TOPOLOGICAL AND GEOMETRIC PROPERTIES OF REFINABLE FUNCTIONS AND MRA AFFINE FRAMES

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Abstract. We investigate some topological and geometric properties of the set $\mathcal{R}$ of all refinable functions in $L^2(\mathbb{R}^d)$, and of the set of all MRA affine frames. We prove that $\mathcal{R}$ is nowhere dense in $L^2(\mathbb{R}^d)$; the unit sphere of $\mathcal{R}$ is path-connected in the L^2-norm; and for any M-dimensional hyperplane generated by L^2-functions $f_0, \ldots, f_M$, either almost all the functions in the hyperplane are refinable or almost all the functions in the hyperplane are not refinable. We show that the set of all MRA affine frames is nowhere dense in $L^2(\mathbb{R}^d)$. We also obtain a new characterization of the L^2-closure $\overline{\mathcal{R}}$ of $\mathcal{R}$, and extend the above topological and geometric results from $\mathcal{R}$ to $\overline{\mathcal{R}}$, and even further to the set of all refinable vectors and its L^2-closure.

1. Introduction

Let $N \in \mathbb{N}$, the set of all positive integers, and $f_1, \ldots, f_N \in L^2 := L^2(\mathbb{R}^d)$. We say that $F := (f_1, \ldots, f_N)^T$ is a *refinable vector* (of length N) if $f_1(\cdot/2), \ldots, f_N(\cdot/2)$ are in the L^2-closure of the linear span of $\{f_n(\cdot - k) : 1 \le n \le N, k \in \mathbb{Z}^d\}$ ([7]). We denote by $\mathcal{R}_N$ the set of all refinable vectors of length N, and by $\overline{\mathcal{R}}_N$ the L^2-closure of $\mathcal{R}_N$. For the scalar case ($N = 1$), a refinable vector is usually known as a *refinable function*, and we use $\mathcal{R}$, instead of $\mathcal{R}_1$, to denote the set of all refinable functions.

The purpose of this paper is to investigate the topological and geometric properties of the set $\mathcal{R}_N$ of refinable vectors, its closure $\overline{\mathcal{R}}_N$, and the set of all MRA affine frames. We will prove that while $\mathcal{R}_N$ is nowhere dense in $(L^2)^N$, the unit sphere of $\mathcal{R}_N$ is path-connected in the L^2-norm. Moreover, any M-dimensional hyperplane generated by $(L^2)^N$-functions $F_0, ..., F_M$ is either “almost” completely contained in $\mathcal{R}_N$, or is “almost” completely contained in $(L^2)^N \setminus \mathcal{R}_N$. In addition, all these results remain valid for $\overline{\mathcal{R}}_N$, the L^2-closure of $\mathcal{R}_N$. We apply our results to obtain that the set of all MRA affine frames (with a fixed number of generators in the scaling space) is nowhere dense.

Date: January 19, 2010.

1991 *Mathematics Subject Classification.* Primary 42C40, 42C15, 41A65, 46E40.

Key words and phrases. refinable functions, path-connectivity, nowhere density, multiresolution analysis, affine frame.

We first investigate the path-connectedness of the set $\mathcal{R}_N$ of refinable vectors of length N. The set $\mathcal{R}_N$ is always path-connected since it is homogeneous. Therefore we are only interested in the path-connectedness of the unit sphere of $\mathcal{R}_N$ (we remark that in general the unit sphere of a path-connected set is not necessarily path-connected). We shall prove the following result in Section 2:

Theorem 1.1. *For every $N \in \mathbb{N}$, the unit sphere of $\mathcal{R}_N$ is a path-connected subset of $(L^2)^N$.*

We remark that the set of all scaling functions, an important subclass of $\mathcal{R}_N$, is shown to be path-connected in [23, 35] (see Section 7 for the precise definition of a scaling function). For other related path-connectedness results for scaling functions, wavelets and affine frames, the reader may refer to [3, 12, 16, 17, 19, 21, 22, 26, 30] etc.

Secondly, we study the geometrical properties of the set $\mathcal{R}_N$ in Sections 3 and 4. In particular, we establish the following $(N+2)$-point rule, hyperplane property, and nowhere dense property.

Theorem 1.2. *Let $N \in \mathbb{N}$, and $F, G \in (L^2)^N$. If $F + \epsilon_q G \in \mathcal{R}_N$ for distinct scalars $\epsilon_q, 1 \le q \le N+2$, then $F + tG \in \mathcal{R}_N$ for all $t \in \mathbb{R}$, except for possibly countably many t's.*

Theorem 1.3. *Let $M, N \in \mathbb{N}$. If $F_0, \ldots, F_M \in (L^2)^N$ such that $\{F_m - F_0, 1 \le m \le M\}$ are linearly independent, then either $g(t_1, \ldots, t_M) := F_0 + \sum_{m=1}^{M} t_m F_m \in \mathcal{R}_N$ for almost all $\mathbf{t} := (t_1, \ldots, t_M) \in \mathbb{R}^M$, or $g(t_1, \ldots, t_M) \notin \mathcal{R}_N$ for almost all $\mathbf{t} := (t_1, \ldots, t_M) \in \mathbb{R}^M$.*

Theorem 1.4. *For every $N \in \mathbb{N}$, the set $\mathcal{R}_N$ is nowhere dense in $(L^2)^N$. Hence the interior of $\mathcal{R}_N$ is the empty set.*

Roughly speaking, the $(N+2)$-point rule in Theorem 1.2 means that given any line in $(L^2)^N$, if there exist $N+2$ points on that line belonging to $\mathcal{R}_N$ then all except for possibly countable many points on that line belong to $\mathcal{R}_N$. On the other hand, the hyperplane property in Theorem 1.3 means that given any finite-dimensional hyperplane in $(L^2)^N$, either it is almost-completely contained in $\mathcal{R}_N$, or it is almost-completely contained in the complement set $(L^2)^N \setminus \mathcal{R}_N$.

The condition and conclusion in the $(N+2)$-point rule for $\mathcal{R}_N$ are optimal for the scalar case $N = 1$. In fact, for any two given distinct scalars ϵ_1, ϵ_2, we can construct two L^2-functions F and G such that $F + tG$ is refinable only when $t = \epsilon_1, \epsilon_2$ (Example 3.1). Also for a given countable subset T of $\mathbb{R}$, we construct two functions F and G such that $F + tG$ is refinable for all real t except $t \in T$ (Example 3.2).

For the set of refinable vectors, we have not seen any result in the literature on its geometric properties, like the $(N+2)$-point rule and hyperplane property in

Theorems 1.2 and 1.3. For the set of wavelets, a result of this nature is discussed in [13, 21, 22], where it is shown that if $\psi, \tilde{\psi}$ are two orthonormal wavelets, then $t\psi + (1-t)\tilde{\psi}$ are (Riesz) wavelets for all real t except possibly when $t = 1/2$.

The nowhere dense property for the set $\mathcal{R}_N$ in Theorem 1.4 follows immediately from the nowhere dense property for its closure $\overline{\mathcal{R}}_N$ (see Theorem 1.7 below). We point out that for the one-dimensional scalar case, i.e., $d = N = 1$, the nowhere density property of the set of all refinable functions was obtained in [33] with a different proof (see also [34]).

Define the Fourier transform $\hat{f}$ of an integrable function f by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx$$

and interpret the Fourier transform of a square-integrable function f as usual. For every $L \in \mathbb{N}$ we denote by $\mathcal{A}_L$ the class of all vectors $F = (f_1, \ldots, f_N)^T \in (L^2)^N$ such that any $L \times L$ submatrix of the $(\mathbb{Z}_N \times \mathbb{Z}_+) \times \mathbb{Z}^d$ matrix

$$\mathcal{F}(\xi) := \Big(\widehat{F}\big(2^j(\xi + 2k\pi)\big)\Big)_{j\in\mathbb{Z}_+, k\in\mathbb{Z}^d} \tag{1.1}$$

has zero determinant for almost all $\xi \in \mathbb{R}^d$, where $\mathbb{Z}_N := \{1, \ldots, N\}$ and $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. For a vector-valued function $F = (f_1, \ldots, f_N)^T$ we use $S(F)$ to denote the shift-invariant space $\overline{\mathrm{span}}\{f_n(\cdot - k) : 1 \le n \le N, k \in \mathbb{Z}^d\}$ generated by F. In general, for a countable set F, we use $S(F)$ to denote the smallest closed subspace of $L^2(\mathbb{R}^d)$ that contains $\{f(\cdot - k) \mid f \in F, k \in \mathbb{Z}^d\}$ (see [1, 2, 6, 7, 8, 11] and references therein for the study of shift-invariant spaces).

The L^2-closure $\overline{\mathcal{R}}_N$ of $\mathcal{R}_N$ differs from the set $\mathcal{R}_N$ (see [31] or Example 3.2 (ii)). Strang and Zhou ([31]) characterized the set $\overline{\mathcal{R}}_1$, the L^2 closure of the set of all refinable functions, which can be stated as follows:

$$\overline{\mathcal{R}}_1 = \mathcal{A}_2.$$

Our characterization below is a generalization of the above Strang-Zhou's result from the scalar case ($N = 1$) to the vector case ($N \ge 1$), with a different proof and additional characterization (see Section 5 for the proof) .

Theorem 1.5. *Let $N \ge 1$. Then $\overline{\mathcal{R}}_N = \mathcal{A}_{N+1} = \cup_{\Phi\in\mathcal{R}_N}(S(\Phi))^N$.*

For $\Psi = (\psi_1, \ldots, \psi_N)^T$, define the Gramian fibers

$$G_\Psi(\xi) = \Big(\sum_{k\in\mathbb{Z}^d} \hat{f}(\xi + 2k\pi)\overline{\hat{g}(\xi + 2k\pi)}\Big)_{f,g\in F_\Psi}, \quad \xi \in \mathbb{R}^d, \tag{1.2}$$

where

$$F_\Psi = \big\{2^{-jd}\psi_n(2^{-j}\cdot) : \ 1 \le n \le N, j \ge 1\big\}.$$

The Gramian fibers have been used for the characterization of many properties of the shift-invariant system

$$E(F_\Psi) = \big\{f(\cdot - k):\ f \in F_\Psi, k \in \mathbb{Z}^d\big\},$$

and the dyadic wavelet system

$$X(\Psi) := \big\{2^{jd/2}\psi_n(2^j \cdot - k):\ 1 \le n \le N, j \in \mathbb{Z}, k \in \mathbb{Z}^d\big\}$$

([10, 27, 28]). Define the *multiplicity function* $M_\Psi : [-\pi, \pi]^d \longmapsto \mathbb{Z}_+$ of the shift-invariant space $S(F_\Psi)$ by

$$M_\Psi(\xi) = \text{rank } G_\Psi(\xi) \tag{1.3}$$

([4, 29]). The multiplicity function depends only on the underlying shift-invariant space $S(F_\Psi)$ ([7]). For any $F = (f_1, \ldots, f_N)^T \in (L^2)^N$, we note that $\mathcal{F}(\xi)$ in (1.1) and the Gramian fibers $G_{F(2\cdot)}(\xi)$ are related by

$$G_{F(2\cdot)}(\xi) = 2^{-d}\mathcal{F}(\xi)(\mathcal{F}(\xi))^T,\ \xi \in \mathbb{R}^d. \tag{1.4}$$

Therefore $F \in \mathcal{A}_{L+1}$ if and only if $M_{F(2\cdot)}(\xi) \le L$ for almost all $\xi \in \mathbb{R}^d$. For $L \ge 0$, define

$$\mathcal{M}_L = \Big\{F = (f_1, \ldots, f_N)^T:\ M_{F(2\cdot)}(\xi) \le L \text{ for almost all } \xi \in \mathbb{R}^d\Big\}. \tag{1.5}$$

Thus by Theorem 1.5 we have the following characterization of $\overline{\mathcal{R}}_N$ via multiplicity functions:

Corollary 1.6. *Let $N \ge 1$. Then $\overline{\mathcal{R}}_N = \mathcal{M}_N$.*

We next apply the characterization in Theorem 1.5 to investigate the topological and geometric properties of $\overline{\mathcal{R}}_N$ in Section 6.

Theorem 1.7. *Let $M, N \in \mathbb{N}$. Then*

- (i) *The unit sphere of the set $\overline{\mathcal{R}}_N$ is a path-connected subset of $(L^2)^N$.*
- (ii) *If $F, G \in (L^2)^N$, and $F + \epsilon_q G \in \overline{\mathcal{R}}_N$ for distinct scalars $\epsilon_q, 1 \le q \le N+2$, then $F + tG \in \overline{\mathcal{R}}_N$ for all $t \in \mathbb{R}$.*
- (iii) *Let $F_0, \ldots, F_M \in (L^2)^N$ such that $\{F_m - F_0, 1 \le m \le M\}$ are linearly independent. Then either $g(t_1, \ldots, t_M) := F_0 + \sum_{m=1}^M t_m F_m \in \overline{\mathcal{R}}_N$ for all $\mathbf{t} := (t_1, \ldots, t_M) \in \mathbb{R}^M$, or $g(t_1, \ldots, t_M) \notin \overline{\mathcal{R}}_N$ for almost all $\mathbf{t} := (t_1, \ldots, t_M) \in \mathbb{R}^M$.*
- (iv) *The set $\overline{\mathcal{R}}_N$ is nowhere dense in $(L^2)^N$.*

In Section 7, we give another application of our new characterization $\overline{\mathcal{R}}_N = \cup_{\Phi \in \mathcal{R}_N}(S(\Phi))^N$ in Theorem 1.5. We establish the nowhere density of the set $\mathcal{F}_{M,N}$ of all MRA affine frames of length M associated with a multiresolution analysis (MRA) having a scaling vector of length N (see Section 7 for the precise definitions).

Theorem 1.8. *For any $M \geq N \geq 1$, the set $\mathcal{F}_{M,N}$ is a nowhere dense subset of $(L^2)^M$.*

It was proved by M. Bownik [9] that the set of all affine frames is dense in L^2. However, the result in Theorem 1.8 for the special $M = N = 1$ case implies that the set of all MRA affine frames is nowhere dense in L^2. This indicates that there are many affine frames that are not MRA affine frames.

2. Path-connectedness for the set of refinable vectors

To prove Theorem 1.1, we need the following characterization of a refinable vector in the Fourier domain ([7]).

Lemma 2.1. *Let $F = (f_1, \dots, f_N)^T \in (L^2)^N$. Then F is refinable if and only if there exists an $N \times N$-matrix-valued 2π-periodic measurable function $m(\xi)$ such that*

$$\widehat{F}(2\xi) = m(\xi)\widehat{F}(\xi) \quad \text{a.e.} \quad \xi \in \mathbb{R}^d. \tag{2.1}$$

Now we start to prove Theorem 1.1.

Proof of Theorem 1.1. Let S be the refinable vector in $\mathcal{R}_N$ whose first component is the Shannon scaling function and whose other components are identically zero, i.e.,

$$\widehat{S}(\xi) = \left((2\pi)^{-d/2}\chi_{[-\pi,\pi]^d}(\xi), 0, \cdots, 0\right)^T.$$

Here χ_E is the characteristic function on a measurable set E. We will establish Theorem 1.1 by constructing a continuous path in the unit sphere of $\mathcal{R}_N$ connecting any given refinable vector in the unit sphere of $\mathcal{R}_N$ to the refinable vector S.

Take any refinable vector $F \in \mathcal{S}_N$, the unit sphere of $\mathcal{R}_N$. By Lemma 2.1,

$$\widehat{F}(2\xi) = m(\xi)\widehat{F}(\xi) \tag{2.2}$$

for some matrix-valued 2π-periodic function $m(\xi)$.

From (2.2) and the assumption $F \in \mathcal{S}_N$ it follows that the restriction of $\widehat{F}(\xi)$ on the torus $[-\pi, \pi]^d$ is not identically zero, i.e.,

$$\widehat{F}\chi_{[-\pi,\pi]^d} \neq 0, \tag{2.3}$$

for otherwise using (2.2) iteratively we will have that $\widehat{F}(\xi) = 0$ for almost all $\xi \in \mathbb{R}^d$, a contradiction to the assumption $F \in \mathcal{S}_N$.

For $0 \le t \le 1$, we define $\Psi_t = (\psi_{1,t}, \ldots, \psi_{N,t})^T$ by

$$\widehat{\Psi}_t(\xi) = \begin{cases} (1-t)^{\sum_{j=1}^{\infty} a(2^{-j}\xi)} \widehat{F}(\xi) & \text{if } 0 \le t < 1 \\ \widehat{F}(\xi)\chi_{[-\pi,\pi]^d}(\xi) & \text{if } t = 1, \end{cases} \tag{2.4}$$

where $a(\xi)$ is a 2π-periodic function whose restriction on $[-\pi,\pi]^d$ is the characteristic function on $[-\pi,\pi]^d \backslash [-\pi/2,\pi/2]^d$. From (2.4), it follows that

$$\Psi_0 = F, \tag{2.5}$$

and

$$\widehat{\Psi}_t(2\xi) = m_t(\xi)\widehat{\Psi}_t(\xi), \quad t \in [0,1], \tag{2.6}$$

where

$$m_t(\xi) = \begin{cases} (1-t)^{a(\xi)} m(\xi) & \text{if } 0 \le t < 1 \\ (1-a(\xi))m(\xi) & \text{if } t = 1. \end{cases}$$

Again from (2.4), we obtain

$$\lim_{t \to t_0} \widehat{\Psi}_t(\xi) = \widehat{\Psi}_{t_0}(\xi) \quad \text{a.e. } \xi \in \mathbb{R}^d \tag{2.7}$$

for all $t_0 \in [0,1]$, and

$$|\widehat{F}(\xi)\chi_{[-\pi,\pi]^d}(\xi)| \le |\widehat{\Psi}_t(\xi)| \le |\widehat{F}(\xi)| \quad \text{a.e. } \xi \in \mathbb{R}^d \tag{2.8}$$

for all $t \in [0,1]$. By (2.3), (2.7) and (2.8),

$$\lim_{t \to t_0} \|\Psi_t - \Psi_{t_0}\|_2 = 0, \ t_0 \in [0,1], \tag{2.9}$$

and

$$0 < \|\widehat{F}(\cdot)\chi_{[-\pi,\pi]^d}(\cdot)\|_2 \le \|\widehat{\Psi}_t\|_2 \le \|\widehat{F}\|_2, \ 0 \le t \le 1. \tag{2.10}$$

Denote the normalization of $\Psi_t, 0 \le t \le 1$, by

$$\Phi_t := \frac{\Psi_t}{\|\Psi_t\|_2} \tag{2.11}$$

which is well-defined by (2.10). From (2.5), (2.6), (2.9) and (2.10), it follows that $\Phi_t, 0 \le t \le 1$, form a continuous path in $\mathcal{S}_N$ connecting $F \in \mathcal{S}_N$ and Φ_1 in (2.11). Also from (2.4), (2.6) and (2.11), the function $\Phi_1 = (\phi_{1,1}, \ldots, \phi_{N,1})^T$ has the following properties:

$$\widehat{\Phi}_1(\xi) = \frac{\widehat{F}(\xi)\chi_{[-\pi,\pi]^d}(\xi)}{\|\widehat{F}(\cdot)\chi_{[-\pi,\pi]^d}\|_2}, \tag{2.12}$$

and

$$\widehat{\Phi}_1(2\xi) = m_1(\xi)\widehat{\Phi}_1(\xi) \tag{2.13}$$

for some matrix-valued 2π-periodic function $m_1(\xi)$.

Define $\Psi_t = (\psi_{1,t}, \ldots, \psi_{N,t})^T$, $1 \le t \le 2$, by

$$\widehat{\Psi}_t(\xi) = \begin{cases} \Big(\frac{\widehat{\phi}_{1,1}(\xi)}{|\widehat{\phi}_{1,1}(\xi)|} \sqrt{|\widehat{\phi}_{1,1}(\xi)|^2 + (-3+4t-t^2)|\widehat{\Phi'_1}(\xi)|^2}, (2-t)\widehat{\Phi'_1}(\xi) \Big)^T & \text{if } \widehat{\phi}_{1,1}(\xi) \ne 0, \\ \big(\sqrt{-3+4t-t^2}|\widehat{\Phi'_1}(\xi)|, (2-t)\widehat{\Phi'_1}(\xi) \big)^T & \text{if } \widehat{\phi}_{1,1}(\xi) = 0, \end{cases} \tag{2.14}$$

where $\widehat{\Phi'_1}(\xi) = (\widehat{\phi}_{2,1}(\xi), \ldots, \widehat{\phi}_{N,1}(\xi))$. From the definition of $\Psi_t, 1 \le t \le 2$, we have that

$$\begin{cases} \Psi_1 = \Phi_1, \\ |\widehat{\Psi}_t(\xi)| = |\widehat{\Phi}_1(\xi)| \text{ for all } t \in [1,2], \\ \|\Psi_t\|_2 = 1 \text{ for all } t \in [1,2], \text{ and} \\ \lim_{t\to t_0} \|\Psi_t - \Psi_{t_0}\|_2 = 0 \ \text{ for all } t_0 \in [1,2]. \end{cases} \tag{2.15}$$

Here we have used the observation that for all $t_0 \in [1,2]$, $\lim_{t\to t_0} \widehat{\Psi}_t(\xi) = \widehat{\Psi}_{t_0}(\xi)$ for almost all $\xi \in \mathbb{R}^d$. We also note that

$$\widehat{\Psi}_t(\xi) = A_t(\xi)\widehat{\Phi}_1(\xi) \tag{2.16}$$

for some matrix-valued 2π-periodic function $A_t(\xi)$ with its restriction on $[-\pi,\pi]^d$ being defined by

$$A_t(\xi) = \begin{cases} \begin{pmatrix} \sqrt{1 + (-3+4t-t^2)\frac{|\widehat{\Phi'_1}(\xi)|^2}{|\widehat{\phi}_{1,1}(\xi)|^2}} & 0 \\ 0 & (2-t)I_{N-1} \end{pmatrix} & \text{if } \widehat{\phi}_{1,1}(\xi) \ne 0, \\ \begin{pmatrix} 1 & \sqrt{-3+4t-t^2}\frac{\overline{\widehat{\Phi'_1}(\xi)}}{|\widehat{\Phi'_1}(\xi)|} \\ 0 & (2-t)I_{N-1} \end{pmatrix} & \text{if } \widehat{\phi}_{1,1}(\xi) = 0, \end{cases}$$

where I_l is the $l \times l$ identity matrix. Combining (2.13) and (2.16), and using the nonsingularity of the matrix $A_t(\xi), 1 \le t < 2$, for almost all $\xi \in \mathbb{R}^d$, we obtain the refinability of $\Psi_t, 1 \le t < 2$,

$$\widehat{\Psi}_t(2\xi) = A_t(2\xi)\widehat{\Phi}_1(2\xi) = A_t(2\xi)m_1(\xi)\widehat{\Phi}_1(\xi) = m_t(\xi)\widehat{\Psi}_t(\xi) \tag{2.17}$$

where

$$m_t(\xi) = A_t(2\xi)m_1(\xi)(A_t(\xi))^{-1}, \quad 1 \le t < 2.$$

Write

$$\widehat{\Psi}_2(\xi) = (\widehat{\psi}_{1,2}(\xi), \ldots, \widehat{\psi}_{N,2}(\xi))^T.$$

Then $\widehat{\psi}_{k,2}(\xi), 2 \le k \le N$, are zero functions. This together with (2.13) and (2.15) proves the refinability of Ψ_t for $t = 2$,

$$\widehat{\Psi}_2(2\xi) = m_2(\xi)\widehat{\Psi}_2(\xi) \tag{2.18}$$

for the matrix-valued 2π-periodic function $m_2(\xi)$ with its restriction on $[-\pi,\pi]^d$ being defined by

$$m_2(\xi)=\begin{cases}\begin{pmatrix}\frac{\widehat{\psi}_{1,2}(2\xi)}{\widehat{\psi}_{1,2}(\xi)} & 0\\ 0 & I_{N-1}\end{pmatrix} & \text{if } |\widehat{\Phi}_1(\xi)|\ne 0,\\ I_N & \text{if } |\widehat{\Phi}_1(\xi)|=0.\end{cases}$$

Therefore it follows from (2.15), (2.17) and (2.18) that the functions $\Phi_t, 1\le t\le 2$, defined by

$$\Phi_t:=\Psi_t, \tag{2.19}$$

form a continuous path in $\mathcal{S}_N$ from Φ_1 in (2.11) to Φ_2 in (2.19).

From the definition of $\Phi_2=(\phi_{1,2},\ldots,\phi_{N,2})^T$, we have that $\phi_{k,2}\equiv 0$ for $2\le k\le N$ and $\phi_{1,2}$ is a nonzero refinable function with its Fourier transform supported in $[-\pi,\pi]^d$. Now we define $\Psi_t=(\psi_t,0,\ldots,0)^T, 2\le t\le 3$, by

$$\widehat{\psi}_t(\xi)=\begin{cases}\frac{\widehat{\phi}_{1,2}(\xi)}{|\widehat{\phi}_{1,2}(\xi)|^{t-2}} & \text{if } \widehat{\phi}_{1,2}(\xi)\ne 0 \text{ and } \xi\in[-\pi,\pi]^d,\\ t-2 & \text{if } \widehat{\phi}_{1,2}(\xi)=0 \text{ and } \xi\in[-\pi,\pi]^d,\\ 0 & \text{if } \xi\notin[-\pi,\pi]^d.\end{cases} \tag{2.20}$$

Clearly,

$$\Psi_2=\Phi_2, \tag{2.21}$$

$$|\widehat{\psi}_{t_0}(\xi)|>0,\ \xi\in[-\pi,\pi]^d \tag{2.22}$$

for every $2<t_0\le 3$, and

$$\begin{cases}\lim_{t\to t_0}\widehat{\psi}_t(\xi)=\widehat{\psi}_{t_0}(\xi), & \xi\in\mathbb{R}^d\\ |\widehat{\psi}_{t_0}(\xi)|\le\max(|\widehat{\phi}_{1,2}(\xi)|,1),\ \xi\in[-\pi,\pi]^d,\\ |\widehat{\psi}_{t_0}(\xi)|\ge\min(|\widehat{\phi}_{1,2}(\xi)|,1),\ \xi\in\operatorname{supp}\widehat{\phi}_{1,2},\end{cases} \tag{2.23}$$

for all $2\le t_0\le 3$. Then

$$\|\widehat{\Psi}_t\|_2\ge\|\min(|\widehat{\phi}_{1,2}(\cdot)|,1)\|_2,\ 2\le t\le 3 \tag{2.24}$$

by (2.21) and (2.23),

$$\lim_{t\to t_0}\|\widehat{\Psi}_t\|_2=\|\widehat{\Psi}_{t_0}\|_2,\ 2\le t_0\le 3, \tag{2.25}$$

by (2.23), and

$$\widehat{\Psi}_t(2\xi)=m_t(\xi)\widehat{\Psi}_t(\xi),\quad 2<t\le 3 \tag{2.26}$$

by (2.22), where m_t is a 2π-periodic function whose restriction on $[-\pi,\pi]^d$ is given by

$$m_t(\xi):=\begin{pmatrix}\widehat{\psi}_t(2\xi)/\widehat{\psi}_t(\xi) & 0\\ 0 & I_{N-1}\end{pmatrix}.$$

Therefore by (2.18), (2.21) and (2.24) –(2.26), the functions $\Phi_t, 2 \le t \le 3$, which are defined by

$$\Phi_t := \frac{\Psi_t}{\|\Psi_t\|_2}, \tag{2.27}$$

form a continuous path in $\mathcal{S}_N$ connecting Φ_2 in (2.19) and Φ_3 in (2.27).

From (2.20) and (2.27), we see that the function $\Phi_3 = (\phi_{1,3}, 0, \ldots, 0)^T$ has the following property:

$$\widehat{\phi}_{1,3}(\xi) = \big((2\pi)^{-d/2}\chi_{[-\pi,\pi]^d}(\xi)e^{i\theta(\xi)}, 0, \ldots, 0\big)^T \tag{2.28}$$

for some real-valued measurable function $\theta(\xi)$. Define $\Psi_t, 3 \le t \le 4$, by

$$\widehat{\Psi}_t(\xi) = \big((2\pi)^{-d/2}\chi_{[-\pi,\pi]^d}(\xi)e^{i(4-t)\theta(\xi)}, 0, \ldots, 0\big)^T. \tag{2.29}$$

Using similar arguments as the those used in the proofs of various properties of $\Psi_t, 2 \le t \le 3$, we have the following properties for $\Psi_t, 3 \le t \le 4$: $\Psi_3 = \Phi_3$, $\lim_{t\to t_0} \|\Psi_t - \Psi_{t_0}\|_2 = 0$ for all $t_0 \in [3,4]$, and $\Psi_t \in \mathcal{S}_N$. Using the above properties of $\Psi_t, 3 \le t \le 4$, we conclude that the functions $\Phi_t, 3 \le t \le 4$, which are defined by

$$\Phi_t := \Psi_t, \tag{2.30}$$

form a continuous path in $\mathcal{S}_N$ connecting Φ_3 in (2.27) and Φ_4 in (2.30).

By (2.29) and (2.30),

$$\Phi_4 = S. \tag{2.31}$$

Hence $\Phi_t, 0 \le t \le 4$, is a continuous path in $\mathcal{S}_N$ connecting F and the fixed element $S \in \mathcal{S}_N$. Therefore the path-connectedness of the unit sphere $\mathcal{S}_N$ follows. □

Remark 2.2. *Let $\mathcal{R}_N([-\pi,\pi]^d)$ be the set of all refinable vectors with their Fourier transforms supported in $[-\pi,\pi]^d$. From the above proof of the path-connectedness of the unit sphere of the set $\mathcal{R}_N$, we have the path-connectedness of the unit sphere of the set $\mathcal{R}_N([-\pi,\pi]^d)$ in the topology induced from $(L^2)^N$.*

3. $(N+2)$-POINT RULE FOR THE SET OF REFINABLE VECTORS

In this section, we study the $(N+2)$-point rule for the set of refinable vectors, and give an elementary proof of Theorem 1.2 for the scalar case ($N=1$). Our proof of Theorem 1.2 for the general case ($N \ge 2$) is much more complicated and different from the scalar case, and will be given in Appendix A. In this section, we also show by two examples that this $(N+2)$-point rule is optimal for the scalar case, $N=1$. In particular, in Example 3.1 given any two distinct scalars ϵ_1 and ϵ_2 we construct two L^2 functions F and G such that $F+tG$ is *not refinable* for all real t except $t=\epsilon_1,\epsilon_2$, while in Example 3.2 given any countable subset

T of $\mathbb{R}$ we construct two functions F and G such that the functions $F+tG$ are *refinable* for all real t except those t in T.

Proof of Theorem 1.2 for $N=1$. Let

$$\tilde T=\big\{t\in\mathbb{R}:\mu\{\xi\in\mathbb{R}^d:\widehat F(\xi)+t\widehat G(\xi)=0 \text{ and } \widehat G(\xi)\neq 0\}>0\big\},$$

where μ is the Lebesgue measure. Then $\tilde T$ is at most countable by Lemma A.1 in Appendix A. We will prove that $F+tG$ is refinable when $t\notin\tilde T\cup\{0\}$.

For each $i=1,2,3$, by the refinability of $F+\epsilon_iG$,

$$\widehat F(2\xi)+\epsilon_i\widehat G(2\xi)=m_i(\xi)(\widehat F(\xi)+\epsilon_i\widehat G(\xi)) \tag{3.1}$$

for some 2π-periodic functions $m_i(\xi)$. This implies that

$$(\epsilon_2-\epsilon_1)\widehat G(2\xi)=(m_2(\xi)-m_1(\xi))\widehat F(\xi)+(\epsilon_2m_2(\xi)-\epsilon_1m_1(\xi))\widehat G(\xi),$$

and

$$(\epsilon_3-\epsilon_1)\widehat G(2\xi)=(m_3(\xi)-m_1(\xi))\widehat F(\xi)+(\epsilon_3m_3(\xi)-\epsilon_1m_1(\xi))\widehat G(\xi).$$

Thus

$$\alpha(\xi)\widehat F(\xi)+\beta(\xi)\widehat G(\xi)=0,\qquad \text{a.e.}\quad \xi\in\mathbb{R}^d, \tag{3.2}$$

where

$$\alpha(\xi)=(\epsilon_3-\epsilon_1)(m_2(\xi)-m_1(\xi))-(\epsilon_2-\epsilon_1)(m_3(\xi)-m_1(\xi)), \tag{3.3}$$

and

$$\beta(\xi)=(\epsilon_3-\epsilon_1)(\epsilon_2m_2(\xi)-\epsilon_1m_1(\xi))-(\epsilon_2-\epsilon_1)(\epsilon_3m_3(\xi)-\epsilon_1m_1(\xi)). \tag{3.4}$$

For $t\notin\tilde T\cup\{0\}$, we let

$$c_t(\xi)=\frac{t-\epsilon_2}{\epsilon_1-\epsilon_2}m_1(\xi)+\frac{\epsilon_1-t}{\epsilon_1-\epsilon_2}m_2(\xi),\ d_t(\xi)=\frac{t-\epsilon_2}{\epsilon_1-\epsilon_2}\epsilon_1m_1(\xi)+\frac{\epsilon_1-t}{\epsilon_1-\epsilon_2}\epsilon_2m_2(\xi).$$

Then by (3.1),

$$\widehat F(2\xi)+t\widehat G(2\xi)=c_t(\xi)\widehat F(\xi)+d_t(\xi)\widehat G(\xi). \tag{3.5}$$

Let

$$\begin{cases} E_1=\{\xi\in\mathbb{R}^d:\ \alpha(\xi)\neq0,\beta(\xi)\neq0\},\\ E_2=\{\xi\in\mathbb{R}^d:\ \alpha(\xi)\neq0,\beta(\xi)=0\},\\ E_3=\{\xi\in\mathbb{R}^d:\ \alpha(\xi)=0,\beta(\xi)\neq0\},\\ E_4=\{\xi\in\mathbb{R}^d:\ \alpha(\xi)=0,\beta(\xi)=0\}.\end{cases}$$

Since α and β are 2π-periodic, we have that

$$\cup_{i=1}^4E_i=\mathbb{R}^d\quad\text{and}\quad E_i+2\pi\mathbb{Z}^d=E_i,\quad i=1,2,3,4. \tag{3.6}$$

For $\xi\in E_1$, we obtain from (3.2) and (3.5) that

$$\widehat F(2\xi)+t\widehat G(2\xi)=\big(c_t(\xi)-d_t(\xi)\alpha(\xi)/\beta(\xi)\big)\widehat F(\xi). \tag{3.7}$$

Define m on E_1 by

$$m(\xi)=\begin{cases} 0 & \text{if } \xi\in E_5\cap E_1,\\ \big(1-t\alpha(\xi)/\beta(\xi)\big)^{-1}\big(c_t(\xi)-d_t(\xi)\alpha(\xi)/\beta(\xi)\big) & \text{if } \xi\in E_1\backslash E_5,\end{cases}$$

where $E_5:=\{\xi\in\mathbb{R}^d:\ (\widehat{G}(\xi+2k\pi))_{k\in\mathbb{Z}^d}=0\}$. The 2π-periodic function m is well-defined on E_1, that is, $1-t\alpha(\xi)/\beta(\xi)\neq 0$ for almost all $\xi\in E_1\backslash E_5$, since

$$(1-t\alpha(\xi)/\beta(\xi))(\widehat{F}(\xi+2k\pi))_{k\in\mathbb{Z}^d}=\big((\widehat{F}+t\widehat{G})(\xi+2k\pi)\big)_{k\in\mathbb{Z}^d}\neq 0\quad \text{a.e.}\quad \xi\in E_1\backslash E_5$$

by (3.2) and the assumption $t\notin\tilde{T}$. Therefore it follows from (3.2) and (3.7) that

$$\widehat{F}(2\xi)+t\widehat{G}(2\xi)=m(\xi)(\widehat{F}(\xi)+t\widehat{G}(\xi)),\qquad \xi\in E_1. \tag{3.8}$$

For $\xi\in E_2$, we obtain from (3.2) that $\widehat{F}(\xi)=0$. This together with (3.5) implies that

$$\widehat{F}(2\xi)+t\widehat{G}(2\xi)=d_t(\xi)\widehat{G}(\xi)=m(\xi)(\widehat{F}(\xi)+t\widehat{G}(\xi)),\ \xi\in E_2, \tag{3.9}$$

where $m(\xi)=\frac{1}{t}d_t(\xi)$ is a 2π-periodic function on E_2 and $t\notin\tilde{T}\cup\{0\}$.

Similarly for $\xi\in E_3$, we obtain from (3.2) that $\widehat{G}(\xi)=0$, and

$$\widehat{F}(2\xi)+t\widehat{G}(2\xi)=m(\xi)(\widehat{F}(\xi)+t\widehat{G}(\xi)),\ \xi\in E_3, \tag{3.10}$$

where $m(\xi)=c_t(\xi)$ is a 2π-periodic function on E_3.

Finally for $\xi\in E_4$, we have that $\alpha(\xi)=\beta(\xi)=0$. Solving the equations (3.3) and (3.4) leads to

$$m_3(\xi)=m_2(\xi)=m_1(\xi),\ \xi\in E_4.$$

This together with (3.5) implies

$$\widehat{F}(2\xi)+t\widehat{G}(2\xi)=m(\xi)(\widehat{F}(\xi)+t\widehat{G}(\xi)),\quad \xi\in E_4, \tag{3.11}$$

where $m(\xi)=m_1(\xi)$ is a 2π-periodic function on E_4.

Combining (3.6) and (3.8) – (3.11) proves the refinability of $F+tG$ with $t\notin\tilde{T}\cup\{0\}$. ☐

Example 3.1. Let ϵ_1,ϵ_2 be two distinct numbers. Define the functions f and g by

$$f=\frac{\epsilon_2 f_0-\epsilon_1 g_0}{\epsilon_2-\epsilon_1}\quad\text{and}\quad g=\frac{g_0-f_0}{\epsilon_2-\epsilon_1},$$

where f_0 is the Haar function $\chi_{[0,1]}$ and $g_0(x):=\max(1-|x|,0)$ is the hat function. Since $f+\epsilon_1 g=f_0$ and $f+\epsilon_2 g=g_0$, both $f+\epsilon_1 g$ and $f+\epsilon_2 g$ are refinable, and hence both belong to $\mathcal{R}_1$. Noting that

$$f+tg=\frac{\epsilon_2-t}{\epsilon_2-\epsilon_1}f_0+\frac{t-\epsilon_1}{\epsilon_2-\epsilon_1}g_0,$$

and using the explicit formulas for the Fourier transform of f_0 and g_0,

$$\widehat{f_0}(\xi) = \frac{1-e^{-i\xi}}{i\xi} \quad \text{and} \quad \widehat{g_0}(\xi) = \Big(\frac{1-e^{-i\xi}}{i\xi}\Big)^2,$$

one can verify that for all $t \neq \epsilon_1, \epsilon_2$,

$$\det\begin{pmatrix} \widehat{f+tg}(\xi+2k\pi) & \widehat{f+tg}(\xi) \\ \widehat{f+tg}(2(\xi+2k\pi)) & \widehat{f+tg}(2\xi) \end{pmatrix} \not\equiv 0$$

for all $k \in \mathbb{Z}\backslash\{0\}$. Therefore $f+tg$ are *not* refinable for all $t \neq \epsilon_1, \epsilon_2$, Moreover, using the characterization of the L^2-closure of the set of all refinable functions (Theorem 1.5), $f+tg$ are *not* included in the L^2-closure of the set of refinable functions for all $t \neq \epsilon_1, \epsilon_2$.

Example 3.2. Let $T = \{t_j\}_{j=1}^L$ be a countable subset of $\mathbb{R}$.
(i) For $L = 0$, we let f be refinable and $g = 0$. Then $f+tg$ is refinable for all $t \in \mathbb{R}$.
(ii) For $L = 1$, one may verify that for the functions f and g defined by $\hat{f} = \chi_{[\pi/2,\pi]} - t_1\chi_{[0,\pi/2]}$ and $\hat{g} = \chi_{[0,\pi/2]}$, $f+tg$ are refinable for all real t except $t = t_1$. This also shows that $\overline{\mathcal{R}}_1 \neq \mathcal{R}_1$.
(iii) If $2 \le L \le +\infty$, we let $\{E_j, 2 \le j \le L\}$ be a partition of the interval $[0,\pi/4]$ with $\mu(E_j) > 0, 2 \le j \le L$, and we define

$$\hat{f}(\xi) = \begin{cases} -t_1 & \text{if } \frac{\pi}{2} \le \xi \le \pi, \\ -t_j & \text{if } \xi \in \pi/4 + E_j, \\ -t_1 & \text{if } 0 \le \xi \le \frac{\pi}{4}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\hat{g}(\xi) = \chi_{[0,\pi]}(\xi).$$

Then for any real t,

$$\hat{f}(\xi) + t\hat{g}(\xi) = \begin{cases} -t_1 + t & \text{if } \frac{\pi}{2} \le \xi \le \pi, \\ -t_j + t & \text{if } \xi \in \frac{\pi}{4} + E_j, \\ -t_1 + t & \text{if } 0 \le \xi \le \frac{\pi}{4}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore for any $t \notin T$, $f+tg$ is refinable since

$$\hat{f}(2\xi) + t\hat{g}(2\xi) = m(\xi)(\hat{f}(\xi) + t\hat{g}(\xi)), \quad \xi \in \mathbb{R},$$

for a 2π-periodic function $m(\xi)$ whose restriction onto $[-\pi,\pi]$ is defined by

$$m(\xi) = \begin{cases} \frac{\hat{f}(2\xi)+t\hat{g}(2\xi)}{\hat{f}(\xi)+t\hat{g}(\xi)} & \text{if } \xi \in [0,\pi], \\ 0 & \text{if } \xi \in [-\pi, 0). \end{cases}$$

For $t = t_1$, $f + tg$ is not refinable since

$$\hat{f}(2\xi) + t\hat{g}(2\xi) = t_1 - t_2 \neq 0 \text{ and } \hat{f}(\xi) + t\hat{g}(\xi) = 0$$

for all $\xi \in (\pi/4 + E_2)/2$. For $t = t_j, 2 \leq j \leq L$, $f + tg$ is not refinable since

$$\hat{f}(2\xi) + t\hat{g}(2\xi) = t_j - t_1 \neq 0 \text{ and } \hat{f}(\xi) + t\hat{g}(\xi) = 0$$

for all $\xi \in \pi/4 + E_j$. Therefore the functions $f + tg$ are refinable for all real t except those $t \in T$.

4. Hyperplane property for the set of refinable vectors

We prove Theorem 1.3 by induction on the dimension M of the hyperplane

$$T(F_0, \ldots, F_M) := \Big\{F_0 + \sum_{m=1}^{M} t_m F_m : \ \mathbf{t} := (t_1, \ldots, t_M) \in \mathbb{R}^M\Big\}. \tag{4.1}$$

By Theorem 1.2, the conclusion in Theorem 1.3 holds for $M = 1$. Inductively assume that the conclusion in Theorem 1.3 holds for $M \geq 1$. Let $F_0, \ldots, F_{M+1}$ be functions in $(L^2)^N$ such that $F_m - F_0, 1 \leq m \leq M + 1$, are linearly independent. For any scalar t_{M+1}, we define

$$R(t_{M+1}) = \Big\{(t_1, ..., t_M) : \ F_0 + t_{M+1}F_{M+1} + \sum_{m=1}^{M} t_m F_m \in \mathcal{R}_N\Big\}. \tag{4.2}$$

By the inductive hypothesis, either $\mu(R(t_{M+1})) = 0$ or $\mu(\mathbb{R}^M \setminus R(t_{M+1})) = 0$, where μ is the Lebesgue measure. If there do not exist distinct numbers $t^q_{M+1}, 1 \leq q \leq N + 2$, such that $\mathbb{R}^M \setminus R(t^q_{M+1})$ has zero measure for any $q = 1, \cdots, N + 2$, then the set

$$E = \Big\{(t_1, ..., t_{M+1}) : \ F_0 + \sum_{m=1}^{M+1} t_m F_m \in \mathcal{R}_N\Big\} \tag{4.3}$$

has measure zero since

$$\mu(E) = \int_{\mathbb{R}} \mu(R(t_{M+1}))dt_{M+1} = 0.$$

Otherwise, there exist distinct scalars $t^1_{M+1}, \ldots, t^{N+2}_{M+1}$ such that

$$\mu\big(\mathbb{R}^M \setminus R(t^q_{M+1})\big) = 0, \ 1 \leq q \leq N + 2. \tag{4.4}$$

By Theorem 1.2, for any $(t_1, ..., t_M) \in T := \cap_{q=1}^{N+2} R(t^q_{M+1})$, we have that $F_0 + \sum_{m=1}^{M} t_m f_m + t f_{M+1} \in \mathcal{R}_N$ for all real t except countable many t's. Thus the set

$$E(t_1, ..., t_M) = \Big\{t \in \mathbb{R} : F_0 + \sum_{m=1}^{M} t_m F_m + tF_{M+1} \in \mathcal{R}_N\Big\}$$

satisfies

$$\mu(\mathbb{R}\setminus E(t_1,...,t_M))=0. \tag{4.5}$$

For the set E in (4.3),

$$\mathbb{R}^{M+1}\setminus E \ \subset\ \Big((\mathbb{R}^M\setminus(\cap_{q=1}^{N+2}R(t^q_{M+1})))\times\mathbb{R}\Big)\cup\tilde{E}, \tag{4.6}$$

where

$$\tilde{E}=\Big\{(t_1,...,t_M,t_{M+1}): t_{M+1}\in\mathbb{R}\setminus E(t_1,...,t_M), (t_1,...,t_M)\in\cap_{q=1}^{N+2}R(t^q_{M+1})\Big\}.$$

By (4.4), (4.5), and (4.6), we obtain

$$\begin{aligned}\mu(\mathbb{R}^{M+1}\setminus E) &\le \mu\Big((\mathbb{R}^M\setminus(\cap_{q=1}^{N+2}R(t^q_{M+1})))\times\mathbb{R}\Big)+\mu(\tilde{E})\\ &= 0+\int_{\cap_{q=1}^{N+2}R(t^q_{M+1})}\mu(\mathbb{R}\setminus E(t_1,...,t_M))dt_1...dt_M=0.\end{aligned}$$

This complete the proof of Theorem 1.3. □

5. L^2-CLOSURE OF THE SET OF REFINABLE VECTORS

In this section, we characterize the L^2-closure $\overline{\mathcal{R}}_N$ of all refinable vectors, and give some related remarks on the sets of all M-refinable vectors and of all polyscale refinable vectors.

Now we start to prove Theorem 1.5, with the arrangement of the proof in such a way that it will be used in the proof of Theorem 1.7.

Proof of Theorem 1.5. We will prove the following inclusions:

$$\overline{\mathcal{R}}_N\subset\mathcal{A}_{N+1}, \tag{5.1}$$

$$\cup_{\Phi\in\mathcal{R}_N}(S(\Phi))^N\subset\mathcal{A}_{N+1}, \tag{5.2}$$

$$\mathcal{A}_{N+1}\subset\overline{\mathcal{R}}_N, \tag{5.3}$$

and

$$\mathcal{A}_{N+1}\subset\cup_{\Phi\in\mathcal{R}_N}(S(\Phi))^N. \tag{5.4}$$

(i) The proof of (5.1). Let $F=(f_1,\dots,f_N)^T\in\overline{\mathcal{R}}_N$. Suppose that the sequence $F_n\in\mathcal{R}_N, n\ge1$, satisfies $\lim_{n\to\infty}\|F_n-F\|_2=0$. By Parseval's formula, $\lim_{n\to\infty}\|\widehat{F}_n-\widehat{F}\|_2=0$. Without loss of generality, we further assume that

$$\lim_{n\to\infty}\widehat{F}_n(\xi)=\widehat{F}(\xi)\quad\text{a.e.}, \tag{5.5}$$

for otherwise we may replace the sequence $\{F_n\}$ by a subsequence satisfying (5.5). As each F_n is in $\mathcal{R}_N$,

$$\widehat{F}_n(2\xi) = m_n(\xi)\widehat{F}_n(\xi) \quad \text{a.e.}$$

for some matrix-valued 2π-periodic function $m_n(\xi)$. For any $j \in \mathbb{N}$, applying the above refinement equation iteratively implies that

$$\widehat{F}_n(2^j\xi) = m_{n,j}(\xi)\widehat{F}_n(\xi) \quad \text{a.e.}$$

for some 2π-periodic function $m_{n,j}(\xi)$. Therefore any $(N+1)\times(N+1)$ submatrix $A_n(\xi)$ of the $(\mathbb{Z}_N \times \mathbb{Z}_+) \times \mathbb{Z}^d$ matrix $\big(\widehat{F}_n(2^j(\xi + 2k\pi))\big)_{j\in\mathbb{Z}_+, k\in\mathbb{Z}^d}$ can be written as

$$A_n(\xi) = B_n(\xi)C_n(\xi),$$

where $B_n(\xi)$ is an $(N+1)\times N$ matrix with entries being 2π-periodic functions, and $C_n(\xi) = (\widehat{F}_n(\xi + 2k_i\pi))_{1\le i\le N+1}$ for some distinct integers $k_i \in \mathbb{Z}^d, 1 \le i \le N+1$. This implies that

$$\det A_n(\xi) = 0 \quad \text{a.e.}$$

Taking limit in the above equality and using (5.5) prove (5.1).

(ii) The proof of (5.2). Take any $F \in \cup_{\Phi\in\mathcal{R}_N}(S(\Phi))^N$. Let $\Phi \in \mathcal{R}_N$ such that

$$\widehat{F}(\xi) = m_1(\xi)\widehat{\Phi}(\xi) \tag{5.6}$$

and

$$\widehat{\Phi}(2\xi) = m_0(\xi)\widehat{\Phi}(\xi) \tag{5.7}$$

for some $N \times N$ matrix-valued 2π-periodic functions $m_0(\xi)$ and $m_1(\xi)$. For any $j \in \mathbb{Z}_+$, using (5.6) and (5.7) repeatedly leads to

$$\widehat{F}(2^j\xi) = m_{2^j}(\xi)\widehat{\Phi}(\xi) \tag{5.8}$$

for some $N \times N$ matrix-valued 2π-periodic functions $m_{2^j}(\xi)$. Therefore any $(N+1)\times(N+1)$ submatrix $A(\xi)$ of the $(\mathbb{Z}_N\times\mathbb{Z}_+)\times\mathbb{Z}^d$ matrix $\big(\widehat{F}(2^j(\xi+2k\pi))\big)_{j\in\mathbb{Z}_+,k\in\mathbb{Z}^d}$ can be written as

$$A(\xi) = B(\xi)C(\xi),$$

(hence $\det A(\xi) = 0$ for almost all $\xi \in \mathbb{R}^d$), where $B(\xi)$ is an $(N+1)\times N$ matrix with entries being 2π-periodic functions, and $C(\xi) = (\widehat{\Phi}(\xi + 2k_i\pi))_{1\le i\le N+1}$ for some distinct $k_i \in \mathbb{Z}^d, 1 \le i \le N+1$. Therefore (5.2) follows.

(iii) The proof of (5.3). Take any function $F = (f_1, \ldots, f_N)^T \in (L^2)^N$ such that $\|F\|_2 = 1$ and any $(N+1)\times(N+1)$ submatrix of the matrix $\mathcal{F}(\xi)$ in (1.1) has zero determinant for almost all $\xi \in \mathbb{R}^d$. Denote by $r_j(F)(\xi)$ the rank of the $((j+1)N)\times\mathbb{Z}^d$ matrix $\big(\widehat{F}(2^i(\xi+2k\pi))\big)_{0\le i\le j,k\in\mathbb{Z}^d}$. We let $Z(F)$ be the set of all

$\xi \in \mathbb{R}^d$ such that $\mathcal{F}(\xi)$ is a zero matrix, and $E_{J,K}(F)$ be the set of all $\xi \in \mathbb{R}^d$ such that

$$r_j(F)(\xi) = \begin{cases} 0 & \text{if } 0 \le j < j_0, \\ k_0 & \text{if } j_0 \le j < j_1, \\ \vdots & \\ k_{M-1} & \text{if } j_{M-1} \le j < j_M, \\ k_M & \text{if } j \ge j_M, \end{cases}$$

where $J := (j_0, j_1, \ldots, j_M)$ and $K := (k_0, k_1, \ldots, k_M)$, with $0 \le j_0 < j_1 < \ldots < j_M$ and $1 \le k_0 < k_1 < \ldots < k_M \le N$, where $0 \le M \le N-1$. From our constructions and assumptions on $\mathcal{F}$, the sets $E_{J,K}(F)$ and $Z(F)$ have the following properties:

1. They are shift-invariant, i.e., $E_{J,K}(F) + 2\pi\mathbb{Z}^d = E_{J,K}(F)$ and $Z(F) + 2\pi\mathbb{Z}^d = Z(F)$.
2. They are mutually disjoint, i.e., $E_{J,K}(F) \cap E_{J',K'}(F)$ and $E_{J,K}(F) \cap Z(F)$ have zero Lebesgue measure for all (J,K) and (J',K') with $(J',K') \ne (J,K)$.
3. They form a decomposition of $\mathbb{R}^d$:

$$\mathbb{R}^d = Z(F) + \cup_{J,K} E_{J,K}(F). \tag{5.9}$$

For $\xi \in E_{J,K}(F)$, we let $A_{j_0}(\xi), \ldots, A_{j_M}(\xi)$ be $N \times N$ permutation matrices such that $A_{j_s}(\xi + 2k\pi) = A_{j_s}(\xi)$ for all $k \in \mathbb{Z}^d$ and $0 \le s \le M$, and $(\widehat{F}_{J,K}(\xi + 2k\pi))_{k \in \mathbb{Z}^d}$ has rank r_M, where we define

$$\widehat{F}_{J,K}(\xi) := (A_{j_0}(\xi))^{-1} \sum_{s=0}^{M} \begin{pmatrix} 0_{k_{s-1}} & 0 & 0 \\ 0 & I_{k_s - k_{s-1}} & 0 \\ 0 & 0 & 0_{N-k_s} \end{pmatrix} A_{j_s}(\xi) \widehat{F}(2^{j_s}\xi), \tag{5.10}$$

denote by I_l the $l \times l$ identity matrix, and set $k_{-1} = 0$. If $j_0 = 0$, we further require that the permutation matrix $A_{j_0}(\xi)$ be so chosen that

$$\begin{aligned} &\left\| \left\{ \begin{pmatrix} I_{k_0} & 0 \\ 0 & 0_{N-k_0} \end{pmatrix} A_{j_0}(\xi) \widehat{F}(\xi + 2k\pi) \right\}_{k \in \mathbb{Z}^d} \right\|_{\ell^2(\mathbb{Z}^d)} \\ &\qquad \ge \frac{1}{N} \left\| \left\{ A_{j_0}(\xi) \widehat{F}(\xi + 2k\pi) \right\}_{k \in \mathbb{Z}^d} \right\|_{\ell^2(\mathbb{Z}^d)} \quad \text{a.e. } \xi \in E_{J,K}(F). \end{aligned} \tag{5.11}$$

The existence of such measurable permutation matrices $A_{j_0}(\xi), \ldots, A_{j_M}(\xi)$ follows from the definition of the set $E_{J,K}(F)$. From the above construction of the function $\widehat{F}_{J,K}$, we see that

$$\begin{aligned} A_{j_0}(\xi)\widehat{F}_{J,K}(\xi) &= \Big(\underbrace{\widehat{f}_{n_{k_{-1}+1}}(2^{j_0}\xi),\dots,\widehat{f}_{n_{k_0}}(2^{j_0}\xi)}_{k_0-k_{-1}},\dots, \\ &\qquad \underbrace{\widehat{f}_{n_{k_{M-1}+1}}(2^{j_M}\xi),\dots,\widehat{f}_{n_{k_M}}(2^{j_M}\xi)}_{k_M-k_{M-1}},\underbrace{0,\dots,0}_{N-k_M}\Big)^T \end{aligned}$$

where $1 \le n_j \le N$ for $1 \le j \le k_M$, and that $A_{j_0}(\xi)(\widehat{F}_{J,K}(\xi+2k\pi))_{k\in\mathbb{Z}^d}$ has rank k_M on $E_{J,K}(F)$. (This implies that for any $1 \le n \le k_M$, the submatrix chosen from the first n rows of the matrix $A_{j_0}(\xi)(\widehat{F}_{J,K}(\xi+2k\pi))_{k\in\mathbb{Z}^d}$ has rank n on $E_{J,K}(F)$.)

For $0 \le t \le 1$, we define $\Psi_t \in (L^2)^N$ by

$$\widehat{\Psi}_t(\xi) = \begin{cases} \widehat{F}(\xi) + t\widehat{F}_{J,K}(\xi) & \text{if } \xi \in E_{J,K}(F), \\ 0 & \text{if } \xi \in Z(F). \end{cases} \tag{5.12}$$

The functions $\Psi_t, 0 \le t \le 1$, are well defined by (5.9) and (5.10). Moreover, one may easily verify that

$$\Psi_0 = F, \tag{5.13}$$

$$\lim_{t\to t_0} \widehat{\Psi}_t(\xi) = \widehat{\Psi}_{t_0}(\xi) \quad \text{a.e.} \tag{5.14}$$

for all $t_0 \in [0,1]$, and

$$|\widehat{\Psi}_t(\xi)| \le 2\sum_{j=0}^{\infty} |\widehat{F}(2^j\xi)| \quad \text{a.e.} \tag{5.15}$$

for all $t \in [0,1]$. By direct computation, we obtain

$$\begin{aligned} \Big\|\sum_{j=0}^{\infty}|\widehat{F}(2^j\xi)|\Big\|_2 &\le \sum_{j=0}^{\infty}\|\widehat{F}(2^j\xi)\|_2 \\ &= \sum_{j=0}^{\infty} 2^{-j/2}\|F\|_2 < \infty. \end{aligned} \tag{5.16}$$

Combining (5.14), (5.15) and (5.16), we conclude by the dominated convergence theorem that

$$\lim_{t\to t_0} \|\Psi_t - \Psi_{t_0}\|_2 = 0, \quad t_0 \in [0,1], \tag{5.17}$$

and

$$\|\Psi_t\|_2 \le (1+\sqrt{2})\|F\|_2,\ t \in [0,1]. \tag{5.18}$$

From the definition of $\Psi_t, 0 \le t \le 1$, and the construction of the function $\widehat{F}_{J,K}(\xi)$ on $E_{J,K}(F)$, we have that

$$\begin{aligned}
\|\Psi_t\|_2^2 &= \sum_{J,K} \int_{E_{J,K}(F)} |\widehat{F}(\xi) + t\widehat{F}_{J,K}(\xi)|^2 d\xi \\
&\ge \sum_{J,K \text{ with } j_0=0} \int_{E_{J,K}(F)} |\widehat{F}(\xi) + t\widehat{F}_{J,K}(\xi)|^2 d\xi \\
&\ge \sum_{J,K \text{ with } j_0=0} \int_{E_{J,K}(F)} (1+t)^2 \Big| \begin{pmatrix} I_{k_0} & 0 \\ 0 & 0_{N-k_0} \end{pmatrix} A_{j_0}(\xi)\widehat{F}(\xi)\Big|^2 d\xi \\
&\ge (1+t)^2 \sum_{J,K \text{ with } j_0=0} \int_{E_{J,K}(F)} \left(\frac{1}{N}\right)^2 |\widehat{F}(\xi)|^2 d\xi = \frac{(1+t)^2\|F\|_2^2}{N^2}. \qquad (5.19)
\end{aligned}$$

Now we prove the refinability of $\Psi_t, 0 < t \le 1$. Since there exist finitely many $N \times N$ permutations, we may divide the set $E_{J,K}(F)$ into the union of finitely many mutually disjoint subsets $E_{J,K,p}(F), p \in P_{J,K}$, such that $E_{J,K,p}(F) + 2\pi\mathbb{Z}^d = E_{J,K,p}(F)$, and the permutation matrices $A_{j_0}(\xi), \ldots, A_{j_M}(\xi)$ in the definition of $\widehat{F}_{J,K}(\xi)$ are constant matrices on $E_{J,K,p}(F)$ for every $p \in P_{J,K}$. Noting that the matrices $(\widehat{\Psi}_t(\xi + 2k\pi))_{k\in\mathbb{Z}^d}$ and $(\widehat{F}(2^j(\xi + 2k\pi)))_{0\le j\in\mathbb{Z}, k\in\mathbb{Z}^d}$ have the same dimension k_M for any $\xi \in E_{J,K,p}(F)$, there exist matrix-valued 2π-periodic functions $m_j^{J,K,p}(\xi), 0 \le j \in \mathbb{Z}$, such that

$$\widehat{F}(2^j\xi) = m_j^{J,K,p}(\xi)\widehat{\Psi}_t(\xi) \quad \text{a.e.} \quad \xi \in E_{J,K,p}(F). \tag{5.20}$$

On the other hand, from the definition of the set $Z(F)$ we have that

$$\widehat{\Psi}_t(\xi) = \widehat{F}(2^j\xi) = 0, \quad \xi \in Z(F). \tag{5.21}$$

Combining (5.20) and (5.21) and using (5.9), we conclude that for any $0 \le j \in \mathbb{Z}$, there exists a matrix-valued 2π-periodic function m_j such that

$$\widehat{F}(2^j\xi) = m_j(\xi)\widehat{\Psi}_t(\xi), \quad 0 \le j \in \mathbb{Z}. \tag{5.22}$$

From the construction of the function Ψ_t and the sets $E_{J,K,p}(F)$, there exist constant matrices $\tilde{m}_j^{J,K,p}$ such that

$$\widehat{\Psi}_t(\xi) = \begin{cases} \sum_{j=0}^M \tilde{m}_j^{J,K,p}\widehat{F}(2^{k_j}\xi) & \xi \in E_{J,K,p}, \\ 0 & \xi \in Z(F). \end{cases} \tag{5.23}$$

Then by (5.22) and (5.23),

$$\widehat{\Psi}_t(2\xi) = m_{J,K,p}(\xi)\widehat{\Psi}_t(\xi) \quad \text{if } 2\xi \in E_{J,K,p}$$

and

$$\widehat{\Psi}_t(2\xi) = m_Z(\xi)\widehat{\Psi}_t(\xi) \text{ if } 2\xi \in Z(F)$$

for some matrix-valued 2π-periodic functions $m_{J,K,p}$ and m_Z. Hence

$$\widehat{\Psi}_t(2\xi) = \Big(\sum_{J,K,p} m_{J,K,p}(\xi)\chi_{E_{J,K,p}(F)/2}(\xi) + m_Z(\xi)\chi_{Z(F)/2}(\xi)\Big)\widehat{\Psi}_t(\xi),$$

where χ_E is the characteristic function on a set E. This proves the refinability of $\Psi_t, 0 < t \le 1$.

From the above arguments, we conclude that the functions $\Phi_t, 0 \le t \le 1$, defined by

$$\Phi_t = \frac{\Psi_t}{\|\Psi_t\|_2}, \tag{5.24}$$

have the following properties:

$$\begin{cases} \Phi_0 = F; \\ \|\Phi_t\|_2 = 1; \\ \Phi_t \text{ is refinable for every } 0 < t \le 1; \text{ and} \\ \lim_{t\to t_0} \|\Phi_t - \Phi_{t_0}\|_2 = 0 \text{ for all } t_0 \in [0,1]. \end{cases} \tag{5.25}$$

This proves that $F \in \overline{\mathcal{R}}_N$, and hence (5.3) follows.

(iv) The proof of (5.4). Take any $F \in \mathcal{A}_{N+1}$ and let $\Phi_t, 0 \le t \le 1$, be as in (5.25). We define an $N \times N$ matrix-valued 2π-periodic function m by

$$m(\xi) = \begin{cases} 0 & \text{if } \xi \in Z(F) \text{ or } \xi \in E_{J,K}(F) \text{ with } j_0 \neq 0, \\ \frac{\|\Psi_1\|_2}{2} A_{J,K}(\xi) \begin{pmatrix} I_{k_0} & 0 \\ 0 & 0 \end{pmatrix} A_{j_0}(\xi) & \text{if } \xi \in E_{J,K}(F) \text{ with } j_0 = 0, \end{cases}$$

where Ψ_1 and $A_{j_0}(\xi)$ are defined as in (5.12) and (5.10) respectively, and the matrix $A_{J,K}(\xi)$ is chosen so that

$$A_{J,K}(\xi) \begin{pmatrix} I_{k_0} & 0 \\ 0 & 0 \end{pmatrix} A_{j_0}(\xi)\widehat{F}(\xi) = \widehat{F}(\xi) \quad \text{a.e.} \quad \xi \in E_{J,K}(F).$$

The existence of such a matrix $A_{J,K}(\xi)$ follows from the fact that both $(\widehat{F}(\xi + 2k\pi))_{k\in\mathbb{Z}^d}$ and $\begin{pmatrix} I_{k_0} & 0 \\ 0 & 0 \end{pmatrix} A_{j_0}(\xi)\big(\widehat{F}(\xi+2k\pi)\big)_{k\in\mathbb{Z}^d}$ have the same rank k_0 for almost all $\xi \in E_{J,K}(F)$.

Let Φ_1 be as in (5.24). For $\xi \in Z(F) \cup (\cup_{J,K \text{ with } j_0\neq 0} E_{J,K}(F))$, we have that $\widehat{F}(\xi + 2k\pi) = 0$, which yields

$$\widehat{F}(\xi) = m(\xi)\widehat{\Phi}_1(\xi) = 0 \quad \text{a.e.} \quad \xi \in Z(F) \cup (\cup_{J,K \text{ with } j_0\neq 0} E_{J,K}(F)). \tag{5.26}$$

For $\xi \in E_{J,K}(F)$ with $j_0 = 0$, from (5.10) it follows that

$$\begin{aligned} m(\xi)\widehat{\Phi}_1(\xi) &= \frac{1}{2} A_{J,K}(\xi) \begin{pmatrix} I_{k_0} & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad \times \Big(A_{j_0}(\xi)\widehat{F}(\xi) + \sum_{s=0}^{M} \begin{pmatrix} 0_{k_{s-1}} & 0 & 0 \\ 0 & I_{k_s - k_{s-1}} & 0 \\ 0 & 0 & 0_{N-k_s} \end{pmatrix} A_{j_s}(\xi)\widehat{F}(2^{j_s}\xi) \Big) \\ &= A_{J,K}(\xi) \begin{pmatrix} I_{k_0} & 0 \\ 0 & 0 \end{pmatrix} A_{j_0}(\xi)\widehat{F}(\xi) \end{aligned}$$

$$= \widehat{F}(\xi). \tag{5.27}$$

Combining (5.9), (5.26) and (5.27) proves $\widehat{F}(\xi) = m(\xi)\widehat{\Phi}_1(\xi)$ for almost all $\xi \in \mathbb{R}^d$. Hence (5.4) follows from the refinability of the function Φ_1. ☐

Remark 5.1. A $d \times d$ matrix M with integer entries is said to be a *dilation* if all its eigenvalues have norm strictly larger than one. We say that $F = (f_1, \ldots, f_N)^T \in (L^2)^N$ is an *M-refinable vector* if it satisfies a refinement equation

$$\widehat{F}(M^T\xi) = m(\xi)\widehat{F}(\xi), \tag{5.28}$$

where $m(\xi)$ is a matrix-valued 2π-periodic function. Using similar arguments, we may extend the result in Theorem 1.5 for $\overline{\mathcal{R}}_N$ to the L^2-closure of the set of all M-refinable vectors.

Theorem 5.2. *Let M be a dilation, $N \in \mathbb{N}$ and $F \in (L^2)^N$. Then the following statements are equivalent:*

- (i) *F is in the $(L^2)^N$-closure of the set of all M-refinable vectors.*
- (ii) *Any $(N+1)\times(N+1)$ submatrix of the $(\mathbb{Z}_N \times \mathbb{Z}_+)\times\mathbb{Z}^d$ matrix $\big(\widehat{F}\big((M^T)^j(\xi + 2k\pi)\big)\big)_{j\in\mathbb{Z}_+, k\in\mathbb{Z}^d}$ has zero determinant for almost all $\xi \in \mathbb{R}^d$.*
- (iii) *There exists a M-refinable vector Φ and a matrix-valued 2π-periodic function $m(\xi)$ such that*

$$\widehat{F}(\xi) = m(\xi)\widehat{\Phi}(\xi) \quad \text{a.e.} \quad \xi \in \mathbb{R}^d.$$

Remark 5.3. For a vector $F = (f_1, \ldots, f_N)^T \in (L^2)^N$, we say that F is *poly-scale M-refinable* if there exist 2π-periodic functions $m_i, 1 \le i \le I$, such that

$$\widehat{F}(\xi) = \sum_{i=1}^{I} m_i(B^{-i}\xi)\widehat{F}(B^{-i}\xi), \tag{5.29}$$

where $B = M^T$. Clearly the poly-scale M-refinability becomes the M-refinability when the scale I becomes one (see [15, 32] for the poly-scale refinability and its applications). It is known that if F is poly-scale refinable, then the vector $\tilde{F}$, whose Fourier transform is given by

$$\hat{\tilde{F}}(\xi) = \begin{pmatrix} \hat{F}(B^{I-1}\xi) \\ \vdots \\ \hat{F}(\xi) \end{pmatrix},$$

satisfies the following refinement equation

$$\hat{\tilde{F}}(B\xi) = \tilde{H}(\xi)\hat{\tilde{F}}(\xi), \tag{5.30}$$

where

$$\tilde{H}(\xi) = \begin{pmatrix} m_1(B^{I-1}\xi) & m_2(B^{I-2}\xi) & \cdots & m_{I-1}(B\xi) & m_I(\xi) \\ I_N & 0 & \cdots & 0 & 0 \\ 0 & I_N & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_N & 0 \\ 0 & 0 & \cdots & 0 & I_N \end{pmatrix}.$$

Denote the set of all poly-scale M-refinable vectors by $\mathcal{R}_{N,I}$, and its L^2-closure by $\overline{\mathcal{R}}_{N,I}$. Using similar arguments as in the proof of Theorem 1.5, we have the following result for the set $\overline{\mathcal{R}}_{N,I}$:

Theorem 5.4. *Let $N, I \geq 1$. If $F \in \overline{\mathcal{R}}_{N,I}$, then any $(NI+1)\times(NI+1)$ submatrix of the $(\mathbb{Z}_N \times \mathbb{Z}_+) \times \mathbb{Z}^d$ matrix $\big(\hat{F}\big((M^T)^j(\xi+2k\pi)\big)\big)_{j\in\mathbb{Z}_+, k\in\mathbb{Z}^d}$ has zero determinant for almost all $\xi \in \mathbb{R}^d$.*

6. TOPOLOGICAL AND GEOMETRICAL PROPERTIES OF $\overline{\mathcal{R}}_N$

In this section, we give the proof of Theorem 1.7.

Proof of Theorem 1.7. (i) For any $F = (f_1, \ldots, f_N)^T \in \overline{\mathcal{R}}_N$ with $\|F\|_2 = 1$, define $\Phi_0 = F$ and let $\Phi_t, 0 < t \leq 1$, be as in (5.24). By (5.25), the functions $\Phi_t, 0 \leq t \leq 1$, form a continuous path in the unit sphere of $\overline{\mathcal{R}}_N$ connecting the function F in the unit sphere of $\overline{\mathcal{R}}_N$ and Φ_1 in the unit sphere of $\mathcal{R}_N$. This together with the path-connectedness of the unit sphere of $\mathcal{R}_N$ (Theorem 1.1) proves the path-connectedness of the unit sphere of $\overline{\mathcal{R}}_N$.

(ii) For $J_N = \{j_1, \ldots, j_{N+1}\} \subset \mathbb{Z}_+ \times \{1, \ldots, N\}$ and $K_N = \{k_1, \ldots, k_{N+1}\} \subset \mathbb{Z}^d$ with cardinality $N+1$, we denote by $A_{J_N,K_N}(\xi, t)$ the $(N+1)\times(N+1)$ submatrix by taking the j_1-th, $\ldots, j_{N+1}$-th rows and the k_1-th, $\ldots$, k_{N+1}-th columns of the matrix $\big((\hat{F} + t\hat{G})(2^j(\xi+2k\pi))\big)_{j\in\mathbb{Z}_+, k\in\mathbb{Z}^d}$. Then the determinant of $A_{J_N,K_N}(\xi, t)$, denoted by $P_{J_N,K_N}(\xi, t)$, is a polynomial in t with degree at most $N+1$. By Theorem 1.5 and the assumption that $F + \epsilon_1 G, \ldots, F + \epsilon_{N+2} G \in \overline{\mathcal{R}}_N$, we have

$$P_{J_N,K_N}(\xi, \epsilon_i) = 0 \quad \text{a.e. } \xi \in \mathbb{R}^d$$

for $1 \le i \le N+2$. Therefore $P_{J_N,K_N}(\xi,t)$ is a zero polynomial in t. Hence

$$F + tG \in \mathcal{R}_{J_N,K_N} \quad \text{for all} \quad t \in \mathbb{R}, \tag{6.1}$$

where

$$\mathcal{R}_{J_N,K_N} = \{H = (h_1,\ldots,h_N)^T \in (L^2)^N : \ \det A_{J_N,K_N}(\xi) = 0, \ \text{a. e. } \xi \in \mathbb{R}^d\}, \tag{6.2}$$

and $A_{J_N,K_N}(\xi)$ is the $(N+1)\times(N+1)$ submatrix by taking the j_1-th, …, j_{N+1}-th rows and k_1-th, …, k_{N+1}-th columns of the matrix $\big(\widehat{H}(2^j(\xi+2k\pi))\big)_{j\in\mathbb{Z}_+,k\in\mathbb{Z}^d}$. By Theorem 1.5, we have

$$\overline{\mathcal{R}}_N = \cap_{J_N,K_N}\mathcal{R}_{J_N,K_N}. \tag{6.3}$$

Therefore $F + tG \in \overline{\mathcal{R}}_N$ by (6.1) and (6.3).

(iii) We prove the conclusion by induction on the dimension M of the hyperplane $T(F_0,\ldots,F_M)$ in (4.1). For $M=1$ the conclusion follows from the $(N+2)$-point rule for $\overline{\mathcal{R}}_N$, the second conclusion of this theorem. Inductively we assume that the conclusion holds for $M \ge 1$. Let $T(F_0,F_1,...,F_{M+1})$ be the hyperplane generated by $F_0,\ldots,F_{M+1}$. Then by the inductive hypothesis, for any $t_{M+1}\in\mathbb{R}$, the set $R(t_{M+1})$ in (4.2) either has zero Lebesgue measure (i.e., $\mu(R(t_{M+1}))=0$), or is the whole space (i.e., $R(t_{M+1})=\mathbb{R}^M$).

If there do not exist $(N+2)$ distinct numbers $t^n_{M+1}, 1\le n\le N+2$, such that $R(t^n_{M+1})=\mathbb{R}^M, 1\le n\le N+2$, then the set

$$E = \Big\{(t_1,...,t_{M+1}) : \ F_0 + \sum_{m=1}^{M+1} t_m F_m \in \overline{\mathcal{R}}_N\Big\}$$

has Lebesgue measure zero since

$$\mu(E) = \int_{t_{M+1}} \mu(R(t_{M+1}))dt_{M+1} = 0.$$

Otherwise there exist $(N+2)$ distinct numbers $t^n_{M+1}, 1\le n\le N+2$, such that $R(t^n_{M+1})=\mathbb{R}^M, 1\le n\le N+2$. Then by the $(N+2)$-point rule for $\overline{\mathcal{R}}_N$ we see that any function $F = F_0+\sum_{m=1}^{N+1} t_m F_m$ is in $\overline{\mathcal{R}}_N$. Hence $T(F_0,...,F_{M+1})\subset\overline{\mathcal{R}}_N$. This completes the inductive proof.

(iv) Take any $\epsilon>0$ and $F=(f_1,\ldots,f_N)^T\in(L^2)^N$. We need to show that $B(F,\epsilon)\cap((L^2)^N\setminus\overline{\mathcal{R}}_N)\ne\emptyset$, where $B(F,\epsilon)=\{H=(h_1,\ldots,h_N)^T\in(L^2)^N : \|H-F\|_2<\epsilon\}$. Take $G=(g_1,\ldots,g_N)^T\in L^2\setminus\overline{\mathcal{R}}_N$ with $\|G\|_2=1$. (The existence of such a function follows from the obvious observation that $\overline{\mathcal{R}}_N$ is a proper closed subset of $(L^2)^N$, see Example 3.1.) By the $(N+2)$-point rule for $\overline{\mathcal{R}}_N$ (the second conclusion in Theorem 1.7) and the fact that $G\notin\overline{\mathcal{R}}_N$, there exists $0<\delta_1<\epsilon$ such that $H_\delta := \delta F+G\notin\overline{\mathcal{R}}_N$ for all $\delta>(\delta_1)^{-1}$. On the other hand, $\|\delta^{-1}H_\delta-F\|_2=\delta^{-1}\|G\|_2<\epsilon$. Therefore $\delta^{-1}H_\delta\in B(F,\epsilon)\cap((L^2)^N\setminus\overline{\mathcal{R}}_N)$. $\square$

For $J_N = \{j_1, \dots, j_{N+1}\} \subset \mathbb{Z}_+ \times \{1, \dots, N\}$ and $K_N = \{k_1, \dots, k_{N+1}\} \subset \mathbb{Z}^d$, from the proof of Theorem 1.7, we have the following $(N+2)$-point rule and nowhere density for the set $\mathcal{R}_{J_N,K_N}$ in (6.2).

Theorem 6.1. *Let* $J_N \subset \mathbb{Z}_+ \times \{1, \dots, N\}^N$, $K_N \subset \mathbb{Z}^d$, *and* $F, G \in (L^2)^N$. *Then*

(i) ($(N+2)$-point rule) *If there are* $N+2$ *distinct numbers* $\epsilon_1, \dots, \epsilon_{N+2}$ *such that each* $F + \epsilon_i G \in \mathcal{R}_{J_N,K_N}$ *for* $1 \le i \le N+2$, *then* $F + tG \in \mathcal{R}_{J_N,K_N}$ *for* $t \in \mathbb{R}$.

(ii) (Hyperplane property) *Let* $F_0, \dots, F_M \in (L^2)^N$ *such that* $F_1 - F_0, \dots, F_M - F_0$ *are linearly independent. Then either* $g(t_1, \dots, t_m) := F_0 + \sum_{m=1}^M t_m F_m \in \mathcal{R}_{J_N,K_N}$ *for almost all* $\mathbf{t} := (t_1, \dots, t_M) \in \mathbb{R}^M$, *or* $g(t_1, \dots, t_m) \notin \mathcal{R}_{J_N,K_N}$ *for almost all* $\mathbf{t} := (t_1, \dots, t_M) \in \mathbb{R}^M$.

(iii) (Nowhere density property) *The set* $\mathcal{R}_{J_N,K_N}$ *is nowhere dense in* $(L^2)^N$.

7. Nowhere Density of MRA affine frames

In this section, we study the nowhere density of all MRA affine frames and prove Theorem 1.8.

For this purpose, we first recall a few definitions. In this paper, a *multiresolution analysis* (MRA) means a family of subspaces $\{V_j\}_{j \in \mathbb{Z}}$ of L^2 which satisfies the following conditions:

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(ii) $V_j = \{f(2^j \cdot) : f \in V_0\}$ for all $j \in \mathbb{Z}$;

(iii) $\cap_{j \in \mathbb{Z}} V_j = \{0\}$ and $\overline{\cup_{j \in \mathbb{Z}} V_j} = L^2$; and

(iv) There exist a function $\Phi = (\phi_1, \dots, \phi_N)^T$, called a *scaling vector*, such that the core scaling space V_0 is the shift-invariant space $S(\Phi)$, which is defined as the minimal closed subspace of L^2 containing $\{\phi_n(\cdot - k) : 1 \le n \le N, k \in \mathbb{Z}^d\}$.

From the nested property $V_{-1} \subset V_0$ in the condition (i), the scaling vector Φ is a refinable vector of length N.

Remark 7.1. There are several slightly different (but not equivalent) definitions of a multiresolution analysis, especially on the technical condition (iv). For instance, in a standard definition ([14, 18, 24, 25]), the core scaling space V_0 has an orthonormal basis $\{\phi_n(\cdot - k) : k \in \mathbb{Z}^d, 1 \le n \le N\}$ generated by finitely many functions $\phi_1, \dots, \phi_N$, while in the frame MRA (FMRA) theory (cf. [6]), the core scaling space V_0 has a shift-invariant frame $\{\phi_n(\cdot - k) : 1 \le n \le N, k \in \mathbb{Z}^d\}$. More generally, $\{V_j\}_{j \in \mathbb{Z}}$ is called a GMRA (cf. [4, 5]) when the core scaling space V_0 is only assumed to be shift-invariant, that is, $f \in V_0$ if and only if $f(\cdot - k) \in V_0$ for all $k \in \mathbb{Z}^d$.

Let $\psi_1, \dots, \psi_M \in L^2$. We say that $\Psi := (\psi_1, \dots, \psi_M)^T$ is an affine frame of L^2 (or $\psi_1, \dots, \psi_M$ are *affine frames* of L^2) if $\{\psi_{m;j,k} := 2^{jd/2} \psi_m(2^j \cdot - k) : j \in \mathbb{Z}, k \in$

$\mathbb{Z}^d, 1 \le m \le M\}$ is a frame for L^2, that is, there exist positive constants A, B such that

$$A\|f\|_2 \le \Big(\sum_{m=1}^{M} \sum_{j\in\mathbb{Z}, k\in\mathbb{Z}^d} |\langle f, \psi_{m;j,k}\rangle|^2\Big)^{1/2} \le B\|f\|_2 \quad \forall\ f \in L^2. \tag{7.1}$$

MRA-frames of length M, that are associated with a multiresolution analysis having a scaling vector of length N, are those affine frames $\Psi = (\psi_1, \ldots, \psi_M)^T$ such that $\psi_1, \ldots, \psi_M$ belong to the dilated scaling space V_1 for some multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ with a scaling vector $\Phi = (\phi_1, \ldots, \phi_N)^T$ of length N. We denote the set of all such MRA-frames by $\mathcal{F}_{M,N}$.

Now we give the proof of Theorem 1.8, where the new characterization $\overline{\mathcal{R}}_N = \cup_{\Phi\in\mathcal{R}_N}(S(\Phi))^N$ of the set $\overline{\mathcal{R}}_N$ in Theorem 1.5 plays a crucial role.

Proof of Theorem 1.8. By Theorems 1.5 and 1.7, the set $T := \cup_{\Phi\in\mathcal{R}_N}(S(\Phi))^N$ is a nowhere dense subset of $(L^2)^N$. For any vector $\Psi = (\psi_1, \ldots, \psi_M)^T \in \mathcal{F}_{M,N}$, we cut Ψ to a vector of length N, which is denoted by $\tilde{\Psi} = (\psi_1, \ldots, \psi_N)^T$. We denote the set of all such vectors $\tilde{\Psi}$ associated with $\Psi \in \mathcal{F}_{M,N}$ by $\widetilde{\mathcal{F}}_{M,N}$. Since $\widetilde{\mathcal{F}}_{M,N} \subset \{f(2\cdot) : f \in T\}$, we have that $\widetilde{\mathcal{F}}_{M,N}$ is a nowhere dense subset of $(L^2)^N$. On the other hand, one may easily verify that $\mathcal{F}_{M,N} \subset \tilde{\mathcal{F}}_{M,N} \times (L^2)^{M-N}$. Therefore the nowhere density of the set $\mathcal{F}_{M,N}$ in $(L^2)^M$ follows. ☐

In view of the previous results, we make the following conjectures on MRA affine frames.

Conjectures: *(i) The set $\cup_{N=1}^{\infty}\mathcal{F}_{M,N}$ is a nowhere dense subset of $(L^2)^M$ for any $M \ge 1$. (ii) The set $\mathcal{F}_{M,N}$ is path-connected for all $M, N \ge 1$.*

Appendix A. Proof of Theorem 1.2 for general N

In this appendix, we give the proof of Theorem 1.2 for $N \ge 2$. For this purpose, we need several technical lemmas:

Lemma A.1. *Let $M \in \mathbb{N}$, $g_0, \ldots, g_M$ be measurable functions on a subset $E \subset \mathbb{R}^d$ with finite Lebesgue measure, $K = \{\xi \in E : (g_0(\xi), \ldots, g_M(\xi))^T \neq 0\}$, and*

$$T = \Big\{t \in \mathbb{R} : \sum_{m=0}^{M} g_m(\xi)t^m \neq 0 \quad \text{for almost all} \quad \xi \in K\Big\}.$$

Then $T^c := \mathbb{R}\backslash T$ is at most countable.

Proof. Let $K_{m'}, 0 \le m' \le M$, be the subsets of K such that $g_{m'}(\xi) \neq 0$ on $K_{m'}$ but $g_m(\xi) = 0$ for all $m' < m \le M$. Then

$$K = \cup_{m'=0}^{M} K_{m'}, \tag{A.1}$$

and for any $1 \leq m' \leq M$,

$$\sum_{m=0}^{M} g_m(\xi)t^m = g_{m'}(\xi)\prod_{i=1}^{m'}(t - h_{m',i}(\xi)) \quad \text{a.e.} \quad \xi \in K_{m'} \tag{A.2}$$

for some measurable functions $h_{m',1}, \ldots, h_{m',m'}$ on $K_{m'}$. By (A.2), $\sum_{m=0}^{M} g_m(\xi)t^m = 0$ on a subset of $K_{m'}$ that has positive Lebesgue measure if and only if the set

$$K_{m',i}(t) = \{\xi \in K_{m'} : \ h_{m',i}(\xi) = t\}$$

has positive Lebesgue measure for some $1 \leq i \leq m'$. This together with (A.1) leads to

$$\begin{aligned} T^c &= \cup_{1\leq m'\leq M, 1\leq i\leq m'}\{t : \mu(K_{m',i}(t)) > 0\} \\ &= \cup_{1\leq m'\leq M, 1\leq i\leq m', 1\leq n, k\in\mathbb{Z}} E_{m',i}(n,k), \end{aligned} \tag{A.3}$$

where $E_{m',i}(n,k) = \{t \in \mathbb{R} : \ \mu(K_{m',i}(t)) > n^{-1}, k^{-1} \leq |t| \leq k\}$. Since

$$\begin{aligned} \frac{\#(E_{m',i}(n,k))}{n} &\leq k \sum_{t\in E_{m',i}(n,k)} |t|\mu(K_{m',i}(t)) \\ &\leq k \int_{\xi\in K_{m'}} \min\{|h_{m',i}(\xi)|, k\} d\xi \leq \mu(E)k^2 < \infty, \end{aligned}$$

where $\#(S)$ denotes the cardinality of a set S, we have that the set $E_{m',i}(n,k)$ has finite cardinality. This together with (A.3) proves that the set T^c is at most countable. ☐

Lemma A.2. *Let $F, G \in (L^2)^N$ and $H_t := F + tG, t \in \mathbb{R}$. Then there exist a subset T of $\mathbb{R}$ (depending on F and G only), and subsets E_{J_n,K_n} of $[-\pi,\pi]^d$ associated with the index sets $J_n := \{j_1, \ldots, j_n\} \subset \{1, \ldots, N\}$ and $K_n := \{k_1, \ldots, k_n\} \subset \mathbb{Z}^d$ having cardinality n, where $1 \leq n \leq N$, such that:*

- (i) *The set $T^c := \mathbb{R}\backslash T$ is at most countable.*
- (ii) *For any $t \in T$, the $n \times n$ submatrix obtained from taking the j_1-th, ..., j_n-th rows and the k_1-th, ..., k_n-th columns of the $N \times \mathbb{Z}^d$ matrix $(\widehat{H}_t(\xi + 2k\pi))_{k\in\mathbb{Z}^d}$ is nonsingular for almost all $\xi \in E_{J_n,K_n}$.*
- (iii) *For any $t \in \mathbb{R}$, the rank of the matrix $(\widehat{H}_t(\xi + 2k\pi))_{k\in\mathbb{Z}^d}$ is at most n for almost all $\xi \in E_{J_n,K_n}$.*
- (iv) *For any $t \in \mathbb{R}$, $(\widehat{H}_t(\xi + 2k\pi))_{k\in\mathbb{Z}^d} = 0$ holds for almost all $\xi \in E_0 := [-\pi,\pi]^d\backslash \cup_{n=1}^N \cup_{J_n,K_n} E_{J_n,K_n}$.*

Proof. For $1 \leq n \leq N$, $J_n = \{j_1, \ldots, j_n\} \subset \{1, \ldots, N\}$ and $K_n = \{k_1, \ldots, k_n\} \subset \mathbb{Z}^d$ having cardinality n, we denote by $P_{J_n,K_n}(\xi, t)$ the determinant of the $n \times n$ submatrix obtained from taking the j_1-th, ..., j_n-th rows and the k_1-th, ..., k_n-th columns of the $N\times\mathbb{Z}^d$ matrix $(\widehat{H}_t(\xi+2k\pi))_{k\in\mathbb{Z}^d}$. Then

$$P_{J_n,K_n}(\xi,t)=\sum_{m=0}^{n} h_{m,J_n,K_n}(\xi)t^m \tag{A.4}$$

for some measurable functions $h_{0,J_n,K_n},\ldots,h_{n,J_n,K_n}$ on $[-\pi,\pi]^d$. We let

$$\tilde E_{J_n,K_n}=\big\{\xi\in[-\pi,\pi]^d:\ (h_{0,J_n,K_n}(\xi),\ldots,h_{n,J_n,K_n}(\xi))^T\ne 0\big\}, \tag{A.5}$$

$$E_{J_n,K_n}=\tilde E_{J_n,K_n}\backslash(\cup_{n'=n+1}^N\cup_{J_{n'},K_{n'}}\tilde E_{J_{n'},K_{n'}}), \tag{A.6}$$

$$T_{J_n,K_n}\ =\ \{t\in\mathbb{R}:\ P_{J_n,K_n}(\xi,t)\ne 0 \text{ for almost all }\ \xi\in E_{J_n,K_n}\}, \tag{A.7}$$

and

$$T=\cap_{n=1}^N\cap_{J_n,K_n}T_{J_n,K_n}.$$

Now we show that the sets T and E_{J_n,K_n} satisfy all the conclusions in the lemma.

(i) $T^c=\cup_{n=1}^N\cup_{J_n,K_n}(\mathbb{R}\backslash T_{J_n,K_n})$ is a countable set by Lemma A.1.

(ii) From the definition of P_{J_n,K_n}, we see that for any $t\in T$, $P_{J_n,K_n}(\xi,t)\ne 0$ for almost all $\xi\in E_{J_n,K_n}$. Hence the second conclusion follows.

(iii) From the definition of the sets E_{J_n,K_n}, any $(n+1)\times(n+1)$ submatrix of $(\widehat{H}_t(\xi+2k\pi))_{k\in\mathbb{Z}^d}$ has zero determinant on E_{J_n,K_n}. This implies that for any $t\in\mathbb{R}$, the matrix $(\widehat{H}_t(\xi+2k\pi))_{k\in\mathbb{Z}^d}$ has rank at most n for almost all $\xi\in E_{J_n,K_n}$.

(iv) Since $E_0=[-\pi,\pi]^d\backslash(\cup_{n=1}^N\cup_{J_n,K_n}E_{J_n,K_n})=[-\pi,\pi]^d\backslash(\cup_{n=1}^N\cup_{J_n,K_n}\tilde E_{J_n,K_n})$, the polynomial $P_{J_1,K_1}(\xi,t)$ is identically zero on E_0 for any J_1 and K_1. This proves that for any $t\in\mathbb{R}$, $(\widehat{H}_t(\xi+2k\pi))_{k\in\mathbb{Z}^d}=0$ for almost all $\xi\in E_0$. □

Denote by I_j the identity matrix of size $j\times j$.

Lemma A.3. *Let $N,M\ge 1$, and $A(\xi),B(\xi)$ be $N\times M$ matrices whose entries are measurable functions on a measurable set E. Suppose that there exists $\{j_1,\ldots,j_M\}\subset\{1,\ldots,N\}$ such that the determinant of the $M\times M$ submatrix, which is obtained by taking the j_1-th, …, j_M-th rows of the matrix $A(\xi)+tB(\xi)$, is not a zero polynomial in $t\in\mathbb{R}$ for almost all $\xi\in E$. Then there exist a finite index set $\mathcal{P}$, a partition $\{E_p\}_{p\in\mathcal{P}}$ of the set E, i.e., $\cup_{p\in\mathcal{P}}E_p=E$ and $E_p\cap E_{p'}=\emptyset$ for any $p\ne p'$, and for each $p\in\mathcal{P}$, an integer $n_0(p)\in[0,M]$ and matrices $P_p(\xi)$ of size $N\times N$, $Q_p(\xi)$ of size $M\times M$ and $D_p(\xi)$ of size $n_0(p)\times n_0(p)$, $X_{1,p}(\xi)$ and $Y_{1,p}(\xi)$ of size $n_0(p)\times(M-n_0(p))$, $X_{2,p}(\xi)$ and $Y_{2,p}(\xi)$ of size $(M-n_0(p))\times(M-n_0(p))$, $X_{3,p}(\xi)$ and $Y_{3,p}(\xi)$ of size $(N-n_0(p))\times(M-n_0(p))$, such that*

$$\det P_p(\xi)\det Q_p(\xi)\ne 0, \tag{A.8}$$

$$\det(X_{2,p}(\xi)+tY_{2,p}(\xi))=1\quad\text{when } n_0(p)<M, \tag{A.9}$$

and

$$P_p(\xi)(A(\xi)+tB(\xi))Q_p(\xi) = \begin{pmatrix} tI_{n_0(p)}+D_p(\xi) & X_{1,p}(\xi)+tY_{1,p}(\xi) \\ 0 & X_{2,p}(\xi)+tY_{2,p}(\xi) \\ 0 & X_{3,p}(\xi)+tY_{3,p}(\xi) \end{pmatrix} \tag{A.10}$$

for almost all $\xi \in E_p$*, where* $t \in \mathbb{R}$*.*

Proof. For any matrices $A(\xi)$ and $B(\xi)$ of size $N \times M$, we will show that there exist a finite partition of E, say $\{E_p, p \in \mathcal{P}\}$, and matrices $P_p(\xi), Q_p(\xi), C_p(\xi)$ and $D_p(\xi)$ on E_p whose entries are measurable functions such that

$$\det P_p(\xi) \det Q_p(\xi) \neq 0 \quad \text{a.e.} \quad \xi \in E_p, \tag{A.11}$$

and

$$\begin{aligned} & P_p(\xi)(A(\xi)+tB(\xi))Q_p(\xi) \\ & = \begin{pmatrix} C_p(\xi) & tI_{n_0(p)}+D_p(\xi) & X_{01,t}(\xi) & X_{02,t}(\xi) & \cdots & \cdots & X_{0l,t}(\xi) \\ 0 & 0 & \tilde{X}_{01,t}(\xi) & \tilde{X}_{02,t}(\xi) & \cdots & \cdots & \tilde{X}_{0l,t}(\xi) \\ 0 & 0 & I_{n_1(p)} & X_{12,t}(\xi) & \cdots & \cdots & X_{1l,t}(\xi) \\ 0 & 0 & 0 & \tilde{X}_{12,t}(\xi) & \cdots & \cdots & \tilde{X}_{1l,t}(\xi) \\ 0 & 0 & 0 & I_{n_2(p)} & \ddots & \ddots & X_{2l,t}(\xi) \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \tilde{X}_{(l-1)l,t}(\xi) \\ 0 & 0 & 0 & 0 & \cdots & \cdots & I_{n_l(p)} \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix} \end{aligned} \tag{A.12}$$

for almost all $\xi \in E_p$, where $n_0(p), n_1(p), \ldots, n_l(p) \in \mathbb{Z}_+ := \{0\} \cup \mathbb{N}$, and $X_{ij,t}(\xi)$ and $\tilde{X}_{ij,t}(\xi)$ are matrices of the form $X(\xi)+tY(\xi)$ for some matrices $X(\xi)$ and $Y(\xi)$, which may be different at different occurrence.

We will prove, by induction on $N \geq 1$, the decomposition (A.12) for any matrix $A(\xi)+tB(\xi)$ of size $N \times M$. Indeed, it suffices to prove the conclusion under the additional assumption that $B(\xi)$ has constant rank m on the set E; for otherwise, we partition the set E into a finite union of $E_m, 0 \leq m \leq \min(N, M)$, so that the matrix $B(\xi)$ has rank m on E_m, and prove the conclusion for each E_m.

First we prove the conclusion for $N = 1$. In this case, $M \geq 1$ and either $m = 0$ or $m = 1$.

(i) $m = 0$: In this case, $A(\xi)+tB(\xi) = A(\xi)$. We may further assume that $A(\xi)$ has constant rank m' for almost all $\xi \in E$; for otherwise we partition the set E into a finite union of $E_{m'}, 0 \leq m' \leq \min(N, M) = 1$, so that the matrix $A(\xi)$ has constant rank m' on $E_{m'}$. Therefore by the above assumption on $A(\xi)$ and $B(\xi)$, there exist nonsingular matrices $P(\xi)$ of size 1×1 and $Q(\xi)$ of size $M\times M$ such that

$$P(\xi)(A(\xi)+tB(\xi))Q(\xi)=P(\xi)A(\xi)Q(\xi)=\begin{cases}(0\mid 1) & \text{if } m'=1\\ 0 & \text{if } m'=0\end{cases} \tag{A.13}$$

holds for almost all $\xi\in E$.

(ii) $m=1$: In this case, we choose nonsingular matrices $P(\xi)$ of size 1×1 and $Q(\xi)$ of size $M\times M$ so that

$$P(\xi)B(\xi)Q(\xi)=(0\mid 1),$$

which implies that

$$P(\xi)(A(\xi)+tB(\xi))Q(\xi)=(C(\xi)\mid tI_1+D(\xi)) \tag{A.14}$$

holds for some matrix-valued measurable functions $C(\xi)$ of size $1\times(M-1)$ and $D(\xi)$ of size 1×1.

Then the decomposition (A.12) for $N=1$ follows from (A.13) and (A.14).

Inductively, we assume that the decomposition (A.12) holds for any matrix $A(\xi)+tB(\xi)$ of size $n\times M$, where $1\le n\le N$. Consider any matrices $A(\xi),B(\xi)$ of size $(N+1)\times M$ with $B(\xi)$ having constant rank m on E without loss of generality. Clearly, either $m=N+1$ or $m\le N$.

(i) $m=N+1$: Let $P(\xi)$ and $Q(\xi)$ be nonsingular matrices so chosen that $P(\xi)B(\xi)Q(\xi)=(0\mid I_m)$. Then the decomposition (A.12) follows in this case, since

$$P(\xi)(A(\xi)+tB(\xi))Q(\xi)=(C(\xi)\mid tI_m+D(\xi)) \tag{A.15}$$

for some matrices $C(\xi)$ and $D(\xi)$ of sizes $m\times(M-m)$ and $m\times m$ respectively.

(ii) $m\le N$: Choose a nonsingular matrix $P_1(\xi)$ so that

$$P_1(\xi)B(\xi)=\begin{pmatrix}B_1(\xi)\\ 0\end{pmatrix},$$

where $B_1(\xi)$ is a matrix of size $m\times M$. Then

$$P_1(\xi)(A(\xi)+tB(\xi))=\begin{pmatrix}A_1(\xi)+tB_1(\xi)\\ A_2(\xi)\end{pmatrix} \tag{A.16}$$

for some matrices $A_1(\xi),A_2(\xi)$ whose entries are measurable functions on E. Moreover, without loss of generality, we may assume that the rank of $A_2(\xi)$ is a constant m' on E, for otherwise we partition the set E into finite subsets so that $A_2(\xi)$ has constant rank for almost all ξ in every subset of that partition. There are two subcases:

(a) $m' = 0$. In this case,

$$P_1(\xi)(A(\xi)+tB(\xi)) = \begin{pmatrix} A_1(\xi)+tB_1(\xi) \\ 0 \end{pmatrix}. \tag{A.17}$$

Then the decomposition (A.12) for $A(\xi)+tB(\xi)$ follows by using (A.17) and applying the inductive hypothesis to the $m \times M$ matrix $A_1(\xi)+tB_1(\xi)$ since $m \le N$.

(b) $m' \ge 1$: In this case, we select nonsingular matrices $P_2(\xi)$ and $Q_2(\xi)$ so that

$$P_2(\xi)A_2(\xi)Q_2(\xi) = \begin{pmatrix} 0 & I_{m'} \\ 0 & 0 \end{pmatrix}. \tag{A.18}$$

Combining (A.16) and (A.18), we obtain that

$$\begin{aligned} &\begin{pmatrix} I_m & 0 \\ 0 & P_2(\xi) \end{pmatrix} P_1(\xi)(A(\xi)+tB(\xi))Q_1(\xi)Q_2(\xi) \\ &= \begin{pmatrix} A_3(\xi)+tB_2(\xi) & A_4(\xi)+tB_3(\xi) \\ 0 & I_{m'} \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{A.19}$$

for some matrices $A_3(\xi), A_4(\xi), B_2(\xi), B_3(\xi)$ with measurable entries on E. Therefore the decomposition (A.12) for $A(\xi)+tB(\xi)$ follows by using (A.19) and applying the inductive hypothesis for the matrix $A_3(\xi)+tB_2(\xi)$ of size $(N-m')\times(M-m')$.

This completes the inductive proof of (A.12).

Denote by $A_1(\xi)+tB_1(\xi)$ the $M\times M$ submatrix obtained by taking the j_1-th, …, j_M-th rows of the matrix $A(\xi)+tB(\xi)$. Then by the assumption on $A(\xi)$ and $B(\xi)$, the determinant of $A_1(\xi)+tB_1(\xi)$ is not a zero polynomial in $t\in\mathbb{R}$ for almost all $\xi\in E$. Therefore by Lemma A.1, there exists a set T such that $\mathbb{R}\backslash T$ is at most a countable set and that for any $t\in T$ the matrix $A_1(\xi)+tB_1(\xi)$ is nonsingular for almost all $\xi\in E$. This implies that for those $t\in T$, the matrix $A(\xi)+tB(\xi)$ has rank M for almost all $\xi\in E$. Similarly for the matrix $tI_{n_0(p)}+D_p(\xi)$ in (A.12), we can find a set T_1 such that $\mathbb{R}\backslash T_1$ is at most a countable set and $tI_{n_0(p)}+D_p(\xi)$ is nonsingular for almost all $\xi\in E$ when $t\in T_1$. Therefore it follows from (A.12) that

$$n_0(p)+n_1(p)+\cdots+\cdots+n_l(p) = M$$

and the matrix $C_p(\xi)$ has size $n_0(p)\times 0$. Hence (A.10) follows. □

Now we start to prove Theorem 1.2 for $N\ge 2$.

Proof of Theorem 1.2 for $N \ge 2$. We let $H_t := (f_1+tg_1,\ldots,f_N+tg_N)^T =: (h_{1,t},\ldots,h_{N,t})^T$, the sets T and E_{J_n,K_n} be as in Lemma A.2, and

$E_0=[-\pi,\pi]^d\backslash(\cup_{n=1}^N\cup_{J_n,K_n}E_{J_n,K_n})$, where $t\in\mathbb{R}, 1\le n\le N$, and $J_n=\{j_1,\ldots,j_n\}\subset\{1,\ldots,N\}$ and $K_n=\{k_1,\ldots,k_n\}\subset\mathbb{Z}^d$ have cardinalities n. By Lemma A.2, $T^c:=\mathbb{R}\backslash T$ is at most countable. Therefore it suffices to prove that for any $t\in T$ there exist measurable functions m_0^t on E_0 and m_{J_n,K_n}^t on $E_{J_n,K_n}, 1\le n\le N$, such that for all $k\in\mathbb{Z}^d$,

$$\widehat{H}_t(2(\xi+2k\pi))=m_0^t(\xi)\widehat{H}_t(\xi+2k\pi)\quad\text{a.e.}\quad \xi\in E_0,\tag{A.20}$$

and

$$\widehat{H}_t(2(\xi+2k\pi))=m_{J_n,K_n}^t(\xi)\widehat{H}_t(\xi+2k\pi)\quad\text{a.e.}\quad \xi\in E_{J_n,K_n}.\tag{A.21}$$

On the set E_0, it follows from Lemma A.2 that for all $k\in\mathbb{Z}^d$,

$$\widehat{F}(\xi+2k\pi)=\widehat{G}(\xi+2k\pi)=0\quad\text{a.e.}\quad \xi\in E_0.\tag{A.22}$$

By the refinability of $F+\epsilon_qG, 1\le q\le N+2$, we have that for all $k\in\mathbb{Z}^d$,

$$\widehat{F}(2(\xi+2k\pi))+\epsilon_q\widehat{G}(2(\xi+2k\pi))=0\quad\text{a.e.}\quad \xi\in E_0,$$

which implies that for all $k\in\mathbb{Z}^d$,

$$\widehat{F}(2(\xi+2k\pi))=\widehat{G}(2(\xi+2k\pi))=0\quad\text{a.e.}\ \xi\in E_0.\tag{A.23}$$

Then (A.20) follows from (A.22) and (A.23) by letting $m_0^t(\xi)=0$ on E_0.

We prove (A.21) first for the case $n=N$. In this case, $J_N=\{1,\ldots,N\}$. For any $1\le n'\le N, K_N=\{k_1,\ldots,k_N\}\subset\mathbb{Z}^d$, and $k\in\mathbb{Z}^d\backslash K_N$, define

$$G_{n',k,t}(\xi):=\begin{pmatrix}\widehat{H}_t(\xi+2k_1\pi) & \cdots & \widehat{H}_t(\xi+2k_N\pi) & \widehat{H}_t(\xi+2k\pi)\\ \widehat{h}_{n',t}(2(\xi+2k_1\pi)) & \cdots & \widehat{h}_{n',t}(2(\xi+2k_N\pi)) & \widehat{h}_{n',t}(2(\xi+2k\pi))\end{pmatrix}$$

on E_{J_N,K_N}. Then

$$\det G_{n',k,t}(\xi)=\sum_{i=0}^{N+1}t^iP_{n',k,i}(\xi)$$

for some measurable functions $P_{n',k,i}(\xi), 0\le i\le N+1$, on E_{J_N,K_N}. By the refinability of the functions $H_{\epsilon_1},\cdots,H_{\epsilon_{N+2}}$, there exist some vector-valued 2π-periodic functions $m_{n',\epsilon_q}, 1\le q\le N+2$, such that

$$\widehat{h}_{n',\epsilon_q}(2(\xi+2l\pi))=m_{n',\epsilon_q}(\xi)\widehat{H}_{\epsilon_q}(\xi+2l\pi),\ l\in\mathbb{Z}^d,$$

which implies that for $t=\epsilon_q, 1\le q\le N+2$,

$$\sum_{i=0}^{N+1}t^iP_{k,n',i}(\xi)=0\quad\text{a.e.}\quad \xi\in E_{J_N,K_N}.\tag{A.24}$$

This leads to the *crucial* conclusion that (A.24) holds for all $t\in\mathbb{R}$, which in turn yields that for any real t, the rank of the matrix $G_{n',k,t}(\xi)$ is at most N for almost all $\xi\in E_{J_N,K_N}$. On the other hand, for $t\in T$, the $N\times N$ submatrix

$(\widehat{H}_t(\xi+2k_1\pi)\mid\cdots\mid\widehat{H}_t(\xi+2k_N\pi))$ of $G_{n',k,t}(\xi)$ has rank N for almost all $\xi\in E_{J_n,K_n}$. Therefore for $1\le n'\le N$ and $t\in T$, there exist vector-valued measurable functions $m_{n',t}(\xi)$ such that for all $k\in K_N$, and hence by (A.24) for all $k\in\mathbb{Z}^d$,

$$\widehat{h}_{n',t}(2(\xi+2k\pi))=m_{n',t}(\xi)\widehat{H}_t(\xi+2k\pi)\quad\text{a.e.}\quad \xi\in E_{J_N,K_N}. \tag{A.25}$$

The equation (A.21) then follows for the case $n=N$.

Now we prove (A.21) for the case $1\le n\le N-1$. By Lemma A.2, for any $t\in T$ the $n\times n$ submatrix obtained by taking the j_1-th, …, j_n-th rows of the $N\times n$ matrix $\widehat{H}_{t,K_n}(\xi):=\big(\widehat{H}_t(\xi+2k_1\pi)\mid\cdots\mid\widehat{H}_t(\xi+2k_n\pi)\big)$ is nonsingular for almost all $\xi\in E_{J_n,K_n}$, which implies that the determinant of the above $n\times n$ submatrix is not a zero polynomial in $t\in\mathbb{R}$ for almost all $\xi\in E_{J_n,K_n}$. Hence applying Lemma A.3 with $E=E_{J_n,K_n}$ and $A(\xi)+tB(\xi)=\widehat{H}_{t,K_n}(\xi)$ and without loss of generality, assuming no partition of the set E_{J_n,K_n} necessary, we can find an integer $0\le n_0\le n$, matrices $P_{J_n,K_n}(\xi),Q_{J_n,K_n}(\xi),D_{J_n,K_n}(\xi)$, and $X_{J_n,K_n,i}(\xi),Y_{J_n,K_n,i}(\xi),1\le i\le 3$, of different sizes such that for almost all $\xi\in E_{J_n,K_n}$,

$$\det P_{J_n,K_n}(\xi)\det Q_{J_n,K_n}(\xi)\ne 0, \tag{A.26}$$

$$\det\big(X_{J_n,K_n,2}(\xi)+tY_{J_n,K_n,2}(\xi)\big)=1\quad\text{when } n_0<n, \tag{A.27}$$

and

$$P_{J_n,K_n}(\xi)\widehat{H}_{t,K_n}(\xi)Q_{J_n,K_n}(\xi)=\begin{pmatrix} tI_{n_0}+D_{J_n,K_n}(\xi) & X_{J_n,K_n,1}(\xi)+tY_{J_n,K_n,1}(\xi)\\ 0 & X_{J_n,K_n,2}(\xi)+tY_{J_n,K_n,2}(\xi)\\ 0 & X_{J_n,K_n,3}(\xi)+tY_{J_n,K_n,3}(\xi)\end{pmatrix} \tag{A.28}$$

where $t\in\mathbb{R}$. Then it follows from the second conclusion in Lemma A.2 that for any $t\in T$,

$$\det\big(tI_{n_0}+D_{J_n,K_n}(\xi)\big)\ne 0\quad\text{for almost all } \xi\in E_{J_n,K_n}. \tag{A.29}$$

For any $k\in\mathbb{Z}^d\backslash K_n$, we write

$$\begin{aligned} &P_{J_n,K_n}(\xi)\Big(\widehat{H}_{t,K_n}(\xi)\mid\widehat{H}_t(\xi+2k\pi)\Big)\begin{pmatrix} Q_{J_n,K_n}(\xi) & 0\\ 0 & 1\end{pmatrix}\\ &=\begin{pmatrix} tI_{n_0}+D_{J_n,K_n}(\xi) & X_{J_n,K_n,1}(\xi)+tY_{J_n,K_n,1}(\xi) & \sum_{i=0}^1 t^iC_{1,i,k}(\xi)\\ 0 & X_{J_n,K_n,2}(\xi)+tY_{J_n,K_n,2}(\xi) & \sum_{i=0}^1 t^iC_{2,i,k}(\xi)\\ 0 & X_{J_n,K_n,3}(\xi)+tY_{J_n,K_n,3}(\xi) & \sum_{i=0}^1 t^iC_{3,i,k}(\xi)\end{pmatrix}\end{aligned} \tag{A.30}$$

for some vectors $C_{i,j,k}(\xi),1\le i\le 3,0\le j\le 1$, with components being measurable functions on E_{J_n,K_n}. We need the following claim to establish (A.21).

Claim: *For any $t\in\mathbb{R}$, there exists a matrix $R_{J_n,K_n,t}(\xi)$ such that*

$$\det R_{J_n,K_n,t}(\xi)=1 \tag{A.31}$$

and

$$
\begin{aligned}
&R_{J_n,K_n,t}(\xi)P_{J_n,K_n}(\xi)\Big(\widehat{H}_{t,K_n}(\xi)\mid \widehat{H}_t(\xi+2k\pi)\Big)\begin{pmatrix} Q_{J_n,K_n}(\xi) & 0\\ 0 & 1\end{pmatrix}\\
&= \begin{pmatrix} tI_{n_0}+D_{J_n,K_n}(\xi) & X_{J_n,K_n,1}(\xi)+tY_{J_n,K_n,1}(\xi) & \sum_{i=0}^1 t^iC_{1,i,k}(\xi)\\ 0 & X_{J_n,K_n,2}(\xi)+tY_{J_n,K_n,2}(\xi) & \sum_{i=0}^1 t^iC_{2,i,k}(\xi)\\ 0 & 0 & 0\end{pmatrix} \qquad \text{(A.32)}
\end{aligned}
$$

for almost all $\xi\in E_{J_n,K_n}$.

Proof of the Claim. First we assume that $1\le n_0\le n-1$. By (A.29), for every $t\in T$ the matrix $tI_{n_0}+D_{J_n,K_n}(\xi)$ has nonzero determinant for almost all $\xi\in E_{J_n,K_n}$. By Lemma A.2, for every $t\in\mathbb{R}$ any $(n+1)\times(n+1)$ submatrix of the $N\times(n+1)$ matrix $(\widehat{H}_{t,K_n}(\xi)\mid\widehat{H}_t(\xi+2k\pi))$ has zero determinant for almost all $\xi\in E_{J_n,K_n}$. The above two observations together with (A.27) lead to the following conclusion: for all $t\in T$,

$$
\begin{aligned}
\sum_{i=0}^1 t^iC_{3,i,k}(\xi) &= \Big(X_{J_n,K_n,3}(\xi)+tY_{J_n,K_n,3}(\xi)\Big)\\
&\quad\times\Big(X_{J_n,K_n,2}(\xi)+tY_{J_n,K_n,2}(\xi)\Big)^{-1}\Big(\sum_{i=0}^1 t^iC_{2,i,k}(\xi)\Big)\\
&= \Big(X_{J_n,K_n,3}(\xi)+tY_{J_n,K_n,3}(\xi)\Big)\\
&\quad\times\mathrm{adj}\Big(X_{J_n,K_n,2}(\xi)+tY_{J_n,K_n,2}(\xi)\Big)\times\Big(\sum_{i=0}^1 t^iC_{2,i,k}(\xi)\Big) \qquad \text{(A.33)}
\end{aligned}
$$

for almost all $\xi\in E_{J_n,K_n}$, where $\mathrm{adj}(A)$ is the adjoint of a matrix A. Note that for every $\xi\in E_{J_n,K_n}$, both sides of the above equation (A.33) are polynomials in t by (A.27). Hence for all $t\in\mathbb{R}$, (A.33) holds for almost all $\xi\in E_{J_n,K_n}$. Therefore for $1\le n_0\le n-1$, (A.31) and (A.32) hold by letting

$$
R_{J_n,K_n,t}(\xi)=\begin{pmatrix} I_{n_0} & 0 & 0\\ 0 & I_{n-n_0} & 0\\ 0 & S_{J_n,K_n,t}(\xi) & I_{N-n}\end{pmatrix}
$$

and

$$
S_{J_n,K_n,t}(\xi)=-(X_{J_n,K_n,3}(\xi)+tY_{J_n,K_n,3}(\xi))\,\mathrm{adj}\big(X_{J_n,K_n,2}(\xi)+tY_{J_n,K_n,2}(\xi)\big).
$$

For $n_0=0$, (A.33) follows directly from (A.27), (A.30) and the fact that the rank of the matrix

$$
\begin{pmatrix} X_{J_n,K_n,2}(\xi)+tY_{J_n,K_n,2}(\xi) & \sum_{i=0}^1 t^iC_{2,i,k}(\xi)\\ X_{J_n,K_n,3}(\xi)+tY_{J_n,K_n,3}(\xi) & \sum_{i=0}^1 t^iC_{3,i,k}(\xi)\end{pmatrix}
$$

is equal to n for almost all $\xi \in E_{J_n,K_n}$. In particular for $n_0 = 0$, (A.31) and (A.32) hold by letting

$$R_{J_n,K_n,t}(\xi) = \begin{pmatrix} I_n & 0 \\ S_{J_n,K_n,t}(\xi) & I_{N-n} \end{pmatrix}.$$

For $n_0 = n$, we have that $n - n_0 = 0$ and hence

$$\begin{aligned} & P_{J_n,K_n}(\xi)\big(\widehat{H}_{t,K_n}(\xi) \mid \widehat{H}_t(\xi + 2k\pi)\big) \begin{pmatrix} Q_{J_n,K_n}(\xi) & 0 \\ 0 & 1 \end{pmatrix} \\ = & \begin{pmatrix} tI_n + D_{J_n,K_n}(\xi) & \sum_{i=0}^1 t^i C_{1,i,k}(\xi) \\ 0 & \sum_{i=0}^1 t^i C_{3,i,k}(\xi) \end{pmatrix} \end{aligned}$$

by (A.28) and (A.30). Then for all $t \in T$ (hence for all $t \in \mathbb{R}$),

$$\sum_{i=0}^{1} t^i C_{3,i,k}(\xi) = 0 \quad \text{for almost all } \xi \in E_{J_n,K_n},$$

as for all $t \in T$ the rank of the matrix $\begin{pmatrix} tI_n + D_{J_n,K_n}(\xi) & \sum_{i=0}^1 t^i C_{1,i,k}(\xi) \\ 0 & \sum_{i=0}^1 t^i C_{3,i,k}(\xi) \end{pmatrix}$ is the same as that of the matrix $\big(\widehat{H}_{t,K_n}(\xi) \mid \widehat{H}_t(\xi + 2k\pi)\big)$ by (A.26), and hence is at most n for almost all $\xi \in E_{J_n,K_n}$ by Lemma A.2. Therefore for $n_0 = n$, (A.31) and (A.32) hold by letting $R_{J_n,K_n,t}(\xi) = I_N$.

Let us return to the proof of the equation (A.21). For any $1 \le n' \le N$ and $k \in \mathbb{Z}^d \backslash K_n$, we define

(A.34)

$$U_{n',k,t}(\xi) = \begin{pmatrix} tI_{n_0} + D_{J_n,K_n}(\xi) & X_{J_n,K_n,1}(\xi) + tY_{J_n,K_n,1}(\xi) & \sum_{i=0}^1 t^i C_{1,i,k}(\xi) \\ 0 & X_{J_n,K_n,2}(\xi) + tY_{J_n,K_n,2}(\xi) & \sum_{i=0}^1 t^i C_{2,i,k}(\xi) \\ D_{1,n',t}(\xi) & D_{2,n',t}(\xi) & \widehat{h}_{n',t}(2(\xi + 2k\pi)) \end{pmatrix}$$

and

$$(D_{1,n',t}(\xi) \mid D_{2,n',t}(\xi)) = \big(\widehat{h}_{n',t}(2(\xi + 2k_1\pi)) \mid \cdots \mid \widehat{h}_{n',t}(2(\xi + 2k_n\pi))\big) Q_{J_n,K_n}(\xi),$$

where $t \in \mathbb{R}$. From the refinability of $H_{\epsilon_q}, 1 \le q \le N+2$, there exists a vector-valued measurable function $m_{n',\epsilon_q}(\xi)$ for $1 \le q \le N+2$ such that for any $t = \epsilon_q, 1 \le q \le N+2$,

$$\begin{aligned} & \big(\widehat{h}_{n',t}(2(\xi + 2k_1\pi)) \mid \cdots \mid \widehat{h}_{n',t}(2(\xi + 2k_n\pi)) \mid \widehat{h}_{n',t}(2(\xi + 2k\pi))\big) \\ = \;& m_{n',t}(\xi)\big(\widehat{H}_{t,K_n}(\xi) \mid \widehat{H}_t(\xi + 2k\pi)\big). \end{aligned}$$

This together with (A.32) implies that for almost all $\xi \in E_{J_n,K_n}$,

$$\begin{aligned}
&\big(\widehat{h}_{n',t}(2(\xi+2k_1\pi)) \mid \cdots \mid \widehat{h}_{n',t}(2(\xi+2k_n\pi))| \, \widehat{h}_{n',t}(2(\xi+2k\pi))\big) \begin{pmatrix} Q_{J_n,K_n}(\xi) & 0 \\ 0 & 1 \end{pmatrix} \\
= \ & m_{n',t}(\xi)\big(\widehat{H}_{t,K_n}(\xi) \mid \widehat{H}_t(\xi+2k\pi)\big) \begin{pmatrix} Q_{J_n,K_n}(\xi) & 0 \\ 0 & 1 \end{pmatrix} \\
= \ & m_{n',t}(\xi)(P_{J_n,K_n}(\xi))^{-1}(R_{J_n,K_n,t}(\xi))^{-1} \\
& \times \left\{ R_{J_n,K_n,t}(\xi)P_{J_n,K_n}(\xi)\big(\widehat{H}_{t,K_n}(\xi) \mid \widehat{H}_t(\xi+2k\pi)\big) \begin{pmatrix} Q_{J_n,K_n}(\xi) & 0 \\ 0 & 1 \end{pmatrix} \right\} \\
= \ & m_{n',t}(\xi)(P_{J_n,K_n}(\xi))^{-1}(R_{J_n,K_n,t}(\xi))^{-1} \\
& \times \begin{pmatrix} tI_{n_0} + D_{J_n,K_n}(\xi) & X_{J_n,K_n,1}(\xi) + tY_{J_n,K_n,1}(\xi) & \sum_{i=0}^1 t^i C_{1,i,k}(\xi) \\ 0 & X_{J_n,K_n,2}(\xi) + tY_{J_n,K_n,2}(\xi) & \sum_{i=0}^1 t^i C_{2,i,k}(\xi) \\ 0 & 0 & 0 \end{pmatrix} \\
= \ & m_{n',t}(\xi)(P_{J_n,K_n}(\xi))^{-1}(R_{J_n,K_n,t}(\xi))^{-1} \begin{pmatrix} I_{n_0} & 0 \\ 0 & I_{n-n_0} \\ 0 & 0 \end{pmatrix} \\
& \times \begin{pmatrix} tI_{n_0} + D_{J_n,K_n}(\xi) & X_{J_n,K_n,1}(\xi) + tY_{J_n,K_n,1}(\xi) & \sum_{i=0}^1 t^i C_{1,i,k}(\xi) \\ 0 & X_{J_n,K_n,2}(\xi) + tY_{J_n,K_n,2}(\xi) & \sum_{i=0}^1 t^i C_{2,i,k}(\xi) \end{pmatrix} \\
=: \ & (m_{1,n',t} \mid m_{2,n',t}) \begin{pmatrix} tI_{n_0} + D_{J_n,K_n}(\xi) & X_{J_n,K_n,1}(\xi) + tY_{J_n,K_n,1}(\xi) & \sum_{i=0}^1 t^i C_{1,i,k}(\xi) \\ 0 & X_{J_n,K_n,2}(\xi) + tY_{J_n,K_n,2}(\xi) & \sum_{i=0}^1 t^i C_{2,i,k}(\xi) \end{pmatrix},
\end{aligned}$$

where $t = \epsilon_q, 1 \le q \le N+2$. In other words, for any $t = \epsilon_q, 1 \le q \le N+2$, there exist some vector-valued measurable functions $m_{1,n',t}(\xi)$ and $m_{2,n',t}(\xi)$ on E_{J_n,K_n}, which is independent of $k \in \mathbb{Z}^d \backslash K_n$, such that

(A.35)

$$\begin{cases} D_{1,n',t}(\xi) = m_{1,n',t}(\xi)(tI_{n_0} + D_{J_n,K_n}(\xi)) \\ D_{2,n',t}(\xi) = m_{1,n',t}(\xi)\big(X_{J_n,K_n,1}(\xi) + tY_{J_n,K_n,1}(\xi)\big) \\ \qquad\qquad +m_{2,n',t}(\xi)\big(X_{J_n,K_n,2}(\xi) + tY_{J_n,K_n,2}(\xi)\big) \\ \widehat{h}_{n',t}(2(\xi+2k\pi)) = m_{1,n',t}(\xi)\Big(\sum_{i=0}^1 t^i C_{1,i,k}(\xi)\Big) + m_{2,n',t}(\xi)\Big(\sum_{i=0}^1 t^i C_{2,i,k}(\xi)\Big) \end{cases}$$

hold for almost all $\xi \in E_{J_n,K_n}$. Hence for $t = \epsilon_q, 1 \le q \le N+2$,

$$\det U_{n',k,t}(\xi) = 0 \quad \text{a.e.} \quad \xi \in E_{J_n,K_n}. \tag{A.36}$$

Note that for every $\xi \in E_{J_n,K_n}$, $\det U_{n',k,t}(\xi)$ is a polynomial of degree at most $n+1 \le N+1$. This together with (A.36) leads to the *crucial* conclusion that for all $t \in \mathbb{R}$,

$$\det U_{n',k,t}(\xi) = 0 \quad \text{a.e.} \quad \xi \in E_{J_n,K_n}, \tag{A.37}$$

or equivalently that for all $t \in \mathbb{R}$, $U_{n',k,t}(\xi)$ is a singular matrix for almost all $\xi \in E_{J_n,K_n}$. For any $t \in T$, define

$$\begin{aligned} m_{n',t}(\xi) &= (D_{1,n',t}(\xi) \mid D_{2,n',t}(\xi)) \begin{pmatrix} tI_{n_0} + D_{J_n,K_n}(\xi) & X_{J_n,K_n,1}(\xi) + tY_{J_n,K_n,1}(\xi) \\ 0 & X_{J_n,K_n,2}(\xi) + tY_{J_n,K_n,2}(\xi) \end{pmatrix}^{-1} \\ &\quad \times (I_n\, 0) P_{J_n,K_n}(\xi), \end{aligned}$$

which is well defined by (A.27) and (A.29). Then

$$\begin{aligned} & m_{n',t}(\xi)\big(\widehat{H}_t(\xi + 2k_1\pi) \mid \cdots \mid \widehat{H}_t(\xi + 2k_n\pi)\big) \\ = \; & (D_{1,n',t}(\xi) \mid D_{2,n',t}(\xi)) \begin{pmatrix} tI_{n_0} + D_{J_n,K_n}(\xi) & X_{J_n,K_n,1}(\xi) + tY_{J_n,K_n,1}(\xi) \\ 0 & X_{J_n,K_n,2}(\xi) + tY_{J_n,K_n,2}(\xi) \end{pmatrix}^{-1} (I_n\, 0) \\ & \times \Big\{ P_{J_n,K_n}(\xi)\big(\widehat{H}_t(\xi + 2k_1\pi) \mid \cdots \mid \widehat{H}_t(\xi + 2k_n\pi)\big) Q_{J_n,K_n}(\xi) \Big\} \big(Q_{J_n,K_n}(\xi)\big)^{-1} \\ = \; & (D_{1,n',t}(\xi) \mid D_{2,n',t}(\xi)) \begin{pmatrix} tI_{n_0} + D_{J_n,K_n}(\xi) & X_{J_n,K_n,1}(\xi) + tY_{J_n,K_n,1}(\xi) \\ 0 & X_{J_n,K_n,2}(\xi) + tY_{J_n,K_n,2}(\xi) \end{pmatrix}^{-1} (I_n\, 0) \\ & \times \begin{pmatrix} tI_{n_0} + D_{J_n,K_n}(\xi) & X_{J_n,K_n,1}(\xi) + tY_{J_n,K_n,1}(\xi) \\ 0 & X_{J_n,K_n,2}(\xi) + tY_{J_n,K_n,2}(\xi) \\ 0 & X_{J_n,K_n,3}(\xi) + tY_{J_n,K_n,3}(\xi) \end{pmatrix} \big(Q_{J_n,K_n}(\xi)\big)^{-1} \\ = \; & (D_{1,n',t}(\xi) \mid D_{2,n',t}(\xi)) \big(Q_{J_n,K_n}(\xi)\big)^{-1} \\ = \; & \big(\widehat{h}_{n',t}(2(\xi + 2k_1\pi)) \mid \cdots \mid \widehat{h}_{n',t}(2(\xi + 2k_n\pi))\big), \end{aligned}$$

which implies that

$$\widehat{h}_{n',t}(2(\xi + 2k\pi)) = m_{n',t}(\xi)\widehat{H}_t(\xi + 2k\pi) \text{ for almost all } \xi \in E_{J_n,K_n} \tag{A.38}$$

where $k \in K_n$. By (A.37), for all $t \in T$,

$$\begin{aligned} \widehat{h}_{n',t}(2(\xi + 2k\pi)) &= (D_{1,n',t}(\xi) \mid D_{2,n',t}(\xi)) \\ &\quad \times \begin{pmatrix} tI_{n_0} + D_{J_n,K_n}(\xi) & X_{J_n,K_n,1}(\xi) + tY_{J_n,K_n,1}(\xi) \\ 0 & X_{J_n,K_n,2}(\xi) + tY_{J_n,K_n,2}(\xi) \end{pmatrix}^{-1} \\ &\quad \times \begin{pmatrix} \sum_{i=0}^{1} t^i C_{1,i,k}(\xi) \\ \sum_{i=0}^{1} t^i C_{2,i,k}(\xi) \end{pmatrix} \end{aligned}$$

where $k \notin K_n$. Therefore

$$\widehat{h}_{n',t}(2(\xi + 2k\pi)) = m_{n',t}(\xi)\widehat{H}_t(\xi + 2k\pi) \text{ for almost all } \xi \in E_{J_n,K_n} \tag{A.39}$$

where $k \notin K_N$. Combining (A.38) and (A.39) proves (A.21). This completes the proof.

Acknowledgement. The authors would like to thank the reviewers and the associate editor for their many helpful comments. The third author's research was partially supported by the Academic Research Fund No. R-146-000-073-112, National University of Singapore.

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