le \|\tilde{P}_0 f\|_2 + \Big(\sum_{j=0}^{\infty} M^{2j\alpha}\|Q_j f\|_2^2\Big)^{1/2} \le C\|f\|_{2,\alpha}, \qquad (25)$$

and

$$C^{-1}\|f\|_{2,\alpha} \le \|\tilde{P}_0 f\|_{2,\alpha} + \Big(\sum_{j=0}^{\infty}\|Q_j f\|_{2,\alpha}^2\Big)^{1/2} \le C\|f\|_{2,\alpha},\ f\in H^\alpha. \qquad (26)$$

REMARK 3.5. If $\tilde{P}_j$ are projectors, i.e., $\tilde{P}_j^2 = \tilde{P}_j$, then the estimate (25) follows from inequalities of Bernstein and Jackson type. We refer to [5, 11, 13] for a detailed presentation of such a mechanism. Note that if $\tilde{P}_j$ are projectors, then $Q_j = \tilde{P}_{j+1} - \tilde{P}_j$ are also projectors, and this implies that both $\tilde{P}_j L^2$ and $Q_j L^2$ are closed subspaces of L^2. Thus, the scaling function ϕ of the corresponding MRA $\{V_j\}_{j\in\mathbf{Z}}$ has orthonormal shifts by Theorem 4.1, $Q_j L^2$ is the orthogonal complement of V_j in V_{j+1}, and Q_j are projectors on the wavelet spaces obtained from the MRA $\{V_j\}_{j\in\mathbf{Z}}$. As a consequence, if $\tilde{P}_j$ are projectors, then the frame decomposition (23) becomes essentially the usual orthonormal wavelet decomposition.

For the tight affine frame Ψ associated with an MRA $\{V_j\}_{j\in\mathbf{Z}}$, we have the following result on finite frame decomposition with uniform stability in Sobolev space norm.

THEOREM 3.6. *Let $\beta, \alpha, \phi, \Psi, S(\xi), \tilde{P}_j$ be as in Theorem 3.5. In addition, assume that the function $S(\xi)$ in (10) associated with the affine tight frame Ψ satisfies*

$$S(\xi) \neq 0 \quad \forall\, \xi \in \mathbf{R}. \tag{27}$$

Then any function $f \in V_L, L \geq 1$, has the following finite frame decomposition,

$$\begin{aligned} f &= \tilde{P}_0 h_L + Q_0 h_L + \cdots + Q_{L-1} h_L \\ &= \sum_{k\in\mathbf{Z}} \langle h_L, \tilde{\phi}_{0,k}\rangle \phi_{0,k} + \sum_{j=0}^{L-1} \sum_{\psi\in\Psi} \sum_{k\in\mathbf{Z}} \langle h_L, \psi_{j,k}\rangle \psi_{j,k}, \end{aligned} \tag{28}$$

where $h_L = (\tilde{P}_L)^{-1} f \in V_L$. Furthermore, there exists a positive constant C independent of $L \geq 1$ and $f \in V_L$, so that

$$\begin{aligned} C^{-1}\|f\|_{2,\alpha} &\leq \Big(\sum_{k\in\mathbf{Z}} |\langle h_L, \tilde{\phi}_{0,k}\rangle|^2\Big)^{1/2} \\ &\quad + \Big(\sum_{j=0}^{L-1} M^{2j\alpha} \langle Q_j h_L, h_L\rangle\Big)^{1/2} \leq C\|f\|_{2,\alpha}; \end{aligned} \tag{29}$$

$$C^{-1}\|f\|_{2,\alpha} \leq \|\tilde{P}_0 h_L\|_2 + \Big(\sum_{j=0}^{L-1} M^{2j\alpha} \|Q_j h_L\|_2^2\Big)^{1/2} \leq C\|f\|_{2,\alpha}; \tag{30}$$

and

$$C^{-1}\|f\|_{2,\alpha} \leq \|\tilde{P}_0 h_L\|_{2,\alpha} + \Big(\sum_{j=0}^{L-1} \|Q_j h_L\|_{2,\alpha}^2\Big)^{1/2} \leq C\|f\|_{2,\alpha}. \tag{31}$$

REMARK 3.6. The multiscale techniques have become indispensable tools in several areas of mathematical applications, such as in the numerical treatment of differential (or integral) equations. The task is usually formulated to approximating an (implicitly) given function (e.g., a unknown solution of a different equation) in some infinite dimensional function space B by some subspaces $S_j \subset B$ at different levels, such as the spaces $V_j, j \in \mathbf{Z}$, in an MRA [5, 11, 13]. Corresponding to the above approximating spaces S_j are the approximating operators P_j, that are the projectors from B to S_j. In the affine frame setting, an operator similar to the projector P_j is the operator $\tilde{P}_j$ in (11), which is no longer a projector, but is still an approximating identity. So for affine frame decomposition, we use operator approximation of the identity instead of space approximation of the whole space. Theorems 3.5 and 3.6 assure uniform stability over all levels of the affine frame decomposition in view of the operator approach to approximation of the identity on a Sobolev space.

REMARK 3.7. Given finite collections $\Psi := \{\psi_1, \ldots, \psi_N\}$ and $\tilde{\Psi} := \{\tilde{\psi}_1, \ldots, \tilde{\psi}_N\}$ of L^2 functions. We say that Ψ and $\tilde{\Psi}$ generate a *bi-frame* of $L^2(\mathbf{R}^d)$ if both $\mathcal{F} := \{\psi_{n;j,k}\}_{1\le n\le N, j\in\mathbf{Z}, k\in\mathbf{Z}^d}$ and $\tilde{\mathcal{F}} := \{\tilde{\psi}_{n;j,k}\}_{1\le n\le N, j,k\in\mathbf{Z}}$ are frames of $L^2(\mathbf{R}^d)$, and if

$$f = \sum_{n=1}^{N} \sum_{j\in\mathbf{Z}, k\in\mathbf{Z}^d} \langle f, \psi_{n;j,k}\rangle \tilde{\psi}_{n;j,k} = \sum_{n=1}^{N} \sum_{j\in\mathbf{Z}, k\in\mathbf{Z}^d} \langle f, \tilde{\psi}_{n;j,k}\rangle \psi_{n;j,k} \quad \text{for all } f \in L^2(\mathbf{R}^d),$$

where $\psi_{n;j,k} = M^{jd/2}\psi_n(M^j \cdot - k)$ ([8, 9, 16, 36]). We remark that all results in Theorems 3.1, 3.5 and 3.6 can be generalized to the bi-frame case with standard modification: the tight frame assumption for Ψ by the bi-frame assumption for $\Psi := \{\psi_1, \ldots, \psi_N\}$ and $\tilde{\Psi} := \{\tilde{\psi}_1, \ldots, \tilde{\psi}_N\}$; the regularity assumption (16) and vanishing moment assumption (17) for Ψ by the same assumptions for both Ψ and $\tilde{\Psi}$; the affine frame operator Q_j associated with the tight affine frame Ψ by the affine frame operator R_j associated with the bi-frame Ψ and $\tilde{\Psi}$,

$$R_j = \sum_{n=1}^{N} \sum_{k\in\mathbf{Z}^d} \langle f, \psi_{n;j,k}\rangle \tilde{\psi}_{n;j,k} \quad \forall\ f \in L^2;$$

and $\langle Q_j f, f\rangle$ in Theorems 3.1, 3.5 and 3.6 by $\sum_{n=1}^{N} \sum_{k\in\mathbf{Z}^d} |\langle f, \psi_{n;j,k}\rangle|^2$.

3.1. Proof of Theorem 3.1

To prove Theorem 3.1, we need the following two lemmas.

LEMMA 3.7. *Let $\beta > 0$ and $|\alpha| < \beta$. Assume that ψ satisfies the regularity condition (16) and the vanishing moment condition (17). Then there exists a positive constant C, such that, for all functions $g_j = \sum_{k\in\mathbf{Z}} c_{j,k}\psi_{j,k}$ with $\{c_{j,k}\}_{k\in\mathbf{Z}} \in \ell^2, j \in \mathbf{Z}$,*

$$|\langle \mathcal{J}_\alpha g_j, \mathcal{J}_\alpha g_{j'}\rangle| \le C M^{-(\beta-|\alpha|)|j-j'|} M^{(j_+ + j'_+)\alpha} \Big(\sum_{k\in\mathbf{Z}} |c_{j,k}|^2\Big)^{1/2} \Big(\sum_{k\in\mathbf{Z}} |c_{j',k}|^2\Big)^{1/2},\ j, j' \in \mathbf{Z}. \tag{32}$$

Proof. Without loss of generality, we assume that $j \le j'$. Let $A_j(\xi) = \sum_{k\in\mathbf{Z}} c_{j,k} e^{-ik\xi}$ be the Fourier series of the sequences $\{c_{j,k}\}_{k\in\mathbf{Z}}, j \in \mathbf{Z}$. Now, since

$$\widehat{\mathcal{J}_\alpha g_j}(\xi) = (1+|\xi|^2)^{\alpha/2} A_j(M^{-j}\xi) M^{-j/2}\hat{\psi}(M^{-j}\xi),$$

we have

$$\begin{aligned}
& |\langle \mathcal{J}_\alpha g_j, \mathcal{J}_\alpha g_{j'}\rangle| \qquad (33)\\
= \ & M^{-(j+j')/2} \int_{\mathbf{R}} (1+|\xi|^2)^\alpha A_j(M^{-j}\xi)\overline{A_{j'}(M^{-j'}\xi)}\\
& \quad \hat{\psi}(M^{-j}\xi)\overline{\hat{\psi}(M^{-j'}\xi)} d\xi\\
\le \ & M^{(j_+ + j'_+)\alpha + (j-j')\delta_0}\\
& \times \Big(\int_{\mathbf{R}} |A_j(\xi)|^2 (M^{-2j_+} + M^{2j_-}|\xi|^2)^\alpha |\xi|^{2\delta_0} |\hat{\psi}(\xi)|^2 d\xi\Big)^{1/2}\\
& \times \Big(\int_{\mathbf{R}} |A_{j'}(\xi)|^2 (M^{-2j'_+} + M^{2j'_-}|\xi|^2)^\alpha |\xi|^{-2\delta_0} |\hat{\psi}(\xi)|^2 d\xi\Big)^{1/2},
\end{aligned}$$

where $\delta_0 = \beta - |\alpha|$ and $x_- = \min(0,x)$. For $\xi \in [-\pi,\pi]$, it follows from (16) and (17) that

$$\begin{aligned}
& \sum_{k\in\mathbf{Z}} (M^{-2j_+} + M^{2j_-}|\xi+2k\pi|^2)^\alpha |\xi+2k\pi|^{2\delta_0} |\hat{\psi}(\xi+2k\pi)|^2 \qquad (34)\\
\le \ & C_1 + C_1 \sum_{0\ne k\in\mathbf{Z}} (1+|\xi+2k\pi|^2)^\alpha |\xi+2k\pi|^{2\delta_0} |\hat{\psi}(\xi+2k\pi)|^2 \le C_2
\end{aligned}$$

for $0 \le j \in \mathbf{Z}$, and

$$\begin{aligned}
& \sum_{k\in\mathbf{Z}} (M^{-2j_+} + M^{2j_-}|\xi+2k\pi|^2)^\alpha |\xi+2k\pi|^{2\delta_0} |\hat{\psi}(\xi+2k\pi)|^2 \qquad (35)\\
\le \ & C_3 \sum_{k\in\mathbf{Z}} (1+|\xi+2k\pi|^2)^{\alpha_+} |\xi+2k\pi|^{2\delta_0} |\hat{\psi}(\xi+2k\pi)|^2 \le C_4
\end{aligned}$$

for $0 \ge j \in \mathbf{Z}$, where C_1, C_2, C_3, C_4 are positive constants independent of $j \in \mathbf{Z}$ and $\xi \in [-\pi,\pi]$. Similarly by (16) and (17), we have

$$\sum_{k\in\mathbf{Z}} (M^{-2j'_+} + M^{-2j'_-}|\xi+2k\pi|^2)^\alpha |\xi+2k\pi|^{-2\delta_0} |\hat{\psi}(\xi+2k\pi)|^2 \le C_5 \qquad (36)$$

for all $j \in \mathbf{Z}$ and $\xi \in [-\pi,\pi]$, where C_5 is a positive constant independent of j and ξ. Combining (33), (34), (35) and (36), we obtain

$$\begin{aligned}
& |\langle \mathcal{J}_\alpha g_j, \mathcal{J}_\alpha g_{j'}\rangle|\\
\le \ & M^{(j_+ + j'_+)\alpha + (j-j')\delta_0} \Big(\int_{-\pi}^{\pi} |A_j(\xi)|^2 d\xi\Big)^{1/2} \Big(\int_{-\pi}^{\pi} |A_{j'}(\xi)|^2 d\xi\Big)^{1/2}\\
& \times \Big(\sup_{|\xi|\le\pi} \sum_{k\in\mathbf{Z}} (M^{-2j_+} + M^{2j_-}|\xi+2k\pi|)^\alpha |\xi+2k\pi|^{2\delta_0} |\hat{\psi}(\xi+2k\pi)|^2\Big)^{1/2}\\
& \times \Big(\sup_{|\xi|\le\pi} \sum_{k\in\mathbf{Z}} (M^{-2j'_+} + M^{2j'_-}|\xi+2k\pi|^2)^\alpha |\xi+2k\pi|^{-2\delta_0} |\hat{\psi}(\xi+2k\pi)|^2\Big)^{1/2}
\end{aligned}$$

$$\leq\ CM^{(j_++j'_+)\alpha+(j-j')\delta_0}\Big(\sum_{k\in\mathbf{Z}}|c_{j,k}|^2\Big)^{1/2}\Big(\sum_{k\in\mathbf{Z}}|c_{j',k}|^2\Big)^{1/2},$$

for some positive constant C independent of $j, j' \in \mathbf{Z}$, and $\{c_{j,k}\}, \{c_{j',k}\} \in \ell^2$. Hence, (32) follows. ■

LEMMA 3.8. *Let $\beta > 0$, $\alpha \in (-\beta, \beta)$, and let Ψ be a finite collection of L^2-functions such that any function $\psi \in \Psi$ satisfies the regularity condition (16) and the vanishing moment condition (17). Define*

$$g_j = \sum_{\psi\in\Psi}\sum_{k\in\mathbf{Z}} c_{\psi;j}(k)\psi_{j,k}$$

for some ℓ^2-sequences $\{c_{\psi;j}(k)\}_{k\in\mathbf{Z}}, \psi \in \Psi, j \in \mathbf{Z}$. Then for any $\epsilon > 0$, there exists a positive constant C_ϵ such that

$$\|g_j\|_{2,\alpha} \leq C_\epsilon M^{j_+\alpha}\|g_j\|_2 + \epsilon M^{j_+\alpha}\Big(\sum_{\psi\in\Psi}\sum_{k\in\mathbf{Z}}|c_{\psi;j}(k)|^2\Big)^{1/2},\ j \in \mathbf{Z}. \tag{37}$$

Proof. Let h_s be the characteristic function of the annulus $\{s \leq |\xi| \leq s^{-1}\}$, where s is some sufficiently small positive number to be assigned later. Note that

$$\widehat{g_j}(\xi) = M^{-j/2}\sum_{\psi\in\Psi} A_{\psi;j}(M^{-j}\xi)\widehat{\psi}(M^{-j}\xi),$$

where $A_{\psi;j}$ is the Fourier series of the sequence $\{c_{\psi;j}(k)\}_{k\in\mathbf{Z}}$. Then

$$\big\|\mathcal{F}^{-1}\big(\widehat{g_j}(\cdot)\widehat{h_s}(M^{-j}\cdot)\big)\big\|_{2,\alpha} \leq Cs^{-|\alpha|}M^{j_+\alpha}\|g_j\|_2 \tag{38}$$

for some positive constant C independent of $j \in \mathbf{Z}$ and $s \in (0, 1)$. By (16) and (17), we obtain

$$\begin{aligned}
&\|\mathcal{F}^{-1}(\widehat{g_j}(\cdot)(1-\widehat{h_s}(M^{-j}\cdot))\|_{2,\alpha}^2 \qquad (39)\\
=\ &\Big(\int_{|\xi|\leq M^js}+\int_{|\xi|\geq M^js^{-1}}\Big)|\widehat{g_j}(\xi)|^2(1+|\xi|^2)^\alpha d\xi\\
\leq\ &C_1\sum_{\psi\in\Psi}\int_{|\xi|\leq s}|A_{\psi;j}(\xi)|^2|\widehat{\psi}(\xi)|^2(1+M^{2j}|\xi|^2)^\alpha d\xi\\
&+C_1\sum_{\psi\in\Psi}\int_{|\xi|\geq s^{-1}}|A_{\psi;j}(\xi)|^2|\widehat{\psi}(\xi)|^2(1+M^{2j}|\xi|^2)^\alpha d\xi\\
\leq\ &C_2\max_{|\xi|\leq s}(1+M^{2j}|\xi|^2)^\alpha|\xi|^{2\beta}\times\int_{|\xi|\leq\pi}\sum_{\psi\in\Psi}|A_{\psi;j}(\xi)|^2d\xi\\
&+C_2M^{2j_+\alpha}\sum_{\psi\in\Psi}\int_{|\xi|\geq s^{-1}}|A_{\psi;j}(\xi)|^2|\widehat{\psi}(\xi)|^2(1+|\xi|^2)^{|\alpha|}d\xi\\
\leq\ &C_3M^{2j_+\alpha}s^{2(\beta-|\alpha|)}\sum_{\psi\in\Psi}\int_{-\pi}^{\pi}|A_{\psi;j}(\xi)|^2d\xi,
\end{aligned}$$

where C_1, C_2, C_3 are positive constants independent of $s \in (0,1)$ and $j \in \mathbf{Z}$. Combining (38) and (39), we have, for sufficiently small s, the estimate (37). ■

Proof (Proof of Theorem 3.1). First we establish the inequalities on the right-hand side of (19), (20) and (21). Recall that if $\sum_{k\in\mathbf{Z}} |\widehat{h}(\xi+2k\pi)|^2$ is bounded, there exists a positive constant C so that

$$\Big\| \sum_{k\in\mathbf{Z}} c_k h(\cdot - k)\Big\|_2^2 \le C \sum_{k\in\mathbf{Z}} |c_k|^2$$

for all ℓ^2 sequence $\{c_k\}$. Therefore the inequality on the right-hand side of (20) follows from the inequality on the right-hand side of (19). Clearly, the inequality on the right-hand side of (21) follows from Lemma 3.8 and the inequalities on the right-hand side of (19) and (20). Therefore it suffices to establish the second inequality in (19). This, in turn, depends on the estimate:

$$\sum_{j,k\in\mathbf{Z}} M^{2j+\alpha} |\langle f, \psi_{j,k}\rangle|^2 \le C\|f\|_{2,\alpha}^2,\ f \in H^\alpha, \tag{40}$$

for any function ψ that satisfies (16) and (17), for some positive constant C independent of f. For any compactly supported function ψ, we have

$$\begin{aligned}
&\sum_{k\in\mathbf{Z}} |\langle f, \psi_{j,k}\rangle|^2 \\
= \ & M^j \int_{-\pi}^{\pi} \Big| \sum_{k\in\mathbf{Z}} \widehat{f}(M^j(\xi+2k\pi))\overline{\widehat{\psi}(\xi+2k\pi)}\Big|^2 d\xi \\
\le \ & C_1 M^j \int_{\mathbf{R}} |\widehat{f}(M^j\xi)|^2 |\widehat{\psi}(\xi)|^2 d\xi \\
& + C_1 M^{j-2j_+\alpha} \int_{-\pi}^{\pi} d\xi \Big(\sum_{k\neq 0} |\widehat{f}(M^j(\xi+2k\pi))|^2 (1+M^{j_+}|\xi+2k\pi|)^{2\alpha} \\
& \quad (1+|\xi+2k\pi|)^{-2(\alpha+\beta)}\Big) \times \Big(\sum_{k\neq 0} |\widehat{\psi}(\xi+2k\pi)|^2 (1+|\xi+2k\pi|)^{2\beta}\Big),
\end{aligned}$$

where C_1 is a positive constant independent of f. Thus, for any function ψ satisfying (16) and (17), we obtain

$$\begin{aligned}
&\sum_{j,k\in\mathbf{Z}} M^{2j+\alpha} |\langle f, \psi_{j,k}\rangle|^2 \\
\le \ & C_2 \int_{\mathbf{R}} |\widehat{f}(\xi)|^2 \sum_{j=-\infty}^{\infty} M^{2j+\alpha} |\widehat{\psi}(M^{-j}\xi)|^2 d\xi \\
& + C_2 \sum_{j=-\infty}^{\infty} \int_{|\xi|\ge M^j\pi} |\widehat{f}(\xi)|^2 (1+M^{j_+-j}|\xi|)^{2\alpha} (1+M^{-j}|\xi|)^{-2(\alpha+\beta)} d\xi
\end{aligned}$$

$$\le \ C_3 \int_{\mathbf{R}} |\widehat{f}(\xi)|^2 \sum_{j=-\infty}^{\infty} M^{2j+\alpha} \min(|M^{-j}\xi|^{2\beta}, |M^{-j}\xi|^{-2\beta}) d\xi$$

$$
\begin{aligned}
&+C_3\int_{\mathbf R}|\widehat{f}(\xi)|^2\sum_{j=-\infty}^{\ln_M(|\xi|/\pi)}(1+M^{j_+-j}|\xi|)^{2\alpha}(1+M^{-j}|\xi|)^{-2(\alpha+\beta)}d\xi\\
\le\ & C_4\|f\|_{2,\alpha}^2,
\end{aligned}
$$

where C_2, C_3, C_4 are positive constants independent of $f\in H^\alpha$. This completes the proof of (40), and hence the second inequality in (19).

Next, we establish the first inequalities in (19), (20), and (21). The first inequality in (20) follows from the first inequality of (21), Lemma 3.8, and the second inequality in (19). On the other hand, the first inequality in (19) follows from the first inequality in (20) and the trivial estimate $\|Q_jf\|_2^2\le C\langle Q_jf,f\rangle$. Therefore, it suffices to prove the validity of the first inequality in (21). In this situation, we recall $f=\sum_{j\in\mathbf Z}Q_jf=:\sum_{j\in\mathbf Z}g_j$. By Lemma 3.7 and the second inequality in (19), we obtain

$$
\begin{aligned}
\|f\|_{2,\alpha}^2 &= \sum_{j,j'\in\mathbf Z}\langle\mathcal J_\alpha g_j,\mathcal J_\alpha g_{j'}\rangle\\
&\le C\sum_{|j-j'|>L}M^{-\delta_0|j-j'|}M^{(j_++j'_+)\alpha}\sum_{\psi\in\Psi}\Big(\sum_{k\in\mathbf Z}|\langle f,\psi_{j,k}\rangle|^2\Big)^{1/2}\\
&\quad\times\Big(\sum_{k\in\mathbf Z}|\langle f,\psi_{j',k}\rangle|^2\Big)^{1/2}+\sum_{|j-j'|\le L}\|\mathcal J_\alpha g_j\|_2\|\mathcal J_\alpha g_{j'}\|_2\\
&\le CM^{-\delta_0L}\|f\|_{2,\alpha}^2+CL\sum_{j\in\mathbf Z}\|\mathcal J_\alpha g_j\|_2^2,
\end{aligned}
$$

where $\delta_0=\beta-|\alpha|$, $L\ge1$, and C is a positive constant independent of L. Hence, for sufficient large L in the above estimate, we obtain

$$
\|f\|_{2,\alpha}^2\le C\sum_{j\in\mathbf Z}\|g_j\|_{2,\alpha}^2 \tag{41}
$$

for some positive constant C. This completes the proof of the first inequality in (21), and hence all the inequalities in (19), (20), and (21) are established.

Finally, we prove the unconditional convergence of the affine frame decomposition (18). By Lemma 3.7, we have

$$
\Big\|\sum_{j,k\in\mathbf Z}a_{j,k}\psi_{j,k}\Big\|_{2,\alpha}^2\le C\sum_{j,k\in\mathbf Z}M^{2j_+\alpha}|a_{j,k}|^2 \tag{42}
$$

for some constant C when ψ satisfies the regularity condition (16) and the vanishing moment condition (17). Hence, the unconditional convergence of the frame decomposition (18) follows directly from (19) and (42). ■

3.2. Proof of Proposition 3.2

Clearly if ψ satisfies (16) then $\psi\in H^\beta$. Conversely if ψ is a compactly supported function in H^β then ψ satisfies (16). Indeed, for any $\xi\in\mathbf R$,

$$
\sum_{k\in\mathbf Z}|\widehat{\psi}(\xi+2k\pi)|^2(1+|\xi+2k\pi|^2)^\beta
$$

$$
\begin{aligned}
&\leq C_1 \int_{\mathbf{R}} \Big(\sum_{k\in\mathbf{Z}} |\widehat{h}(\xi+2k\pi-\eta)|(1+|\xi+2k\pi|^2)^\beta \Big) |\widehat{\psi}(\eta)|^2 d\eta \\
&\leq C_2 \int_{\mathbf{R}} |\widehat{\psi}(\eta)|^2 (1+|\eta|^2)^\beta d\eta < \infty,
\end{aligned}
$$

where h is a compactly supported C^∞ function h with $\psi = h\psi$, and C_1, C_2 are positive constants independent of ξ. Therefore a compactly supported function ψ satisfies (16) if and only if $\psi \in H^\beta$. From this, the proof of (22) reduces to showing that

$$
\mathrm{Lip}_0\, \alpha \subset H^\beta \text{ for all } \alpha > \beta \geq 0. \tag{43}
$$

Let $\psi \in \mathrm{Lip}_0\, \alpha, \alpha > \beta \geq 0$, g be a compactly supported C^∞ satisfying $\hat{g}(\xi) = O(\xi^{\alpha+1})$ as $\xi \to 0$ and $\hat{g}(\xi) \neq 0$ as $1/2 \leq |\xi| \leq 1$, and $g_j(x) = 2^j g(2^j x)$ for $j \geq 1$. Then the functions $g_j * \psi, j \geq 1$, obtained by the convolution between g_j and ψ are supported in a bounded set K (independent of j), and their L^∞-norm are bounded by $C2^{-j\alpha}$ for some constant C independent of $j \geq 1$, namely,

$$
\|g_j * \psi\|_\infty \leq C 2^{-j\alpha},\ j \geq 1.
$$

By the standard Littlewood-Paley decomposition of compactly supported Hölder continuous functions [20], we see that

$$
\begin{aligned}
&\int_{\mathbf{R}} (1+|\xi|^2)^\beta |\widehat{\psi}(\xi)|^2 d\xi \\
&\leq 2^\beta \|\psi\|_2^2 + 2^\beta \sum_{j=1}^\infty 2^{2j\beta} \int_{2^{j-1}\leq|\xi|\leq 2^j} |\widehat{\psi}(\xi)|^2 d\xi \\
&\leq C\|\psi\|_2^2 + C\sum_{j=1}^\infty 2^{2j\beta} \|g_j * \psi\|_2^2 \\
&\leq C'\|\psi\|_\infty^2 + C'\sum_{j=1}^\infty 2^{2j\beta} \|g_j * \psi\|_\infty^2 < \infty
\end{aligned}
$$

where C, C' are positive constants. This proves (43) and completes the proof of the Proposition.

3.3. Proof of Corollary 3.3

From its Fourier transform formulation

$$
\widehat{N_m}(\xi) = \left(\frac{1-e^{-i\xi}}{i\xi} \right)^{m+1},
$$

we see that the $m^{\rm th}$ order cardinal B-spline N_m satisfies the regularity condition (18) for any $0 < \beta < m + 1/2$. Hence, the conclusion follows from Theorem 3.1.

3.4. Proof of Corollary 3.4

The conclusion follows directly from (19) and (42).

3.5. Proof of Theorem 3.5

Since ϕ is compactly supported, we have that $\phi = h\phi$ for some compactly supported C^∞ function h. Taking the Fourier transform on both sides of $\phi = h\phi$ and noting that

$\phi \in H^\beta$, we have

$$\begin{aligned} & \sum_{k\in\mathbf{Z}} |\hat{\phi}(\xi+2k\pi)|^2(1+|\xi+2k\pi|^2)^\beta \qquad (44) \\ \le\ & C_1 \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}} (1+|\xi+2k\pi-\eta|^2)^{-2\beta-1}|\hat{\phi}(\eta)|^2 d\eta \times (1+|\xi+2k\pi|^2)^\beta \\ \le\ & C_2\|\phi\|_{2,\beta}^2, \quad \xi\in\mathbf{R}, \end{aligned}$$

for some positive constants C_1, C_2. This proves that ϕ satisfies the regularity condition (16).

By the Hölder inequality, we have

$$\begin{aligned} \sum_{k\in\mathbf{Z}} |\langle f, \phi_{0,k}\rangle|^2 \ \le\ & \int_{-\pi}^{\pi} \Big(\sum_{k\in\mathbf{Z}} |\widehat{f}(\xi+2k\pi)||\widehat{\phi}(\xi+2k\pi)|\Big)^2 d\xi \\ \le\ & \int_{-\pi}^{\pi} \Big(\sum_{k\in\mathbf{Z}} |\widehat{f}(\xi+2k\pi)|^2(1+|\xi+2k\pi|^2)^\alpha\Big) \\ & \times\Big(\sum_{k'\in\mathbf{Z}} |\widehat{\phi}(\xi+2k'\pi)|^2(1+|\xi+2k'\pi|^2)^{-\alpha}\Big) d\xi. \end{aligned}$$

This, together with (44), implies that

$$\sum_{k\in\mathbf{Z}} |\langle f, \phi_{0,k}\rangle|^2 \le C\|f\|_{2,\alpha}^2, \qquad (45)$$

for all $f\in H^\alpha, \alpha\in(-\beta,\beta)$, where C is a positive constant.

By (44) and the assumption that ϕ has linear independent shifts, there exists a positive constant C such that

$$C^{-1} \le \sum_{k\in\mathbf{Z}} |\widehat{\phi}(\xi+2k\pi)|^2 \le C,$$

and

$$C^{-1} \le \sum_{k\in\mathbf{Z}} |\widehat{\phi}(\xi+2k\pi)|^2(1+|\xi+2k\pi|^2)^\alpha \le C, \quad \xi\in\mathbf{R}.$$

For any $f_0\in V_0$, we have that $\widehat{f_0}(\xi) = a(\xi)\hat{\phi}(\xi)$ for some 2π-periodic function $a(\xi)$. Thus,

$$C^{-1}\int_{-\pi}^{\pi} |a(\xi)|^2 d\xi \le \|f_0\|_2^2 \le C\int_{-\pi}^{\pi} |a(\xi)|^2 d\xi,$$

and

$$C^{-1}\int_{-\pi}^{\pi} |a(\xi)|^2 d\xi \le \|f_0\|_{2,\alpha}^2 \le C\int_{-\pi}^{\pi} |a(\xi)|^2 d\xi.$$

This proves that

$$C^{-1}\|f_0\|_2 \le \|f_0\|_{2,\alpha} \le C\|f_0\|_2, \quad f_0\in V_0, \qquad (46)$$

for some positive constant C.

By (44) and the assumption on Ψ, we see that any function $\psi \in \Psi$ satisfies (16). Therefore the inequalities on the right-hand sides of (23), (25) and (26) follow directly from (45), (46), and the inequalities on the right-hand sides of (19), (20) and (21). The inequalities on the left-hand sides of (23), (25) and (26) follow by using a similar method as in the proof of Theorem 3.1. We can therefore safely omit the details of the proof here.

3.6. Proof of Theorem 3.6

For any $f_0 \in V_0$, we have $\widehat{f_0}(\xi) = a(\xi)\hat{\phi}(\xi)$ for some square-integrable 2π-periodic function a. Thus,

$$\widehat{\tilde{P}_0 f_0}(\xi) = a(\xi)S(\xi)\Phi(\xi)\widehat{\phi}(\xi) = S(\xi)\Phi(\xi)\widehat{f_0}(\xi),$$

where $\Phi(\xi) = \sum_{k\in\mathbf{Z}} |\widehat{\phi}(\xi + 2k\pi)|^2$. This, together with strict positivity of $S(\xi)$ and $\Phi(\xi)$, implies that $\tilde{P}_0$ has a bounded inverse on V_0. Hence, we obtain, from dilation invariance, that

$$\|\tilde{P}_L f\|_2 \geq C\|f\|_2,\ f \in V_L, \tag{47}$$

for some positive constant C independent of $L \geq 0$. By (13) and (47), the following finite frame decomposition property holds for any $f \in V_L$:

$$\begin{aligned} f &= Q_{L-1}h_L + \cdots + Q_0 h_L + \tilde{P}_0 h_L \\ &= \sum_{j=0}^{L-1} \sum_{\psi\in\Psi} \sum_{k\in\mathbf{Z}} \langle h_L, \psi_{j,k}\rangle \psi_{j,k} + \sum_{k\in\mathbf{Z}} \langle h_L, \tilde{\phi}_{0,k}\rangle \phi_{0,k}, \end{aligned}$$

where $h_L = \tilde{P}_L^{-1} f \in V_L$. The estimates in (29), (30) and (31) can be proved by using a similar method as in the proof of Theorems 3.1 and 3.5. It is then safe to omit the details of the proof here.

4. RANGES OF THE OPERATORS $\tilde{P}_J$ AND Q_J

We have shown that by Theorem 3.5, for a tight affine frame Ψ associated with an MRA, the identity operator on the Sobolev space has a stable decomposition. Corresponding to the operator decomposition of the identity operator is the decomposition of the Sobolev space H^α, namely:

$$H^\alpha = \tilde{P}_0 H^\alpha + \sum_{j=0}^{\infty} Q_j H^\alpha.$$

An interesting question that arises then is whether or not the subspaces $\tilde{P}_0 H^\alpha$ and $Q_j H^\alpha, j \in \mathbf{Z}$, are Hilbert subspaces of H^α.

We say that a subspace V of L^2 is a *shift-invariant space* if $f(\cdot - k) \in V$ for any $f \in V$ and $k \in \mathbf{Z}$. For a tight affine frame Ψ associated with an MRA, the ranges $\tilde{P}_0 L^2$ and $Q_0 L^2$ are shift-invariant subspaces of L^2. If both the scaling function ϕ of the MRA $\{V_j\}_{j\in\mathbf{Z}}$ and the tight affine frame Ψ associated with this MRA satisfy the regularity condition (16), then following the proof of (45), we have that $\tilde{P}_0 H^\alpha$ is a shift-invariant subspace of $V_0 = V^2(\phi)$ and that $Q_0 H^\alpha$ is a shift-invariant subspace of $V^2(\Psi)$. This motivates our study of the ranges $\tilde{P}_0 H^\alpha$ and $Q_0 H^\alpha$ via the theory of shift-invariant spaces.

THEOREM 4.1. *Let $\beta > 0, \alpha \in (-\beta, \beta)$, and let $\{V_j\}_{j\in\mathbf{Z}}$ be an MRA with compactly supported scaling function $\phi \in H^\beta$ that has linear independent shifts. Assume that Ψ is a finite family of compactly supported L^2-functions in V_1 that generate a tight affine frame of L^2, and that any function $\psi \in \Psi$ satisfies the vanishing moment condition (17). Then the following statements are equivalent:*

(i) Both $\tilde{P}_0 H^\alpha$ and $Q_j H^\alpha, j \in \mathbf{Z}$, are closed in L^2 (or equivalently in H^α).
(ii) $Q_0 H^\alpha$ is the orthogonal complement of V_0 in $V_1 \subset L^2$, and ϕ has orthonormal shifts, i.e., $\langle \phi, \phi(\cdot - k)\rangle = \delta_{k0}, k \in \mathbf{Z}$.
(iii) $\{\psi(\cdot - k) : \psi \in \Psi, k \in \mathbf{Z}\}$ is a tight frame of the shift-invariant space $V^2(\Psi)$.

For a tight frame Ψ associated with an MRA $\{V_j\}_{j\in\mathbf{Z}}$, we note that if the scaling function ϕ of this MRA has orthonormal shifts, and the range $Q_j H^\alpha$ of the frame operator Q_j at each level is the same as the wavelet space at the corresponding level, then the affine frame decomposition of a function f in H^α becomes essentially the usual orthogonal wavelet decomposition. So by Theorem 4.1, the ranges $\tilde{P}_0 H^\alpha$ and $Q_j H^\alpha$ are not closed in L^2 in general.

Recall that for a compactly supported scaling function $\phi \in H^\alpha$, there exists a $\delta > 0$ so that $\phi \in H^{\alpha+\delta}$ (see for instance [31]). Therefore, by Theorem 4.1, we have the following result, which generalizes a result in [21].

COROLLARY 4.2. *Let $\{V_j\}_{j\in\mathbf{Z}}$ be an MRA with compactly supported scaling function ϕ that has linear independent shifts, and let Ψ be a finite family of compactly supported L^2-functions in V_1. Assume that Ψ generates a tight affine frame of L^2, and also a tight frame of the shift-invariant space $V^2(\Psi)$. Then ϕ has orthonormal shifts.*

The rest of this section is divided into three parts. In the first and second parts, we give various characterizations of the topological property of closedness for $\tilde{P}_0 H^\alpha$ and $Q_j H^\alpha$, respectively. The proof of Theorem 4.1 is given in the last part of this section.

4.1. Range of the operator $\tilde{P}_0$

In this subsection, we study the topological property of closedness of the range of the operator $\tilde{P}_0$ in the Sobolev space H^α. We remark that in the following result, the function ϕ needs not be a scaling function and that S needs not be the vanishing moment recovery function of a tight affine frame, even though we use the same notation as before.

THEOREM 4.3. *Let $\beta > 0$, $\alpha \in (-\beta, \beta)$, and $\phi \in H^\beta$ be a compactly supported function that has linear independent shifts. Assume that $S(\xi)$ is a nontrivial trigonometric polynomial, and define the operator $\tilde{P}_0$ on H^α by*

$$\tilde{P}_0 f = \sum_{k\in\mathbf{Z}} \langle f, \tilde{\phi}_{0,k}\rangle \phi_{0,k},$$

where $\widehat{\tilde{\phi}}(\xi) = S(\xi)\widehat{\phi}(\xi)$. Then the following statements are equivalent:

(i) $\tilde{P}_0 H^\alpha$ is closed in L^2 (or equivalently in H^α).
(ii) $S(\xi) \neq 0$ for all $\xi \in \mathbf{R}$.

(iii) $\tilde{P}_0 H^\alpha = V^2(\phi)$.

Proof. First we prove (i) $\Longrightarrow$ (ii). Suppose, on the contrary, that $S(\xi_0) = 0$ for some $\xi_0 \in \mathbf{R}$. Then there exist a positive constant $\delta_0 > 0$ and a function $A(\xi) \in L^2_{2\pi}$ supported in $[\xi_0, \xi_0 + \delta_0]$ so that $S(\xi) \neq 0$ for all $\xi \in (\xi_0, \xi_0 + \delta_0]$, and $A(\xi)S(\xi)^{-1} \notin L^2_{2\pi}$. Here, $L^2_{2\pi}$ denotes, as usual, the space of all square-integrable 2π-periodic functions. For any $\delta \in (0, \delta_0)$, we introduce the functions f_δ and g_δ by setting

$$\widehat{f_\delta}(\xi) = A(\xi)\chi_{E_\delta}(\xi)\widehat{\phi}(\xi),$$

and

$$\widehat{g}_\delta(\xi) = A(\xi)S(\xi)^{-1}\chi_{E_\delta}(\xi)\Big(\sum_{k\in\mathbf{Z}} |\widehat{\phi}(\xi + 2k\pi)|^2\Big)^{-1}\widehat{\phi}(\xi),$$

where $E_\delta = [\xi_0 + \delta, \xi_0 + \delta_0] + 2\pi\mathbf{Z}$. Then g_δ belongs to H^α and satisfies $\tilde{P}_0 g_\delta = f_\delta$, which, in turn, implies that $f_\delta \in \tilde{P}_0 H^\alpha$. Also we note that f_δ tends to f_0 as δ tends to zero, where $\widehat{f_0}(\xi) = A(\xi)\widehat{\phi}(\xi)$. Therefore, since the space $\tilde{P}_0 H^\alpha$ is closed, $f_0 = \tilde{P}_0 g_0$ for some $g_0 \in H^\alpha$. Taking the Fourier transform of both sides, we have, by the property of linear independent shifts of the scaling function ϕ,

$$\sum_{k\in\mathbf{Z}} \widehat{g}_0(\xi + 2k\pi)\overline{\widehat{\phi}(\xi + 2k\pi)} = A(\xi)S(\xi)^{-1},$$

which leads to a contradiction, since the left-hand side belongs to $L^2_{2\pi}$ but the right-hand side does not.

Next, we prove (ii) $\Longrightarrow$ (iii). Let $S(\xi) \neq 0$ for all $\xi \in \mathbf{R}$. Following the proof of (47), we see that the restriction of the operator $\tilde{P}_0$ on $V^2(\phi)$ has a bounded inverse. Recall that $V^2(\phi) \subset H^\alpha$ by the assumption on ϕ. Therefore the above two observations together lead to the assertion that $\tilde{P}_0 H^\alpha = V^2(\phi)$.

Finally, the implication (iii) $\Longrightarrow$ (i) follows easily since the space $V^2(\phi)$ is closed in L^2 as well as in H^α. ■

REMARK 4.1. For functions ψ_n and $\tilde{\psi}_n, 1 \le n \le N$, satisfying (16) with $\beta = 0$, we define the operator R on L^2 by

$$Rf = \sum_{n=1}^{N}\sum_{k\in\mathbf{Z}} \langle f, \psi_n(\cdot - k)\rangle \tilde{\psi}_n(\cdot - k) \quad \forall\, f \in L^2. \tag{48}$$

Using the Fourier technique (c.f. [2, 3]), one may prove the following results for an operator R of the form (48): RL^2 is closed in L^2 if and only if there exists a positive constant C such that

$$C^{-1}(A_\Psi(\xi))^{1/2}A_{\tilde{\Psi}}(\xi)(A_\Psi(\xi))^{1/2} \le \Big((A_\Psi(\xi))^{1/2}A_{\tilde{\Psi}}(\xi)(A_\Psi(\xi))^{1/2}\Big)^2 \le CA_\Psi^{1/2}(\xi)A_{\tilde{\Psi}}(\xi)A_\Psi^{1/2}(\xi) \tag{49}$$

holds for almost all $\xi \in \mathbf{R}^d$; and $RV^2(\Phi)$ is closed in L^2 if and only if there exists a positive constant C such that

$$C^{-1}A_{\Phi,\Psi}(\xi)A_{\tilde{\Psi}}(\xi)A_{\Psi,\Phi}(\xi) \le \Big(A_{\Phi,\Psi}(\xi)A_{\tilde{\Psi}}(\xi)A_{\Psi,\Phi}(\xi)\Big)^2$$

$$\leq CA_{\Phi,\Psi}(\xi)A_{\tilde{\Psi}}(\xi)A_{\Psi,\Phi}(\xi) \quad (50)$$

holds for almost all $\xi \in \mathbf{R}^d$, where $\Phi = \{\phi_n, 1 \leq n \leq N'\}$ satisfies (16) with $\beta = 0$ for every $\phi_n \in \Phi$. Here the correlation matrix $A_{\Psi,\Phi}(\xi)$ is defined by

$$A_{\Psi,\Phi}(\xi) = \sum_{k\in\mathbf{Z}} \widehat{\Psi}(\xi + 2k\pi)\overline{\widehat{\Phi}(\xi + 2k\pi)}^T$$

and the auto-correlation matrix $A_\Phi(\xi) := A_{\Phi,\Phi}(\xi)$. If we further assume that $\{\psi_n(\cdot - k) : 1 \leq n \leq N, k \in \mathbf{Z}\}$ for a Riesz basis for its generating space $V^2(\Psi)$ and any function $\psi \in \Psi$ satisfies (16) with $\beta = \alpha$, then RH^α is closed in L^2 if and only if there exists a positive constant C such that

$$C^{-1}A_{\tilde{\Psi}}(\xi) \leq (A_{\tilde{\Psi}}(\xi))^2 \leq CA_{\tilde{\Psi}}(\xi) \quad (51)$$

holds for almost all $\xi \in \mathbf{R}^d$. This characterization for the closedness of RH^α simply implies the equivalence of the statements (i) and (ii) in Theorem 4.3.

4.2. Range of the operator Q_0

In this subsection, we consider the problem of whether or not Q_0H^α is closed in L^2 (or equivalently in H^α). Thus we establish some connections among the topological property of closedness of Q_0H^α, the frame property of the shifts of functions in Ψ, and the existence of tight affine frames with a minimal number of generators.

THEOREM 4.4. *Let $\beta > 0, \alpha \in (-\beta, \beta)$, and let $\{V_j\}_{j\in\mathbf{Z}}$ be an MRA with compactly supported scaling function $\phi \in H^\beta$ that has linear independent shifts. Assume that $\Psi \subset V_1$ be a finite collection of compactly supported functions that generates a tight affine frame of L^2. Write*

$$\widehat{\psi}(\xi) = H_\psi\Big(\frac{\xi}{M}\Big)\widehat{\phi}\Big(\frac{\xi}{M}\Big),\ \psi \in \Psi, \quad (52)$$

set

$$\mathbf{H}(\xi) = \Big(H_\psi\Big(\xi + \frac{2m\pi}{M}\Big)\Big)_{\psi\in\Psi, 0\leq m\leq M-1}, \quad (53)$$

and let $S(\xi)$ be defined as in (10). Then the following statements are equivalent:

(i) Q_0H^α is a closed subspace of L^2.
(ii) $Q_0H^\alpha = V^2(\Psi)$.
(iii) $\{\psi(\cdot - k) : \psi \in \Psi, k \in \mathbf{Z}\}$ is a frame of $V^2(\Psi)$.
(iv) There exist compactly supported functions $\psi_1^*, \ldots, \psi_{M-1}^*$ in V_1 such that $\Psi^* := \{\psi_1^*, \ldots, \psi_{M-1}^*\}$ generates a tight affine frame of L^2 and that $\{\psi_m^*(\cdot - k) : 1 \leq m \leq M-1, k \in \mathbf{Z}\}$ is a Riesz basis of $V^2(\Psi)$.
(v) The rank of $\mathbf{H}(\xi)$ is $M - 1$ for all $\xi \in \mathbf{R}$.
(vi) $S(\xi)$ satisfies

$$\sum_{m=0}^{M-1} \frac{S(M\xi)}{S(\xi + 2m\pi/M)}\Big|H_0\Big(\xi + \frac{2m\pi}{M}\Big)\Big|^2 = 1,\ \xi \in \mathbf{R}. \quad (54)$$

(vii) The rank of $(\widehat{\psi}(\xi+2k\pi))_{\psi\in\Psi,k\in\mathbf{Z}}$ is independent of ξ in a neighborhood of the origin.

(viii) There exist a positive constant C and negative integer L_1, such that

$$\sum_{j=L_1}^{0}\langle Q_j f, f\rangle \geq C\|f\|_2^2,\ f\in V^2(\Psi). \tag{55}$$

REMARK 4.2. By the dilation invariance of the Sobolev space H^α, the space Q_0H^α is closed in L^2 if and only if all the subspaces $Q_jH^\alpha, j\in\mathbf{Z}$, are closed in L^2. Thus, the condition (i) in Theorem 4.4 is equivalent to the closedness of the subspaces $Q_jH^\alpha, j\in\mathbf{Z}$, in L^2 (or in H^α by (46)).

REMARK 4.3. A finite family Ψ of compactly supported functions may generate different shift-invariant subspaces for different purposes, such as, Q_0L^2 for the theory of frames, $V^2(\Psi)$ in (14) for sampling theory [1, 2], and $S^2(\Psi)$ for approximation theory [4, 23, 26]. Here, $S^2(\Psi)$ is the L^2-closure of the algebraic span of the shifts of functions in Ψ. Clearly, we have

$$Q_0L^2\subset V^2(\Psi)\subset S^2(\Psi).$$

In [2], it is shown that $V^2(\Psi)=S^2(\Psi)$ if and only if $V^2(\Psi)$ is a closed subspace of L^2. This, together with Theorem 4.4, implies that either

$$Q_0L^2=V^2(\Psi)=S^2(\Psi),$$

or

$$Q_0L^2\neq V^2(\Psi)\neq S^2(\Psi).$$

REMARK 4.4. If the scaling function ϕ has orthonormal shifts, then the corresponding symbol H_0 satisfies

$$\sum_{m=0}^{M-1}\Big|H_0\Big(\xi+\frac{2m\pi}{M}\Big)\Big|^2=1. \tag{56}$$

The converse does not hold, as can be seen from the example that $H_0(\xi)=(1+e^{-3i\xi})/2$ for $M=2$ satisfies (56) but the corresponding refinable function $\chi_{[0,3]}$ does not have orthonormal shifts. It is shown in [30] that the function $\psi:=\chi_{[0,3/2]}-\chi_{[3/2,3]}$ generates a tight affine frame of L^2. On the other hand, one may easily verify that ψ has linear independent shifts. The family $\Psi^*:=\{\psi_1^*,\dots,\psi_{M-1}^*\}$ in (iv) of Theorem 4.4 has similar properties, namely: $\{\psi_{j,k}: k\in\mathbf{Z}\}$ generates a Riesz basis for every $j\in\mathbf{Z}$, but $\cup_{j\in\mathbf{Z}}\{\psi_{j,k}: k\in\mathbf{Z}\}$ is a tight affine frame of L^2.

To prove Theorem 4.4, we recall a result on tight frames with $M-1$ generators, given in [8] for $M=2$ and [9] for $M\geq 2$.

LEMMA 4.5. *Let $\{V_j\}_{j\in\mathbf{Z}}$ be an MRA with compactly supported scaling function ϕ that has linear independent shifts, and let H_0 be the symbol of the scaling function ϕ. Assume that $\Psi := \{\psi_1, \ldots, \psi_{M-1}\} \subset V_1$ generates a tight affine frame of L^2. Then*

$$\sum_{m=0}^{M-1} \frac{S(M\xi)}{S(\xi+2m\pi/M)} \Big| H_0\Big(\xi + \frac{2m\pi}{M}\Big)\Big|^2 = 1, \tag{57}$$

where the function $S(\xi)$ is defined as in (10). Conversely, if the trigonometric polynomial S satisfies (57), $S(0) \neq 0$, $S(\xi) \geq 0$ and $S(\xi) = S(-\xi)$ for all $\xi \in \mathbf{R}$, then there exists $\Psi := \{\psi_1, \ldots, \psi_{M-1}\} \subset V_1$ such that Ψ generates a tight affine frame of L^2, and $S(\xi)$ is defined as in (10) with the above tight affine frame Ψ.

To prove Theorem 4.4, we also need a result about dense subspaces of a shift-invariant space.

LEMMA 4.6. *Let $\beta > 0$, $\alpha \in (-\beta, \beta)$, and let $\phi_1, \ldots, \phi_L$ be compactly supported functions that have linear independent shifts and satisfy (16). Also, let $\psi_1, \ldots, \psi_N$ be in the algebraic span of $\{\phi_l(\cdot - k) : 1 \leq l \leq L, k \in \mathbf{Z}\}$, and define*

$$Q_0 f = \sum_{n=1}^{N} \sum_{k\in\mathbf{Z}} \langle f, \psi_n(\cdot - k)\rangle \psi_n(\cdot - k), \ f \in H^\alpha.$$

If the rank of the $N \times \mathbf{Z}$ matrix $\big(\widehat{\psi_n}(\xi_0 + 2k\pi)\big)_{1\leq n\leq N, k\in\mathbf{Z}}$ is L for some $\xi_0 \in \mathbf{R}$, then the closure of $Q_0 H^\alpha$ in L^2 is $V^2(\phi_1, \ldots, \phi_L)$. The converse also holds.

Proof. First we prove the density of $Q_0 H^\alpha$ in $V^2(\phi_1, \ldots, \phi_L)$. Write $\widehat{\psi}_n(\xi) = \sum_{l=1}^{L} H_{n,l}(\xi)\widehat{\phi}_l(\xi)$. Since $\big(\widehat{\phi}_l(\xi + 2k\pi)\big)_{1\leq l\leq L, k\in\mathbf{Z}}$ has rank L for all $\xi \in \mathbf{R}$ by the linear independent shifts of $\phi_1, \ldots, \phi_L$ [28, 35], it follows that the rank of the $N \times \mathbf{Z}$ matrix $\big(\widehat{\psi}_n(\xi + 2k\pi)\big)_{1\leq n\leq N, k\in\mathbf{Z}}$ is the same as that of the $N \times L$ matrix $\mathbf{H}(\xi) := (H_{n,l}(\xi))_{1\leq n\leq N, 1\leq l\leq L}$. By the assumption on $\psi_n, 1 \leq n \leq N$, $\mathbf{H}(\xi_0)$ is of full rank, and hence $\mathbf{H}(\xi)$ is of full rank except for finitely many points, say in the set $\Xi = \{\xi_1, \ldots, \xi_s\}$, since all entries of $\mathbf{H}(\xi)$ are trigonometric polynomials. For any function f in the shift-invariant space $V^2(\phi_1, \ldots, \phi_L)$ generated by $\phi_1, \ldots, \phi_L$, $\widehat{f}(\xi) = \sum_{l=1}^{L} A_l(\xi)\widehat{\phi}_l(\xi)$ for some $A_l \in L^2_{2\pi}, 1 \leq l \leq L$. Clearly, the functions f_ϵ defined by $\widehat{f_\epsilon}(\xi) = \widehat{f}(\xi)\chi_{\mathbf{R}\backslash E_\epsilon}$ tends to f in L^2 as ϵ tends to zero, where $E_\epsilon = \cup_{s'=1}^{s}\big(\xi_{s'} + (-\epsilon, \epsilon) + 2\pi\mathbf{Z}\big)$. Therefore, it suffices to prove that $f_\epsilon \in Q_0 H^\alpha$ for all $\epsilon \in (0, \epsilon_0)$, where ϵ_0 is a sufficiently small positive number so chosen that the matrix $\overline{\mathbf{H}(\xi)}^T \mathbf{H}(\xi)$ is nonsingular and its inverse is bounded for all $\xi \in \mathbf{R}\backslash E_\epsilon$. Define $A_{1,\epsilon}(\xi), \ldots, A_{L,\epsilon}(\xi)$ by

$$\begin{aligned}(A_{1,\epsilon}(\xi), \ldots, A_{L,\epsilon}(\xi))^T &= \big(\overline{\mathbf{H}(\xi)}^T \mathbf{H}(\xi)\big)^{-1} \\ &\quad \times (A_1(\xi), \ldots, A_L(\xi))^T \chi_{E_\epsilon}(\xi),\end{aligned} \tag{58}$$

and define g_ϵ by $\widehat{g}_\epsilon(\xi) = \sum_{l=1}^{L} A_{l,\epsilon}(\xi)\widehat{\tilde{\phi}}_l(\xi)$, where $\tilde{\phi}_l \in V^2(\phi_1, \ldots, \phi_L), 1 \leq l \leq L$, is some bi-orthogonal dual of $\{\phi_1, \ldots, \phi_L\}$, i.e., $\langle \phi_l, \tilde{\phi}_{l'}(\cdot - k)\rangle = \delta_{ll'}\delta_{k0}$ for all $1 \leq l, l' \leq L$

and $k \in \mathbf{Z}$. Then we have

$$\begin{aligned}\widehat{Q_0 g_\epsilon}(\xi) &= \sum_{n=1}^{N} \sum_{l,l',l''=1}^{L} A_{l',\epsilon}(\xi)\overline{H_{n,l}(\xi)} H_{n,l''}(\xi) \\ &\qquad \times \Big(\sum_{k\in\mathbf{Z}} \widehat{\tilde{\phi}}_{l'}(\xi+2k\pi)\overline{\widehat{\phi}_l(\xi+2k\pi)}\Big)\widehat{\phi}_{l''}(\xi) \\ &= \sum_{l''=1}^{L} A_{l''}(\xi)\widehat{\phi}_{l''}(\xi)\chi_{E_\epsilon}(\xi) = \widehat{f}_\epsilon(\xi).\end{aligned}$$

Hence, $Q_0 g_\epsilon = f_\epsilon$. This, together with $g_\epsilon \in H^\alpha$, proves that $Q_0 H^\alpha$ is dense in $V^2(\phi_1, \dots, \phi_L)$.

To establish the converse, it suffices to show that if the rank of the $N \times \mathbf{Z}$ matrix $\big(\widehat{\psi_n}(\xi+2k\pi)\big)_{1\le n\le N, k\in\mathbf{Z}}$ is strictly less than L for all $\xi \in \mathbf{R}$, then there exists a function $g_0 \in V^2(\phi_1, \dots, \phi_L)$, which does not belong to the L^2-closure of $Q_0 H^\alpha$. Let $\mathbf{H}$ and $\tilde{\phi}_l, 1 \le l \le L$, be as in the proof of the previous conclusion. Then the rank of $\mathbf{H}(\xi)$ is at most $L-1$. Therefore there exists a nonzero vector $\mathbf{A}(\xi) = (a_1(\xi), \dots, a_L(\xi))^T$ with trigonometric polynomial entries so that $\mathbf{H}(\xi)\mathbf{A}(\xi) = 0$ for all $\xi \in \mathbf{R}$. One may verify that the function $\tilde{\phi}$ defined by $\widehat{\tilde{\phi}}(\xi) = \sum_{l=1}^{L} \overline{a_l(\xi)}\widehat{\tilde{\phi}}_l(\xi)$ satisfies $\langle \psi_n(\cdot - k), \tilde{\phi}\rangle = 0$ for all $1 \le n \le N, k \in \mathbf{Z}$, and hence $\langle f, \tilde{\phi}\rangle = 0$ for all $f \in Q_0 H^\alpha$. On the other hand, the function $g_0 \in V^2(\phi_1, \dots, \phi_L)$ defined by $\widehat{g}_0(\xi) = \sum_{l=1}^{L} a_l(\xi)\widehat{\phi}_l(\xi)$ satisfies $\langle g_0, \tilde{\phi}\rangle \ne 0$. This proves that g_0 is not in the L^2-closure of $Q_0 H^\alpha$ and hence the conclusion follows. ■

Now we start to prove Theorem 4.4.

Proof (Proof of Theorem 4.4.). We set $\Psi := \{\psi_1, \dots, \psi_N\}$ and divide the proof into the following steps: (viii) $\Longrightarrow$ (vii) $\Longrightarrow$ (vi) $\Longrightarrow$ (v) $\Longrightarrow$ (iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (viii), (ii) $\Longleftrightarrow$ (v), (i)$\Longrightarrow$(vi), and (ii)$\Longrightarrow$(i).

(Proof of (viii)$\Longrightarrow$(vii)): This proof is by indirect argument. Suppose, on the contrary, that the rank of $(\widehat{\psi}_n(\xi+2k\pi))_{1\le n\le N, k\in\mathbf{Z}}$ depends on ξ in any small neighborhood of the origin. Denote the rank of $(\widehat{\psi}_n(2k\pi))_{1\le n\le N, k\in\mathbf{Z}}$ by k_0. Therefore, there exists a nonsingular matrix P such that the matrix $(\widehat{\psi_n^*}(2k\pi))_{1\le n\le k_0, k\in\mathbf{Z}}$ has rank k_0, and $\widehat{\psi_n^*}(2k\pi) = 0$ for all $k_0 + 1 \le n \le N$ and $k \in \mathbf{Z}$, where $(\psi_1^*, \dots, \psi_N^*)^T := P(\psi_1, \dots, \psi_N)^T$. By the assumption, there exists a function $\psi_{n_0}^*, k_0 + 1 \le n_0 \le N$, such that the vector $(\widehat{\psi_n^*}(\xi+2k\pi))_{k\in\mathbf{Z}}$ is not in the space spanned by $(\widehat{\psi_n^*}(\xi+2k\pi))_{k\in\mathbf{Z}}, 1 \le n \le k_0$, in any small neighborhood of the origin. Define ψ by

$$\widehat{\psi}(\xi) = \widehat{\psi}_{n_0}^*(\xi) - \sum_{n=1}^{k_0} a_n(\xi)\widehat{\psi_n^*}(\xi), \tag{59}$$

where the 2π-periodic functions $a_n(\xi), 1 \le n \le k_0$, are so chosen that their Fourier coefficient sequences are summable and

$$\sum_{k\in\mathbf{Z}} \widehat{\psi}(\xi+2k\pi)\overline{\widehat{\psi_n^*}(\xi+2k\pi)} = 0 \tag{60}$$

for $1 \le n \le k_0$ and $|\xi| \le \delta_0$, for some $\delta_0 > 0$. From the construction of ψ, the vector $(\widehat{\psi}(\xi+2k\pi))_{k\in\mathbf{Z}}$ is not identically zero on any neighborhood of the origin, but

$$\widehat{\psi}(2k\pi) = 0, \ k \in \mathbf{Z}, \tag{61}$$

and

$$\widehat{\psi}(\xi) = m(\xi/M)\widehat{\phi}(\xi/M), \tag{62}$$

for some 2π-periodic function $m(\xi)$ that has summable Fourier coefficient sequence and satisfies $m(0) = 0$. Choose any small positive ϵ, and define f_ϵ by $\widehat{f_\epsilon}(\xi) = a_\epsilon(\xi)\widehat{\psi}(\xi)$, where a_ϵ is a square-integrable 2π-periodic function with support contained in $\{|\xi| \le \delta_0\} + 2\pi\mathbf{Z}$ for some sufficiently small number $\delta := \delta(\epsilon)$ to be assigned later. Clearly $f_\epsilon \in V^2(\Psi)$, and

$$C^{-1}\|f_\epsilon\|_2^2 \le \int_{-\pi}^{\pi} |a_\epsilon(M\xi)m(\xi)|^2 d\xi \le C\|f_\epsilon\|_2^2 \tag{63}$$

by (62) and the assumption that ϕ has linear independent shifts. By (62) and the assumption on ψ, we see that $\sum_{s=0}^{M-1} |m(\xi + 2s\pi/M)|^2$ is not identically zero in any neighborhood of the origin, which together with (63) proves that $f_\epsilon \not\equiv 0$ when the support of a_ϵ is chosen appropriately.

Let $a_{n,n'}$ be so chosen that $\widehat{\psi}_n(2k\pi) - \sum_{n'=1}^{k_0} a_{n,n'}\widehat{\psi^*_{n'}}(2k\pi) = 0$ for all $k \in \mathbf{Z}$. The existence of such functions follows from the nonsingularity of the matrix P and the assumption that the rank of $(\widehat{\psi^*_n}(2k\pi))_{1\le n\le k_0, k\in\mathbf{Z}}$ is k_0. By the equality from the orthogonal property (60), we have

$$\begin{aligned}
&\sum_{k\in\mathbf{Z}} |\langle f_\epsilon, \psi_{n;0,k}\rangle|^2 \qquad (64)\\
= &\int_{-\pi}^{\pi} |a_\epsilon(\xi)|^2 \Big| \sum_{k\in\mathbf{Z}} \widehat{\psi}(\xi + 2k\pi)\overline{\widehat{\psi}_n(\xi + 2k\pi)}\Big|^2 d\xi\\
= &\int_{-\pi}^{\pi} |a_\epsilon(\xi)|^2 \Big| \sum_{k\in\mathbf{Z}} \overline{\widehat{\psi}(\xi + 2k\pi)}\\
&\times\Big(\widehat{\psi}_n(\xi + 2k\pi) - \sum_{n'=1}^{k_0} a_{n,n'}\widehat{\psi^*_{n'}}(\xi + 2k\pi)\Big)\Big|^2 d\xi\\
\le\ & C\int_{-\pi}^{\pi} |a_\epsilon(\xi)|^2 \Big(\sum_{s=0}^{M-1} \Big|m\Big(\frac{\xi + 2s\pi}{M}\Big)\Big|^2\Big)\\
&\times\Big(\sum_{s=0}^{M-1} \Big(\sum_{k\in\mathbf{Z}} \Big|\widehat{\phi}\Big(\frac{\xi + 2s\pi}{M} + 2k\pi\Big)\Big| \times \Big|\widehat{\psi}_n(\xi + 2s\pi + 2kM\pi)\\
&- \sum_{n'=1}^{k_0} a_{n,n'}\widehat{\psi}^*_{n'}(\xi + 2s\pi + 2Mk\pi)\Big|\Big)^2\Big) d\xi\\
\le\ & C\epsilon^2 \int_{-\pi}^{\pi} |a_\epsilon(\xi)|^2 \sum_{s=0}^{M-1} \Big|m\Big(\frac{\xi + 2s\pi}{M}\Big)\Big|^2 d\xi\\
\le\ & C\epsilon^2 \|f_\epsilon\|_2^2,
\end{aligned}$$

where we have used the construction of $a_{n,n'}$ and (63) to obtain the second and third inequalities, respectively. For $L_1 \le j \le -1$, we also have

$$\begin{aligned}
&\sum_{k\in\mathbf{Z}} |\langle f_\epsilon, \psi_{n;j,k}\rangle|^2 \\
= \ & M^j \int_{-\pi}^{\pi} \Big| \sum_{k\in\mathbf{Z}} a_\epsilon(M^j(\xi+2k\pi))\widehat{\psi}(M^j(\xi+2k\pi))\overline{\widehat{\psi}_n(\xi+2k\pi)}\Big|^2 d\xi \\
= \ & M^j \int_{-\pi}^{\pi} |a_\epsilon(M^j\xi)|^2 \Big|\sum_{k\in\mathbf{Z}} \widehat{\psi}(M^j\xi+2k\pi)\overline{\widehat{\psi}_n(\xi+2M^{-j}k\pi)}\Big|^2 d\xi,
\end{aligned}$$

where the fact that a_ϵ is supported in a small neighborhood of $2\pi\mathbf{Z}$ has been used. Hence,

$$\begin{aligned}
\sum_{k\in\mathbf{Z}} |\langle f_\epsilon, \psi_{n;j,k}\rangle|^2 \ \le \ & M^j \int_{-\pi}^{\pi} |a_\epsilon(M^j\xi)|^2 \sum_{s=0}^{M-1} \Big| m\Big(\frac{M^j\xi+2s\pi}{M}\Big)\Big|^2 \\
& \sum_{s=0}^{M-1} \Big(\sum_{k\in\mathbf{Z}} \Big|\widehat{\phi}\Big(M^{j-1}\xi + \frac{2s\pi}{M} + 2k\pi\Big)\Big| \\
& \times \big|\widehat{\psi}_n(\xi + 2M^{-j}(s+kM)\pi)\big|\Big)^2 d\xi \\
\le \ & C\epsilon^2 \int_{-\pi}^{\pi} |a_\epsilon(M^j\xi)|^2 \sum_{s=0}^{M-1} \Big| m\Big(M^{j-1}\xi + \frac{2s\pi}{M}\Big)\Big|^2 d\xi \\
\le \ & C\epsilon^2 \|f_\epsilon\|_2^2, \qquad (65)
\end{aligned}$$

where we have used the estimate:

$$\sum_{k\in\mathbf{Z}} \Big|\widehat{\phi}\Big(M^{j-1}\xi + \frac{2s\pi}{M} + 2kM\pi\Big)\Big| \times |\widehat{\psi}_n(\xi + 2M^{-j}(s+kM)\pi)| \le \epsilon \qquad (66)$$

for all $|\xi| \le M^{-j}\delta$ and sufficiently small δ. Here, the estimate (66) follows, since $\widehat{\psi}_n(0) = 0$ and $\widehat{\psi}_n(2M^{-j}k\pi) = H_n(0)\widehat{\phi}(2M^{-j-1}k\pi) = 0$ for all nonzero integer k. Combining (64) and (64) yields

$$\sum_{j=L_1}^{0} \sum_{k\in\mathbf{Z}} |\langle f_\epsilon, \psi_{n;j,k}\rangle|^2 \le C\epsilon^2\|f_\epsilon\|_2^2 \ne 0 \qquad (67)$$

for some positive constant C independent of f_ϵ and ϵ, which contradicts with the assumption (viii).

(Proof of (vii) $\Longrightarrow$ (vi)): Since $\widehat{\psi}_n(\xi) = H_n(\xi/M)\widehat{\phi}(\xi/M)$ and ϕ has linear independent shifts, the rank of $(\widehat{\psi}_n(\xi+2k\pi))_{1\le n\le N, k\in\mathbf{Z}}$ is the same as that of $\mathbf{H}(\xi)$. On the other hand, the rank of $\mathbf{H}(0)$ is at most $M-1$, because $H_n(0) = 0$ for all $1 \le n \le N$ by the frame property of Ψ. Therefore the rank of $\mathbf{H}(\xi)$ is strictly less than M on a small neighborhood of the origin, which, together with Proposition 2.1, implies that

$$\begin{aligned}
\mathbf{A}(\xi) \ := \ & \mathrm{diag}\Big(S(\xi), \ldots, S(\xi+2(M-1)\pi/M)\Big) \\
& - S(M\xi)\Big(\overline{H_0(\xi+2m\pi/M)}H_0(\xi+2m'\pi/M)\Big)_{0\le m,m'\le M-1} \qquad (68)
\end{aligned}$$

is singular in a small neighborhood of the origin. So

$$\det \mathbf{A}(\xi) \equiv 0, \tag{69}$$

where we have also used the fact that the determinant of $\mathbf{A}$ is a trigonometric polynomial. It is known that for $\mathbf{A} \in \mathbf{C}^{n\times n}$ and $\mathbf{v}, \mathbf{w} \in \mathbf{C}^n$, we have

$$\det(\mathbf{A} - \mathbf{v}\mathbf{w}^T) = \det \mathbf{A} - \mathbf{w}^T \mathbf{A}^{\#}\mathbf{v}, \tag{70}$$

where $\mathbf{A}^{\#}$ denotes the adjoint matrix whose entries $A^{\#}_{i,k}$ are the cofactors $A_{k,i}$ of $\mathbf{A}$. Thus,

$$\begin{aligned}\det \mathbf{A}(\xi) &= \prod_{m=0}^{M-1} S\Big(\xi + \frac{2m\pi}{M}\Big)\\ &\quad -S(M\xi)\sum_{m=0}^{M-1}\Big|H\Big(\xi+\frac{2m\pi}{M}\Big)\Big|^2 \prod_{0\le i\ne m\le M-1} S\Big(\xi+\frac{2i\pi}{M}\Big)\end{aligned}$$

by (67) and (70). Hence (vi) follows from (69).

(Proof of (vi)$\Longrightarrow$(v)): Let $\mathbf{A}(\xi)$ be as in (67). Also, let $S_1(\xi)$ be a trigonometric polynomial with real coefficients and satisfy $|S_1(\xi)|^2 = S(\xi)$. The existence of $S_1(\xi)$ follows from the Riesz Lemma. Then we can write $\mathbf{A}(\xi)$ as

$$\mathbf{A}(\xi) = \overline{\mathbf{D}(\xi)}(I_M - \overline{\alpha_0(\xi)}\alpha_0(\xi))\mathbf{D}(\xi), \tag{71}$$

where $\mathbf{D}(\xi) = \mathrm{diag}(S_1(\xi), \ldots, S_1(\xi+2(M-1)\pi/M))$ and $\alpha_0(\xi) = (\tilde H_0(\xi), \ldots, \tilde H_0(\xi+2(M-1)\pi/M))^T$ and $\tilde H_0(\xi) = S_1(M\xi)H_0(\xi)/S_1(\xi)$. By Proposition 2.1, $\tilde H_0(\xi)$ is continuous on $\mathbf{R}$ and $\mathbf{A}(\xi) = \overline{\mathbf{H}(\xi)}^T\mathbf{H}(\xi)$. Therefore, it suffices to prove that $\mathbf{A}(\xi)$ has rank $M-1$ for all $\xi \in \mathbf{R}$. By the assumption, $\alpha_0(\xi)$ is a unit vector for all ξ, which implies that $I - \overline{\alpha_0(\xi)}\alpha_0(\xi)^T$ has rank $M-1$ for all $\xi \in \mathbf{R}$. For any $\xi_1 \in \mathbf{R}$ such that $S(\xi_1 + 2m\pi/M) \neq 0$ for all $m \in \mathbf{Z}$, $\mathbf{D}(\xi_1)$ is nonsingular and, hence, it follows from (71) that $\mathbf{A}(\xi_1)$ has rank $M-1$. For any $\xi_1 \in \mathbf{R}$ such that $S(\xi_1 + 2m\pi/M) = 0$ for some $m \in \mathbf{Z}$, it follows from the assumption (vi) that $S(M\xi_1) = 0$ and $S(\xi_1 + 2m'\pi/M) \neq 0$ for all $m' - m \notin M\mathbf{Z}$ (see [12, 29]). Therefore, $\mathrm{diag}(S(\xi_1), \ldots, S(\xi_1 + 2(M-1)\pi/M))$ has rank $M-1$, which together with (67), implies that $\mathbf{A}(\xi_1) = \mathrm{diag}\big(S(\xi_1), \ldots, S(\xi_1 + 2(M-1)\pi/M)\big)$ has rank $M-1$ for all those ξ_1 with $S(\xi_1 + 2m\pi/M) = 0$ for some $0 \le m \le M-1$. This completes the proof of the assertion (v).

(Proof of (v)$\Longrightarrow$(iv)): Assume that $\mathbf{H}(\xi)$ has rank $M-1$ for all $\xi \in \mathbf{R}$. Then the matrix $\mathbf{A}(\xi)$ in (67) is singular by Proposition 2.1, which implies (57). Therefore by Lemma 4.5, there exist some trigonometric polynomials $H^*_m, 1 \le m \le M-1$, so that the functions $\psi^*_1, \ldots, \psi^*_{M-1}$, defined by

$$\widehat{\psi}_s(\xi) = H^*_s(\xi/M)\widehat{\phi}(\xi/M),\ 1 \le m \le M-1,$$

generate a tight affine frame of L^2 and have the same fundamental function S as the one of $\psi_1, \ldots, \psi_N$. Note that the rank of the matrix $(\widehat{\psi}^*_m(\xi + 2k\pi))_{1\le m\le M-1, k\in\mathbf{Z}}$ is the same as the rank of $\mathbf{H}(\xi)$, and hence is equal to $M-1$. Therefore the shifts of $\psi^*_m, 1 \le m \le M-1$, form a Riesz basis of the corresponding shift-invariant space $V^2(\psi^*_1, \ldots, \psi^*_{M-1})$. So it suffices to prove that

$$V^2(\psi^*_1, \ldots, \psi^*_{M-1}) = V^2(\psi_1, \ldots, \psi_N). \tag{72}$$

As before, let $S_1(\xi)$ be the trigonometric polynomial with real coefficients so that $|S_1(\xi)|^2 = S(\xi)$. Define $\tilde{H}_0(\xi) = S_1(M\xi)H_0(\xi)/S_1(\xi)$ and $\tilde{H}_n(\xi) = H_n(\xi)/S_1(\xi), 1 \le n \le N$, and similarly define $\tilde{H}_0^*(\xi) = S_1(M\xi)H_0(\xi)/S_1(\xi)$ and $\tilde{H}_s^*(\xi) = H_s^*(\xi)/S_1(\xi), 1 \le s \le M-1$. By Proposition 2.1, the vectors

$$\mathbf{v}_s = \left(\tilde{H}_s^*(\xi), \dots, \tilde{H}_s^*(\xi + 2(M-1)\pi/M)\right)^T,\ 0 \le s \le M-1,$$

form an orthonormal basis of $\mathbf{R}^M$ for any $\xi \in \mathbf{R}$, and the vectors

$$\mathbf{u}_n = \left(\tilde{H}_n(\xi), \dots, \tilde{H}_n(\xi + 2(M-1)\pi/M)\right)^T,\ 0 \le n \le N,$$

form a tight frame of $\mathbf{R}^M$ for any $\xi \in \mathbf{R}$. Thus, we have

$$\mathbf{v}_s = \sum_{n=0}^{N} \langle \mathbf{v}_s, \mathbf{u}_n \rangle \mathbf{u}_n,\ 0 \le s \le M-1,$$

and

$$\mathbf{u}_n = \sum_{s=0}^{M-1} \langle \mathbf{u}_n, \mathbf{v}_s \rangle \mathbf{v}_s,\ 0 \le n \le N.$$

Recall that $\mathbf{u}_0 = \mathbf{v}_0$, which implies that

$$\langle \mathbf{v}_s, \mathbf{u}_0 \rangle = \langle \mathbf{v}_s, \mathbf{v}_0 \rangle = 0,\ 1 \le s \le M-1.$$

By the tight frame property, we have $\sum_{n=0}^{N} \mathbf{u}_n \overline{\mathbf{u}_n^T} = I_M$, where I_M stands for the M-dimensional identity matrix. Thus,

$$|\langle \mathbf{u}_0, \mathbf{u}_0 \rangle|^2 + \sum_{n=1}^{N} |\langle \mathbf{u}_0, \mathbf{u}_n \rangle|^2 = \langle \mathbf{u}_0, \mathbf{u}_0 \rangle,$$

which together with $|\mathbf{u}_0| = 1$, implies that

$$\langle \mathbf{u}_n, \mathbf{v}_0 \rangle = \langle \mathbf{u}_n, \mathbf{u}_0 \rangle = 0,\ 1 \le n \le N.$$

Therefore, we obtain

$$\mathbf{v}_s = \sum_{n=1}^{N} \langle \mathbf{v}_s, \mathbf{u}_n \rangle \mathbf{u}_n,\ 1 \le s \le M-1,$$

and

$$\mathbf{u}_n = \sum_{s=1}^{M-1} \langle \mathbf{u}_n, \mathbf{v}_s \rangle \mathbf{v}_s,\ 1 \le n \le N.$$

We can now formulate the above two identities as

$$\tilde{H}_s^*(\xi) = \sum_{n=1}^{N} b_{sn}(M\xi) \tilde{H}_n(\xi),\ 1 \le s \le M-1, \tag{73}$$

and

$$\tilde{H}_n(\xi) = \sum_{s=1}^{M-1} a_{ns}(M\xi)\tilde{H}_s^*(\xi),\ 1 \le n \le N, \tag{74}$$

where $a_{ns}(\xi)$ and $b_{sn}(\xi)$ are trigonometric polynomials. Multiplying $S_1(\xi)\widehat{\phi}(\xi)$ to both sides of (73) and (74) yields

$$\widehat{\psi}_s^*(\xi) = \sum_{n=1}^{N} b_{sn}(\xi)\widehat{\psi}_n(\xi),\ 1 \le s \le M-1,$$

and

$$\widehat{\psi}_n(\xi) = \sum_{s=1}^{M-1} a_{ns}(\xi)\widehat{\psi}_s^*(\xi),\ 1 \le n \le N.$$

This proves (72), and hence the assertion (iv).

(Proof of (iv)$\Longrightarrow$(iii)): By the assumption (iv), the shift-invariant space $V^2(\Psi)$ is closed. Then the assertion (iii) follows from Proposition 2.2.

(Proof of (iii)$\Longrightarrow$(viii)): This implication is obvious.

(Proof of (ii)$\Longleftrightarrow$(v)): For any $f \in H^\alpha$, we have

$$\begin{aligned}\widehat{Q_0 f}(\xi) &= \sum_{n=1}^{N}\sum_{k\in\mathbf{Z}} \widehat{f}(\xi+2k\pi)\overline{H_n\Big(\frac{\xi+2k\pi}{M}\Big)} \\ &\quad \times \overline{\widehat{\phi}\Big(\frac{\xi+2k\pi}{M}\Big)} H_n\Big(\frac{\xi}{M}\Big)\widehat{\phi}\Big(\frac{\xi}{M}\Big) \\ &= \sum_{n=1}^{N}\sum_{s=0}^{M-1} \overline{H_n\Big(\frac{\xi+2s\pi}{M}\Big)} A\Big(\frac{\xi+2s\pi}{M}\Big) H_n\Big(\frac{\xi}{M}\Big)\widehat{\phi}\Big(\frac{\xi}{M}\Big)\end{aligned}$$

for some $A \in L^2_{2\pi}$. Conversely, for any $A \in L^2_{2\pi}$, the function f_0, defined by

$$\widehat{f}_0(\xi) = A\Big(\frac{\xi}{M}\Big)\Big(\sum_{k\in\mathbf{Z}}\Big|\widehat{\phi}\Big(\frac{\xi}{M}+2k\pi\Big)\Big|^2\Big)^{-1}\widehat{\phi}\Big(\frac{\xi}{M}\Big),$$

belongs to H^α and satisfies

$$\widehat{Q_0 f_0}(\xi) = \sum_{n=1}^{N}\sum_{s=0}^{M-1} \overline{H_n\Big(\frac{\xi+2s\pi}{M}\Big)} A\Big(\frac{\xi+2s\pi}{M}\Big) H_n\Big(\frac{\xi}{M}\Big)\widehat{\phi}\Big(\frac{\xi}{M}\Big).$$

This shows that the space $\widehat{Q_0 H^\alpha} = \{\widehat{g} : g \in Q_0 H^\alpha\}$ is characterized by

$$\begin{aligned}\widehat{Q_0 H^\alpha} &= \Big\{\sum_{n=1}^{N}\sum_{s=0}^{M-1} \overline{H_n\Big(\frac{\xi+2s\pi}{M}\Big)} A\Big(\frac{\xi+2s\pi}{M}\Big) \\ &\qquad \times H_n\Big(\frac{\xi}{M}\Big)\widehat{\phi}\Big(\frac{\xi}{M}\Big) : \quad A \in L^2_{2\pi}\Big\}.\end{aligned} \tag{75}$$

One may easily verify that for the space $V^2(\Psi)$, the corresponding space $\widehat{V^2(\Psi)} := \{\widehat{f} : f \in V^2(\Psi)\}$ in the Fourier domain is

$$\widehat{V^2(\Psi)} = \Big\{ \sum_{n=1}^{N} A_n(\xi) H_n\Big(\frac{\xi}{M}\Big)\widehat{\phi}\Big(\frac{\xi}{M}\Big) :\ A_n \in L^2_{2\pi},\ 1 \le n \le N \Big\}. \tag{76}$$

Since ϕ has linear independent shifts, it follows from (75) and (76) that $Q_0 H^\alpha = V^2(\Psi)$ if and only if for any $A_n \in L^2_{2\pi}, 1 \le n \le N$, there exists $A \in L^2_{2\pi}$ so that

$$\begin{aligned}\sum_{n=1}^{N}\sum_{s=0}^{M-1} \overline{H_n\Big(\frac{\xi+2s\pi}{M}\Big)} A\Big(\frac{\xi+2s\pi}{M}\Big) H_n\Big(\frac{\xi+2s'\pi}{M}\Big) \\ = \sum_{n=1}^{N} A_n(\xi) H_n\Big(\frac{\xi+2s'\pi}{M}\Big) \quad \forall\, 0 \le s' \le M-1;\end{aligned} \tag{77}$$

that is,

$$\begin{aligned}\Big(A_1(\xi), \ldots, A_n(\xi)\Big)\mathbf{H}\Big(\frac{\xi}{M}\Big) \\ = \Big(A\Big(\frac{\xi}{M}\Big), \ldots, A\Big(\frac{\xi+2(M-1)\pi}{M}\Big)\Big)\overline{\mathbf{H}\Big(\frac{\xi}{M}\Big)}^T \mathbf{H}\Big(\frac{\xi}{M}\Big).\end{aligned} \tag{78}$$

By the Smith decomposition, we have that $\mathbf{H}(\xi) = \mathbf{H}_1(\xi)\mathbf{D}(\xi)\mathbf{H}_2(\xi)$, where $\det \mathbf{H}_1(\xi)$ and $\det \mathbf{H}_2(\xi)$ are nonzero monomials and $\mathbf{D}(\xi)$ is a diagonal matrix. This, together with (77) and (78), proves that $Q_0 H^\alpha = V^2(\Psi)$ if and only if the rank of the matrix $\mathbf{H}(\xi)$ is independent of $\xi \in \mathbf{R}$. Therefore, since the rank of $\mathbf{H}(0)$ is $M-1$ by (8) and the fact that $H_0(2m\pi/M) = 0$ for $1 \le m \le M-1$, the equivalence of the assertions (v) and (ii) follows.

(Proof of (i)$\Longrightarrow$(vi)): Let $S_1(\xi), \alpha_0(\xi), \tilde{H}_0(\xi), \mathbf{A}(\xi)$ be as in the proof of (vi)$\Longrightarrow$(v). By (67) and Proposition 2.1, $\tilde{H}_0(\xi)$ is continuous and $\mathbf{A}(\xi) = \overline{\mathbf{H}(\xi)}^T\mathbf{H}(\xi)$. Therefore by (71), it suffices to show that $\mathbf{H}(\xi)$ is not of full rank for any $\xi \in \mathbf{R}$, since this implies that $\alpha_0(\xi)$ is a unit vector for any $\xi \in \mathbf{R}$ and the assertion (vi) then follows. Suppose, on the contrary, that $\mathbf{H}(\xi_0)$ is of full rank for some $\xi_0 \in \mathbf{R}$. By Lemma 4.6, the closure of $Q_0 H^\alpha$ in L^2 is V_1, which together with our assumption (i), leads to $Q_0 H^\alpha = V^2(\phi(M\cdot), \ldots, \phi(M\cdot - M + 1)) = V_1$. Hence, $V^2(\Psi) = Q_0 H^\alpha = V_1$ since $Q_0 H^\alpha \subset V^2(\Psi) \subset V_1$, and then (v) holds by the equivalence of the assertions (ii) and (v), which is a contradiction.

(Proof of (ii)$\Longrightarrow$(i)): By the equivalence of the assertions (ii) and (iv), the space $V^2(\Psi)$ is a closed subspace of L^2. This, together with the assumption (ii), proves the assertion (i). ∎

4.3. Proof of Theorem 4.1

First we prove (i)$\Longrightarrow$(ii). By Theorem 4.4, we have

$$\sum_{m=0}^{M-1} \frac{S(M\xi)}{S(\xi+2m\pi/M)}\Big|H\Big(\xi+\frac{2m\pi}{M}\Big)\Big|^2 = 1,\ \xi \in \mathbf{R}. \tag{79}$$

Therefore,

$$\prod_{m=0}^{M-1} S\Big(\xi+\frac{2m\pi}{M}\Big) \;=\; S(M\xi)\sum_{m=0}^{M-1}\prod_{0\le m'\ne m\le M-1} S\Big(\xi+\frac{2m'\pi}{M}\Big) \times H\Big(\xi+\frac{2m\pi}{M}\Big)H\Big(-\xi-\frac{2m\pi}{M}\Big),\ \xi\in\mathbf{C}. \tag{80}$$

Note that all the roots of S are real, since otherwise the right-hand side of (79) becomes zero at ξ_0/M while the left-hand side does not, where ξ_0 is a root of S with nonzero imaginary part so that the magnitude of the imaginary part of ξ_0 is the minimal root with nonzero imaginary part, and this leads to a contradiction. Also we note from Theorem 4.3 that $S(\xi)\ne 0$ for all $\xi\in\mathbf{R}$. Therefore $S(\xi)$ is a constant function. Substituting this back into (79) and using the linear independence of ϕ yields that ϕ has orthonormal shifts (see [15, 32]). Let $\psi_1^*,\dots,\psi_{M-1}^*$ be the orthonormal wavelets generated from the above multiresolution, which is also a tight affine frame. Moreover, the fundamental function of resolution corresponding to the above tight affine frame is the same as the one with generators $\psi_1,\dots,\psi_N$ since both are equal to one. Using the same method as the one in the proof of the implication (v)$\Longrightarrow$(iv) of Theorem 4.4, the space spanned by the shifts of $\psi_1^*,\dots,\psi_{M-1}^*$ is the same as the one spanned by the shifts of $\psi_1,\dots,\psi_N$. This concludes that $V^2(\Psi)$ is the orthogonal L^2-complement of V_0 in V_1.

Next we prove (ii)$\Longrightarrow$(iii). By the tight frame property, we have $\sum_{j\in\mathbf{Z}} Q_j = I$. Thus, by the orthogonal property of the spaces $W_j := Q_jH^\alpha$ from our assumption (ii), we have

$$\|f\|_2^2=\sum_{j\in\mathbf{Z}}\langle Q_jf,f\rangle=\langle Q_0f,f\rangle$$

for any $f\in W_0$. Hence (iii) is valid.

Finally, we prove (iii)$\Longrightarrow$(i). By Theorem 4.4, the spaces $Q_jH^\alpha, 0\le j\in\mathbf{Z}$, are closed subspaces of L^2. Then by Theorem 4.3, it suffices to prove that $S(\xi)$ is a nonzero constant, where $S(\xi)$ is the fundamental function of resolution of the tight affine frame Ψ. By Theorem 4.4, the function $S(\xi)$ satisfies (54), which implies that $\tilde H_0(\xi) := S_1(M\xi)H_0(\xi)/S_1(\xi)$ is a trigonometric polynomial and satisfies

$$\sum_{m=0}^{M-1}\Big|\tilde H_0\Big(\xi+\frac{2m\pi}{M}\Big)\Big|^2=1, \tag{81}$$

where $S_1(\xi)$ is the trigonometric polynomial with real coefficients so that $|S_1(\xi)|^2=S(\xi)$. By unitary extension, there exist trigonometric polynomials $\tilde H_1^*,\cdots,\tilde H_{M-1}^*$ so that

$$\sum_{m=0}^{M}\tilde H_s\Big(\xi+\frac{2m\pi}{M}\Big)\overline{\tilde H_t\Big(\xi+\frac{2m\pi}{M}\Big)}=\delta_{st},\quad 0\le s,t\le M-1,$$

or in matrix formulation,

$$\overline{\mathbf{U}(\xi)}\mathbf{U}(\xi)^T=I_M, \tag{82}$$

where $\mathbf{U}(\xi)=(\alpha_0(\xi)\quad\dots\quad\alpha_{M-1}(\xi))$ and $\alpha_s(\xi)=\big(\tilde H_s(\xi),\dots,\tilde H_s(\xi+2(M-1)\pi/M)\big)^T, 0\le s\le M-1$. By (15) and the assumption (iii), we have

$$\Big(\mathbf{H}(\xi)\mathrm{diag}\big(\Phi(\xi),\dots,\Phi(\xi+2(M-1)\pi/M)\big)\overline{\mathbf{H}(\xi)}^T\Big)^2$$

$$= C_0\mathbf{H}(\xi)\mathrm{diag}\big(\Phi(\xi),\ldots,\Phi(\xi+2(M-1)\pi/M)\big)\overline{\mathbf{H}(\xi)}^T \qquad (83)$$

for some nonzero constant C_0, where $\mathbf{H}(\xi)$ is defined as in (53), and $\Phi(\xi)=\sum_{k\in\mathbf{Z}}|\widehat{\phi}(\xi+2k\pi)|^2$. By Proposition 2.1, we have

$$\begin{aligned}\overline{\mathbf{H}(\xi)}^T\mathbf{H}(\xi) &= \mathrm{diag}(S_1(\xi),\ldots,S_1(\xi+2(M-1)\pi/M)\big)\\ &\times(I_M-\overline{\alpha_0(\xi)}\alpha_0(\xi)^T)\mathrm{diag}(S_1(\xi),\ldots,S_1(\xi+2(M-1)\pi/M)\big).\end{aligned} \qquad (84)$$

Combining (82) and (83) and using $(I_M-\overline{\alpha_0(\xi)}\alpha_0(\xi)^T)^2=I_M-\overline{\alpha_0(\xi)}\alpha_0(\xi)^T$, we see that the matrix

$$\begin{aligned}\mathbf{B}(\xi) \;:=\; &\big(I_M-\overline{\alpha_0(\xi)}\alpha_0(\xi)^T\big)\mathrm{diag}\big(\tilde\Phi(\xi),\ldots,\tilde\Phi\big(\xi+\frac{2(M-1)\pi}{M}\big)\big)\\ &\times\big(I_M-\overline{\alpha_0(\xi)}\alpha_0(\xi)^T\big).\end{aligned}$$

satisfies

$$\mathbf{B}(\xi)^2=C_0\mathbf{B}(\xi), \qquad (85)$$

where $\tilde\Phi(\xi)=|S_1(\xi)|^2\Phi(\xi)$. On the other hand, we have

$$\begin{aligned}\mathbf{B}(\xi) &= \overline{\mathbf{U}(\xi)}\begin{pmatrix}0 & 0\\ 0 & I_{M-1}\end{pmatrix}\mathbf{U}(\xi)^T\\ &\times\mathrm{diag}\Big(\tilde\Phi(\xi),\ldots,\tilde\Phi\big(\xi+\frac{2(M-1)\pi}{M}\big)\Big)\overline{\mathbf{U}(\xi)}\begin{pmatrix}0 & 0\\ 0 & I_{M-1}\end{pmatrix}\mathbf{U}(\xi)^T\\ &=: \overline{\mathbf{U}(\xi)}\begin{pmatrix}0 & 0\\ 0 & \beta(\xi)\end{pmatrix}\mathbf{U}(\xi)^T.\end{aligned} \qquad (86)$$

Here, $\beta(\xi)$ has rank $M-1$ for almost all ξ, since

$$\begin{aligned}&(\overline{v_1},\ldots,\overline{v_{M-1}})\beta(\xi)(v_1,\ldots,v_{M-1})^T\\ &\qquad=\sum_{m=0}^{M-1}\Big|\sum_{t=1}^{M-1}v_t\tilde H_t\big(\xi+\frac{2m\pi}{M}\big)\Big|^2\tilde\Phi\big(\xi+\frac{2m\pi}{M}\big)\neq 0\end{aligned}$$

for any nonzero vector $(v_1,\ldots,v_{M-1})^T\in\mathbf{R}^{M-1}$ and any ξ satisfying $S(\xi+2m\pi/M)\neq 0, 0\le m\le M-1$. Therefore,

$$\beta(\xi)=C_0I_{M-1} \qquad (87)$$

by (82), (85) and (85). Substituting the above formula of $\beta(\xi)$ into (85) and applying (82) yields

$$\begin{aligned}\mathbf{B}(\xi) &= C_0\overline{\mathbf{U}(\xi)}\mathbf{U}(\xi)^T-C_0\overline{\mathbf{U}(\xi)}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\mathbf{U}(\xi)^T\\ &= C_0(I_M-\overline{\alpha_0(\xi)}\alpha_0(\xi)^T).\end{aligned} \qquad (88)$$

Then comparing the non-diagonal terms of the both sides of (88), we obtain

$$\tilde\Phi\big(\xi+\frac{2\pi s}{M}\big)+\tilde\Phi\big(\xi+\frac{2\pi s'}{M}\big)-\sum_{m=0}^{M-1}\Big|\tilde H_0\big(\xi+\frac{2\pi m}{M}\big)\Big|^2\tilde\Phi\big(\xi+\frac{2\pi m}{M}\big)$$

$$= C_0 \quad 0 \le s \ne s' \le M-1. \tag{89}$$

Now we divide the argument into two cases, $M \ge 3$ and $M = 2$, to show that S is a nonzero constant. For $M \ge 3$, applying (88) with $(s, s') = (0, 1)$ and $(s, s') = (0, 2)$ leads to $\tilde\Phi(\xi) = \tilde\Phi(\xi + 2\pi/M)$. Thus $\tilde\Phi(\xi) = D(M\xi)$ for some trigonometric polynomial D. Substituting this back to the definition of $\mathbf{B}(\xi)$, we obtain $\mathbf{B}(\xi) = D(M\xi)(I_M - \overline{\alpha_0(\xi)}\alpha_0(\xi)^T)$. This, together with (88), yields $D(M\xi) = C_0$, and hence $\tilde\Phi(\xi)$ is a constant function. Recall that $\tilde\Phi(\xi) = \Phi(\xi)|S_1(\xi)|^2$, Therefore, both $\Phi(\xi)$ and $S(\xi) = |S_1(\xi)|^2$ are constant-valued functions.

For $M = 2$, it follows from (81) and (88) that

$$(\tilde\Phi(\xi) - C_0)|\tilde H_0(\xi+\pi)|^2 = -(\tilde\Phi(\xi+\pi) - C_0)|\tilde H_0(\xi)|^2. \tag{90}$$

By (81), the trigonometric polynomials $|\tilde H_0(\xi+\pi)|^2$ and $|\tilde H_0(\xi)|^2$ do not have any common root. These conclusions, along with (90) itself, leads to the existence of a trigonometric polynomial $D(\xi)$, such that

$$\tilde\Phi(\xi) = C_0 + e^{-i\xi} D(2\xi)|\tilde H_0(\xi)|^2. \tag{91}$$

Also, from the definition of $\tilde\Phi$ and the refinement equation $\hat\phi(M\xi) = H_0(\xi)\hat\phi(\xi)$, it follows that

$$\tilde\Phi(2\xi) = |\tilde H_0(\xi)|^2 \tilde\Phi(\xi) + |\tilde H_0(\xi+\pi)|^2 \tilde\Phi(\xi+\pi). \tag{92}$$

Substituting the formulation (91) of $\tilde\Phi$ into (92) and applying (81), we obtain

$$\begin{aligned} e^{-2i\xi} D(4\xi)|\tilde H_0(2\xi)|^2 &= e^{-i\xi} D(2\xi)\big(|\tilde H_0(\xi)|^4 - |\tilde H_0(\xi+\pi)|^4\big) \\ &= e^{-i\xi} D(2\xi)\big(|\tilde H_0(\xi)|^2 - |\tilde H_0(\xi+\pi)|^2\big). \end{aligned} \tag{93}$$

From $\tilde\Phi(-\xi) = \tilde\Phi(\xi)$, it follows that $D(-\xi) = e^{-i\xi} D(\xi)$. Therefore, by (92) and the above "symmetry" of D, we conclude that $D(\xi) \equiv 0$, since otherwise the degree of the trigonometric polynomial of the left-hand side of (92) is strictly larger than that of the right-hand side. Hence, $\tilde\Phi(\xi)$ is a constant function by (91). This proves that $S(\xi)$ is also a nonzero constant function when $M = 2$.

5. ANGLES BETWEEN V_J AND $Q_J H^\alpha$

Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and let H_1, H_2 be its two nontrivial linear subspaces (which are not necessarily Hilbert subspaces). We consider the angle $\theta \in [0, \pi/2]$ between H_1 and H_2, defined by

$$\cos\theta = \sup_{0\ne f\in H_1, 0\ne g\in H_2} \frac{|\langle f, g\rangle|}{\|f\|\|g\|}.$$

By Theorem 3.5, we have the space decomposition property of the Sobolev space as follows:

$$H^\alpha = \tilde P_0 H^\alpha + \sum_{j=0}^{\infty} Q_j H^\alpha.$$

In this section, we study the angles between the spaces $\tilde{P}_0 H^\alpha$ and $Q_j H^\alpha, j \in \mathbf{Z}$. First we give a characterization of whether or not the angles between those spaces are nonzero, and show that those angles are nonzero if and only if $Q_j H^\alpha$ are closed subspaces of L^2.

THEOREM 5.1. *Let $\beta > 0, \alpha \in (-\beta, \beta)$, and let $\{V_j\}_{j\in\mathbf{Z}}$ be an MRA with a compactly supported scaling function $\phi \in H^\beta$ that has linear independent shifts. Assume that $\Psi \subset V_1$ is a finite collection of compactly supported functions, which generates a tight affine frame of L^2. Then the following five statements are equivalent:*

(i) The angle between $\tilde{P}_0 H^\alpha$ and $Q_j H^\alpha$ (as subspaces of L^2) is nonzero for some $j \geq 0$.

(ii) The angle between $Q_j H^\alpha$ and $Q_{j'} H^\alpha$ (as subspaces of L^2) is nonzero for some $j \neq j'$.

(iii) $Q_0 H^\alpha$ is closed in L^2.

(iv) The angles between between $Q_j H^\alpha$ and $Q_{j'} H^\alpha$ (as subspaces of L^2) are nonzero for all $j \neq j'$.

(v) The angles between $\tilde{P}_0 H^\alpha$ and $Q_{j'} H^\alpha$ (as subspaces of L^2) are nonzero for all $j \geq 0$.

REMARK 5.1. We remark that the above characterization holds when the spaces $\tilde{P}_0 H^\alpha$ and $Q_j H^\alpha, 0 \leq j \leq \mathbf{Z}$, are considered as subspaces of the Sobolev space H^α instead of subspaces of L^2. The proof is almost the same as that of Theorem 5.1, and hence we may safely omit its details.

THEOREM 5.2. *Let $\beta > 0, \alpha \in (-\beta, \beta)$, and let $\{V_j\}_{j\in\mathbf{Z}}$ be an MRA with a compactly supported scaling function $\phi \in H^\beta$ that has linear independent shifts. Assume that $\Psi \subset V_1$ is a finite collection of compactly supported functions, which generates a tight affine frame of L^2. Then the following five statements are equivalent:*

(i) The angle between $\tilde{P}_0 H^\alpha$ and $Q_j H^\alpha$ (as subspaces of L^2) is $\pi/2$ for some $j \geq 0$.

(ii) The angle between $Q_j H^\alpha$ and $Q_{j'} H^\alpha$ (as subspaces of L^2) is $\pi/2$ for some $j \neq j'$.

(iii) Both $Q_0 H^\alpha$ and $\tilde{P}_0 H^\alpha$ are closed in L^2.

(iv) The angles between between $Q_j H^\alpha$ and $Q_{j'} H^\alpha$ (as subspaces of L^2) are $\pi/2$ for all $j \neq j'$.

(v) The angles between $\tilde{P}_0 H^\alpha$ and $Q_{j'} H^\alpha$ (as subspaces of L^2) are $\pi/2$ for all $j \geq 0$.

REMARK 5.2. Let ψ be a Schwartz function such that the support of its Fourier transform $\widehat{\psi}$ is contained in $\{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$ and that $\sum_{j\in\mathbf{Z}} |\widehat{\psi}(2^j\xi)|^2 = 1$ for all $0 \neq \xi \in \mathbf{R}$. Then $\{\psi_{j,k}\}_{j,k\in\mathbf{Z}}$ is a tight affine frame of L^2, and can also be used to characterize Sobolev spaces [18, 19, 20]. Let $Q_j, j \in \mathbf{Z}$, be the frame operator on the j-th level corresponding to the above tight affine frame. One may verify that $Q_0 H^\alpha$ is not closed in L^2, and that the angle between $Q_j H^\alpha$ and $Q_{j'} H^\alpha$ is zero when $|j - j'| \leq 1$ and is given by $\pi/2$ otherwise. So it gives rise to a completely different phenomenon as compared to the topological property of closedness of the range $Q_j H^\alpha$ of the frame operator Q_j, and the angle between ranges

$Q_j H^\alpha$ at different levels in Theorems 4.1 and 5.1. We believe that the main reason is that this tight affine frame system is not associated with an MRA.

If the angle between $\tilde{P}_0 H^\alpha$ and $Q_j H^\alpha$ is nonzero, we have the following estimate of this angle via the Sobolev exponent of the scaling function.

THEOREM 5.3. *Let $\beta > 0, |\alpha| < \beta$, and $\{V_j\}_{j\in\mathbf{Z}}$ be an MRA generated by a compactly supported scaling function $\phi \in H^\beta$ that has linear independent shifts. Let $\Psi \subset V_1$ be a finite collection of compactly supported functions in V_1, which generates a tight affine frame of L^2, and assume that $Q_0 H^\alpha$ is closed. Then the angle θ_j between $\tilde{P}_0 H^\alpha$ and $Q_j H^\alpha$ (as subspaces of L^2) and the angle $\theta_{j,j'}$ between $Q_{j'} H^\alpha$ and $Q_j H^\alpha, j, j' \in \mathbf{Z}$ (also as subspaces of L^2) satisfy*

$$|\cos\theta_j| \le CM^{-j\beta},\ 0 \le j \in \mathbf{Z}, \tag{94}$$

and

$$|\cos\theta_{j,j'}| \le CM^{-|j-j'|\beta},\ j, j' \in \mathbf{Z}, \tag{95}$$

respectively, where C is a positive constant independent of j, j'.

REMARK 5.3. The estimate in (5.3) cannot be improved in general. For example, let $\{V_j\}_{j\in\mathbf{Z}}$ be the MRA with the characteristic function $\chi_{[0,1]}$ on the unit interval $[0,1]$ as its scaling function. The function $\psi := \chi_{[0,3/2]} - \chi_{[3/2,3]}$ is a tight affine frame [30]. For $f = \chi_{[0,1]}$ and $g = 2^{j/2}\psi(2^j\cdot -1)$, we see that $\langle f, g\rangle = -2^{-j/2}$, which implies that the angle θ_j between $\tilde{P}_0 H^\alpha$ and $Q_j H^\alpha$ satisfies $|\cos\theta_j| \ge 2^{-j/2}$. On the other hand, $\phi \in H^\beta$ for all $0 < \beta < 1/2$.

In general, we also have the following result on the converse of the above theorem.

THEOREM 5.4. *Let $\{V_j\}_{j\in\mathbf{Z}}$ be an MRA with a compactly supported scaling function $\phi \in L^2$ that has linear independent shifts, and let $\Psi \subset V_1$ be a finite collection of compactly supported functions which generates a tight affine frame of L^2. Assume that $V^2(\Psi)$ is closed. If the angle θ_j between V_0 and $W_j = \{M^{j/2} f(M^j\cdot) : f \in V^2(\Psi)\}$ (as subspaces of L^2) satisfies*

$$0 < |\cos\theta_j| \le CM^{-j\gamma},\ 0 \le j \in \mathbf{Z}, \tag{96}$$

where C and γ are positive constants independent of $j \ge 0$, then $\phi \in H^\beta$ for all $\beta < \gamma$.

REMARK 5.4. The lower bound assumption in (96) cannot be dropped in general, since for a scaling function $\phi \in L^2$ with orthonormal shifts, the angle θ_j between the corresponding spaces V_0 and W_j is always $\pi/2$, or $\cos\theta_j = 0$ for all $j \ge 0$.

REMARK 5.5. For the affine frame operators Q_j, we conclude from Theorems 4.1, 4.4, 5.1, 5.2, 5.3 and 5.4 that under the assumption in Theorem 5.1, there are three possible geometrical structures associated with those affine frame operators:

(i) The angles between different $Q_jH^\alpha, j \in \mathbf{Z}$, are always zero (or equivalently Q_0H^α is not closed in L^2, or equivalently $\{\psi(\cdot - k) : \psi \in \Psi, k \in \mathbf{Z}\}$ is not a frame).

(ii) The angles between different $Q_jH^\alpha, j \in \mathbf{Z}$, are always $\pi/2$ (or equivalently both Q_0H^α and $\tilde{P}_0H^\alpha$ are closed in L^2, or equivalently $\{\psi(\cdot - k) : \psi \in \Psi, k \in \mathbf{Z}\}$ is a tight frame).

(iii) The angles between different $Q_jH^\alpha, j \in \mathbf{Z}$, are always in the open interval $(0, \pi/2)$ (or equivalently Q_0H^α is closed in L^2 but $\tilde{P}_0H^\alpha$ is not closed in L^2, or equivalently $\{\psi(\cdot - k) : \psi \in \Psi, k \in \mathbf{Z}\}$ is a frame, but not a tight frame). In this case, the asymptotic behaviour of those angles is related to the Sobolev exponent of the scaling function ϕ.

5.1. Proof of Theorem 5.1

First we prove (i) $\Longrightarrow$ (iii), and (ii) $\Longrightarrow$ (iii). Suppose, on the contrary, that Q_0H^α is not closed in L^2. By the argument used in the proof of (i)$\Longrightarrow$(vi) of Theorem 4.4, the rank of $(\widehat{\psi}(\xi_0 + 2k\pi))_{\psi\in\Psi,k\in\mathbf{Z}}$ is M for some $\xi_0 \in \mathbf{R}$, which implies that the L^2-closure of Q_jH^α is V_{j+1} by Lemma 4.6. Thus, the angles between $\tilde{P}_0H^\alpha \subset V_0$ and $Q_jH^\alpha, j \geq 0$, and between different Q_jH^α are always zero, since $V_j \subset V_{j+1}$ by the definition of an MRA. This leads to a contradiction.

Next, we prove (iii)$\Longrightarrow$ (iv), and (iii) $\Longrightarrow$ (v). By the property of dilation invariance and the nest condition $V_j \subset V_{j+1}$ in the definition of an MRA, the implications reduce the argument of showing that the angle between V_0 and Q_0H^α is nonzero when Q_0H^α is a closed subspace of L^2. Suppose, on the contrary, that the angle between V_0 and Q_0H^α is zero. Then there exists a nontrivial function f in $V_0 \cap Q_0H^\alpha$, since both V_0 and Q_0H^α are closed subspaces of L^2. Write $\Psi = \{\psi_1, \ldots, \psi_N\}$ and define $H_n, 1 \leq n \leq N$, by $\widehat{\psi}_n(M\xi) = H_n(\xi)\hat{\phi}(\xi)$. By Theorem 4.4, we have

$$\widehat{f}(\xi) = A_0(\xi)H_0\Big(\frac{\xi}{M}\Big)\widehat{\phi}\Big(\frac{\xi}{M}\Big) = \sum_{n=1}^{N} A_n(\xi)H_n\Big(\frac{\xi}{M}\Big)\widehat{\phi}\Big(\frac{\xi}{M}\Big)$$

for some 2π-periodic functions $A_0(\xi), A_1(\xi), \ldots, A_N(\xi)$ in $L^2_{2\pi}$. By the property of linear independent shifts of ϕ, the above identity yields

$$A_0(M\xi)H_0(\xi) = \sum_{n=1}^{N} A_n(M\xi)H_n(\xi). \tag{97}$$

On the other hand, it follows from Proposition 2.1 that

$$\begin{aligned} S(M\xi)H_0\Big(\xi + \frac{2s\pi}{M}\Big)\overline{H_0\Big(\xi + \frac{2s'\pi}{M}\Big)} \\ + \sum_{n=1}^{N} H_n\Big(\xi + \frac{2s\pi}{M}\Big)\overline{H_n\Big(\xi + \frac{2s\pi}{M}\Big)} = S(\xi)\delta_{ss'} \end{aligned} \tag{98}$$

for all $0 \le s, s' \le M-1$. Hence, substituting (97) into (98) leads to

$$S(M\xi)\sum_{n,n'=1}^{N}\Big(A_0(M\xi)\overline{A_0(M\xi)}\delta_{nn'}+A_n(M\xi)\overline{A_{n'}(M\xi)}\Big)$$
$$\times H_n\Big(\xi+\frac{2s\pi}{M}\Big)\overline{H_{n'}\Big(\xi+\frac{2s'\pi}{M}\Big)} = S\Big(\xi+\frac{2s\pi}{M}\Big)A_0(M\xi)\overline{A_0(M\xi)}\delta_{ss'},$$

where $0 \le s, s' \le M-1$, or in matrix formulation,

$$\mathbf{H}(\xi)^T\mathbf{B}(\xi)\overline{\mathbf{H}(\xi)} = |A_0(M\xi)|^2 \mathrm{diag}\Big(S(\xi),\ldots,S\big(\xi+\frac{2(M-1)\pi}{M}\big)\Big), \tag{99}$$

where

$$\mathbf{H}(\xi) = \Big(H_n\big(\xi+\frac{2s\pi}{M}\big)\Big)_{1\le n\le N, 0\le s\le M-1}$$

and

$$\mathbf{B}(\xi) = S(M\xi)|A_0(M\xi)|^2 I_N$$
$$+S(M\xi)\big(A_0(M\xi),\ldots,A_N(M\xi)\big)^T\big(\overline{A_0(M\xi)},\ldots,\overline{A_N(M\xi)}\big).$$

By Theorem 4.4, the rank of $\mathbf{H}(\xi)$ is $M-1$. This, together with (99), implies that $A_0(\xi) = 0$ for almost all $\xi \in \mathbf{R}$. Thus, f is the zero function, which is a contradiction.

Finally, the implications (v)$\Longrightarrow$(i) and (iv)$\Longrightarrow$(ii) are obvious.

5.2. Proof of Theorem 5.2

First we prove (iii)$\Longrightarrow$(iv) and (iii)$\Longrightarrow$(v). By Theorem 4.1, $\tilde{P}_0 H^\alpha = V_0$, and $Q_j H^\alpha$ is the orthogonal complement of V_j in V_{j+1} for any $j \in \mathbf{Z}$. This proves (iv) and (v).

Next, we prove (ii) $\Longrightarrow$ (iii). By dilation invariance, we may assume that $j' = 0$ and $j \ge 1$. By Theorem 5.1, we have that $Q_j H^\alpha$ is a closed subspace of L^2 for every $j \in \mathbf{Z}$. Therefore by Theorem 4.4, without loss of generality, we may assume that $\Psi = \{\psi_1, \ldots, \psi_{M-1}\}$, $\{\psi_s(\cdot - k) : 1 \le s \le M-1, k \in \mathbf{Z}\}$ is a Riesz basis of $V^2(\Psi)$, and $Q_0 H^\alpha = V^2(\Psi)$. Moreover, the matrix $\mathbf{U}(\xi)$, to be defined by

$$\mathbf{U}(\xi) = \big(\tilde{H}_s(\xi + 2m\pi/M)\big)_{0\le s,m\le M-1},$$

is a unitary matrix by Proposition 2.1,

$$\mathbf{U}(\xi)\overline{\mathbf{U}(\xi)}^T = I_M, \tag{100}$$

where $\tilde{H}_s(\xi) = S_1(M\xi)H_s(\xi)/S_1(\xi)$, $S_1(\xi)$ is a trigonometric polynomial with real coefficients such that $|S_1(\xi)|^2 = S(\xi)$, the function H_0 is the symbol of the scaling function ϕ, the functions $H_s, 1 \le s \le M-1$, are defined by $\widehat{\psi}_s(M\xi) = H_s(\xi)\widehat{\phi}(\xi)$, and the trigonometric function S is defined as in (10). From $Q_0 H^\alpha = V_2(\Psi)$ and the Riesz property of $\{\psi_s(\cdot - k) : 1 \le s \le M-1, k \in \mathbf{Z}\}$, we obtain:

$$\widehat{Q_j H^\alpha} = \Big\{\sum_{s=1}^{M-1} B_s(\xi/M^j)\widehat{\psi}_s(\xi/M^j) :\ B_s(\xi), 1 \le s \le M-1, \tag{101}$$
$$\text{are } 2\pi\text{−periodic and square − integrable}\Big\}.$$

By the assumption on the angle between Q_0H^α and Q_jH^α, we have that

$$\begin{aligned} 0 &= \langle \psi_1, g\rangle = \int_{\mathbf R} \widehat{\psi}_1(\xi)\overline{\hat g(\xi)}d\xi \qquad (102)\\ &= \sum_{s=1}^{M-1}\int_{\mathbf R} H_1\Big(\frac{\xi}{M}\Big)H_0\Big(\frac{\xi}{M^2}\Big)\cdots H_0\Big(\frac{\xi}{M^{j+1}}\Big)\overline{B_s\Big(\frac{\xi}{M^j}\Big)H_s\Big(\frac{\xi}{M^{j+1}}\Big)}\Big|\widehat{\phi}\Big(\frac{\xi}{M^{j+1}}\Big)\Big|^2 d\xi\\ &= M^j\sum_{s=1}^{M-1}\int_{-\pi}^{\pi} H_1(M^{j-1}\xi)H_0(M^{j-2}\xi)\cdots H_0(\xi)\overline{B_s(\xi)}R_s(\xi)d\xi, \end{aligned}$$

where $g\in Q_jH^\alpha$ has its Fourier transform being of the form $\sum_{s=1}^{M-1} B_s(\xi/M^j)\widehat{\psi}_s(\xi/M^j)$ for some 2π-periodic square-integrable functions $B_s(\xi), 1\le s\le M-1$,

$$R_s(\xi)=\sum_{m=0}^{M-1} H_0\Big(\frac{\xi+2m\pi}{M}\Big)\overline{H_s\Big(\frac{\xi+2m\pi}{M}\Big)}\Phi\Big(\frac{\xi+2m\pi}{M}\Big), 1\le s\le M-1,$$

and

$$\Phi(\xi)=\sum_{l\in\mathbf Z}|\widehat{\phi}(\xi+2l\pi)|^2.$$

Since $B_s(\xi)$ are arbitrarily chosen, and both H_0 and H_1 are nonzero trigonometric polynomials, we then obtain from (102) that

$$R_s(\xi)=0\quad \forall\,\xi\in\mathbf R,\ 1\le s\le M-1. \qquad (103)$$

This implies that the vector $\mathbf v_0(\xi)$, to be defined by

$$\mathbf v_0(\xi)=\Big(\tilde H_0(\xi+2m\pi/M)S(\xi+2m\pi/M)\Phi(\xi+2m\pi/M)/S(M\xi)\Big)_{0\le m\le M-1},$$

is orthogonal to the vectors $(\tilde H_s(\xi+2m\pi/M))_{0\le m\le M-1}, 1\le s\le M-1$. Hence by (100), there exists a 2π-periodic function R such that $\mathbf v_0(\xi)=R(M\xi)(\tilde H_0(\xi+2m\pi/M))_{0\le m\le M-1}$, which implies that

$$S(\xi)\Phi(\xi)=S(M\xi)R(M\xi):=\tilde R(M\xi). \qquad (104)$$

By the assumption on ϕ, $\Phi(\xi)$ is a trigonometric polynomial and is positive for all $\xi\in\mathbf R$. By Theorem 4.4, all zeros of the trigonometric polynomial $S(\xi)$ lies on the real line if there is. Combining the above two facts for Φ and $S(\xi)$ with (104) implies that either $S(\xi)$ has no zeros, or has a factor of the form $(e^{iM\xi}-e^{-i\xi_0})$ for some $\xi_0\in\mathbf R$. Since the conclusion in the later case contradicts to (54), we then conclude that $S(\xi)$ has not zeros, or equivalently, it is a constant. Hence $\tilde P_0H^\alpha$ is closed in L^2 by Theorem 4.3.

Finally, the implications (i)$\Longrightarrow$(ii), (iv)$\Longrightarrow$(ii), and (v)$\Longrightarrow$(i) are obvious.

5.3. Proof of Theorem 5.3

The estimate (95) follows easily from the estimate (94) and the condition $Q_jH^\alpha\subset V_{j+1}$. So it suffices to prove (94). By Theorem 4.4, we may assume that with $\Psi=\{\psi_1,\dots,\psi_{M-1}\}$, the collection of integer shifts $\{\psi_m(\cdot-k); 1\le m\le M-1, k\in\mathbf Z\}$

is a Riesz basis of Q_0H^α, and that $\widehat{\psi}_m(\xi) = H_m(\xi/M)\widehat{\phi}(\xi/M)$ for some trigonometric polynomials $H_m, 1 \le m \le M-1$.

Let $f := \sum_{k\in\mathbf{Z}} a(k)\phi(\cdot - k) \in V_0$ and $g := \sum_{m=1}^{M-1}\sum_{k\in\mathbf{Z}} d_m(k)\psi_{m;j,k} \in Q_jH^\alpha$, where $\{a(k)\}$ and $\{d_m(k)\}, 1 \le m \le M-1$, are ℓ^2 sequences. By the Riesz basis property of the integer shifts of ϕ, and of $\psi_1, \ldots, \psi_{M-1}$, there exists a positive constant C (independent of f and g), so that

$$C^{-1}\|f\|_2^2 \le \sum_{k\in\mathbf{Z}} |c(k)|^2 \le C\|f\|_2^2 \tag{105}$$

and

$$C^{-1}\|g\|_2^2 \le \sum_{m=1}^{M-1}\sum_{k\in\mathbf{Z}} |d_m(k)|^2 \le C\|g\|_2^2. \tag{106}$$

Setting

$$c_m(l) = \int_{\mathbf{R}} \phi(x)\psi_m(M^jx - l)dx,\ 1 \le m \le M-1,\ l \in \mathbf{Z}, \tag{107}$$

and using the support properties of the functions ϕ and $\psi_m, 1 \le m \le M-1$, we obtain

$$\begin{aligned} |\langle f, g\rangle| \ &\le\ M^{j/2}\sum_{m=1}^{M-1}\sum_{k,k'\in\mathbf{Z}} |a(k)||d_m(k')||\langle\phi(\cdot - k), \psi_m(M^j\cdot - k')\rangle| \\ &\le\ M^{j/2}\sum_{m=1}^{M-1}\sum_{k,k'\in\mathbf{Z}, |M^{-j}k'-k|\le C_0} |a(k)||d_m(k')||c_m(k' - M^jk)| \\ &\le\ M^{j/2}\sum_{m=1}^{M-1}\sum_{k\in\mathbf{Z}} |a(k)|\Big(\sum_{|M^{-j}k'-k|\le C_0} |d_m(k')|^2\Big)^{1/2}\Big(\sum_{l\in\mathbf{Z}}|c_m(l)|^2\Big)^{1/2} \\ &\le\ M^{j/2}\Big(\sum_{m=1}^{M-1}\sum_{l\in\mathbf{Z}}|c_m(l)|^2\Big)^{1/2}\Big(\sum_{k\in\mathbf{Z}}|a(k)|^2\Big)^{1/2} \\ &\qquad \times\Big(\sum_{m=1}^{M-1}\sum_{k\in\mathbf{Z}}\sum_{|M^{-j}k'-k|\le C_0} |d_m(k')|^2\Big)^{1/2}, \end{aligned}$$

where C_0 is a positive constant. This, together with (105) and (106), yields the following estimate of the angle θ_j between $\tilde{P}_0H^\alpha$ and Q_jH^α:

$$\cos\theta_j \le CM^{j/2}\Big(\sum_{m=1}^{M-1}\sum_{k\in\mathbf{Z}}|c_m(k)|^2\Big)^{1/2}, \tag{108}$$

for all $0 \le j \in \mathbf{Z}$, where C is a positive constant independent of j. By (107) and (108), the proof of the estimate (94) reduces to the following estimate:

$$\int_{-\pi}^{\pi}\Big(\sum_{k\in\mathbf{Z}}|\widehat{\phi}(M^j(\xi + 2k\pi))||\widehat{\psi}_m(\xi + 2k\pi)|\Big)^2 d\xi \le CM^{-j(1+\min(\gamma_0, 2\beta))}, \tag{109}$$

for all $1 \le m \le M-1$, where C is an absolute constant.

Write the symbol $H_0(\xi)$ of the scaling function ϕ as

$$H_0(\xi) = \left(\frac{1-e^{-iM\xi}}{M-Me^{-i\xi}}\right)^{\gamma_0} \tilde{H}_0(\xi) \tag{110}$$

for some positive integer γ_0 and some trigonometric polynomial $\tilde{H}_0(\xi)$ not divisible by $(1-e^{-iM\xi})/(1-e^{-i\xi})$. Then it follows from $\phi \in H^\beta$ that

$$\beta \le \gamma_0 \tag{111}$$

(see [24]). By (110), we have

$$H_0(\xi + 2m\pi/M) = O(\xi^{\gamma_0}) \quad \text{as } \xi \to 0,\ 1 \le m \le M-1, \tag{112}$$

Combining (54) and (112), we obtain

$$S(M\xi)|H_0(\xi)|^2 - S(\xi) = O(\xi^{2\gamma_0}) \quad \text{as } \xi \to 0.$$

Thus,

$$H_m(\xi) = O(\xi^{\gamma_0}) \quad \text{as } \xi \to 0,\ 1 \le m \le M-1, \tag{113}$$

by (9). By the property of linear independent shifts of ϕ, there exists a compact set K that contains a neighborhood of the origin, such that $K + 2\pi\mathbf{Z} = \mathbf{R}$ and $|\widehat{\phi}(\xi)|$ is bounded below from zero on the set K [10, 15]. This observation, together with (111), (112), (113), Proposition 2.1, and the refinement relation $\widehat{\phi}(M\xi) = H_0(\xi)\widehat{\phi}(\xi)$, implies that

$$\begin{aligned}
&\int_{-\pi}^{\pi} \Big(\sum_{k\in\mathbf{Z}} |\widehat{\phi}(M^j(\xi+2k\pi))||\widehat{\psi}_m(\xi+2k\pi)|\Big)^2 d\xi \\
= &\int_{-\pi}^{\pi} \prod_{i=0}^{j-1} |H_0(M^i\xi)|^2 \Big(\sum_{k\in\mathbf{Z}} |\widehat{\phi}(\xi+2k\pi)||\widehat{\psi}_m(\xi+2k\pi)|\Big)^2 d\xi \\
\le\ & C_1 \int_{-\pi}^{\pi} |1-e^{-i\xi}|^{2\gamma_0} \prod_{i=0}^{j-1} |H_0(M^i\xi)|^2 d\xi \\
\le\ & C_2 \int_K |\xi|^{2\gamma_0} |\widehat{\phi}(M^j\xi)|^2 d\xi \\
\le\ & C_3 M^{-j(1+2\beta)} \int_{M^jK} (1+|\xi|^2)^\beta |\widehat{\phi}(\xi)|^2 d\xi \\
\le\ & C_4 M^{-j(1+2\beta)},
\end{aligned}$$

where $C_i, 1 \le i \le 4$, are positive constants independent of $0 \le j \in \mathbf{Z}$. This completes the proof of (109), and hence the desired estimate (94).

5.4. Proof of Theorem 5.4

By Theorem 5.1 and the assumption on the angles between V_0 and Q_jH^α, we may conclude that Q_0H^α is a closed subspace of L^2. Therefore by Theorem 4.4, we may assume, without loss of generality, that $\{\psi_m(\cdot - k) : 1 \le m \le M-1, k \in \mathbf{Z}\}$, with $\Psi = \{\psi_1, \ldots, \psi_{M-1}\}$, is a Riesz basis of $V^2(\Psi)$. By the assumption on the angle, we have $\langle \phi, \psi \rangle \neq 0$ for some $\psi \in V^2(\Psi)$, since otherwise the angle between V_0 and $V^2(\Psi)$ is zero. In particular, we may select ψ to be compactly supported, since ϕ has compact support. In the following, we use the bracket product notation:

$$[\widehat{\phi}, \widehat{\psi}](\xi) := \sum_{k\in\mathbf{Z}} \widehat{\phi}(\xi + 2k\pi)\overline{\widehat{\psi}(\xi + 2k\pi)},$$

and consider the function $g \in W_j$, defined by

$$\hat{g}(M^j\xi) = H(M^{j-1}\xi)\cdots H(\xi)[\widehat{\phi}, \widehat{\psi}](\xi)\widehat{\psi}(\xi).$$

Then $g \not\equiv 0$, and

$$\begin{aligned} \|g\|_2^2 &= M^j \int_{-\pi}^{\pi} |H(\xi)\cdots H(M^{j-1}\xi)|^2 |[\widehat{\phi}, \widehat{\psi}](\xi)|^2 [\widehat{\psi}, \widehat{\psi}](\xi) d\xi \\ &\le CM^j \int_{-\pi}^{\pi} |H(\xi)\cdots H(M^{j-1}\xi)|^2 |[\widehat{\phi}, \widehat{\psi}](\xi)|^2 d\xi, \end{aligned} \tag{114}$$

where C is a positive constant independent of j. By direct computation, we also have

$$\langle \phi, g \rangle = M^j \int_{-\pi}^{\pi} |H(M^{j-1}\xi)\cdots H(\xi)|^2 |[\widehat{\phi}, \widehat{\psi}](\xi)|^2 d\xi.$$

This, together with (96) and (113), implies that

$$\int_{-\pi}^{\pi} |H(M^{j-1}\xi)\cdots H(\xi)|^2 |[\widehat{\phi}, \widehat{\psi}](\xi)|^2 d\xi \le CM^{-j(2\gamma+1)}. \tag{115}$$

Let $\delta > 0$ be so chosen that $[\widehat{\phi}, \widehat{\psi}](\xi) \neq 0$ for all $\delta \le |\xi| \le M\delta$, and $|H(\xi) - 1| \le 1/2$ for all $|\xi| \le \delta$. The existence of such a number δ follows from the facts that $H(0) = 1$ and that $[\widehat{\phi}, \widehat{\psi}](\xi)$ is a nonzero trigonometric polynomial. Therefore, it follows from (115) that

$$\int_{\delta \le |\xi| \le M\delta} |\widehat{\phi}(M^j\xi)|^2 d\xi \le CM^{-j(2\gamma+1)} \tag{116}$$

for some positive constant C independent of $0 \le j \le \mathbf{Z}$, which proves that $\phi \in H^\beta$ for all $\beta < \gamma$.

Acknowledgment. The second author acknowledges several helpful discussions with Professors Douglas Hardin and David Larson. The authors also thank the referees for their comments.

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