Localization of Calderon Convolution in the Fourier Domain

Akram Aldroubi, Qiyu Sun and Wai-Shing Tang

Abstract. In this paper, we introduce and study the localization of Calderon convolution for a finitely generated shift-invariant space in the Fourier domain.

§1. Introduction

We say that a linear space V of functions on $\mathbf{R}^d$ is *shift-invariant* if $f \in V$ implies that $f(\cdot - k) \in V$ for all $k \in \mathbf{Z}^d$. For instance, the space of all polynomials of degree at most N, the space of all p-integrable functions, and the space of all band-limited functions in L^2 are shift-invariant spaces. In this paper, we mostly restrict ourselves to the shift-invariant space $V_p(f_1, \ldots, f_n)$ generated by finitely many functions $f_1, \ldots, f_n$,

$$V_p(f_1, \ldots, f_n) := \{F *' D : \ D \in (\ell^p)^{(r)}\},$$

where $F = (f_1, \ldots, f_r)^T$. Here $\ell^p, 1 \le p \le \infty$, is the space of all p-summable sequences on $\mathbf{Z}^d$, $X^{(r)}$ is the direct sum of r copies of a linear space X, and $\|\cdot\|_{\ell^p}$ denotes the usual ℓ^p-norm. Also for $F = (f_1, \ldots, f_r)^T$, the *semi-convolution* $F*'$ on $(\ell^p)^{(r)}$ is defined by

$$F*' : \ (\ell^p)^{(r)} \ni D := \{D(k)\} \longmapsto F *' D := \sum_{k \in \mathbf{Z}^d} D(k)^T F(\cdot - k).$$

We also denote the shift-invariant space $V_p(f_1, \ldots, f_n)$ by $V_p(F)$ when $F = (f_1, \ldots, f_n)^T$. The finitely generated shift-invariant space $V_p(F)$ appears in wavelet analysis ([5, 6, 10, 11]), as well as in sampling theory ([1, 3, 4]). It is well known that the space of all band-limited functions in L^2 is a shift-invariant space generated by the sinc function $\frac{\sin \pi x}{\pi x}$.

Let $L^p, 1 \le p \le \infty$, be the space of all p-integrable functions on $\mathbf{R}^d$, while $\|\cdot\|_p$ denotes the usual L^p-norm. Let $f * g$ denote the usual convolution defined by $f * g(x) = \int_{\mathbf{R}^d} f(x-y)g(y)dy$.

Proceedings Title
XXX (eds.), pp. 1–3.

Copyright © 200x by Nashboro Press, Brentwood, TN.
ISBN 0-9728482-x-x
All rights of reproduction in any form reserved.

Definition 1.1. *Let V be a linear subspace of $L^p, 1 \leq p \leq \infty$. We say that $G = (g_1, \ldots, g_s)^T$ is a stable Calderon convolutor for V if there exist positive constants A and B such that*

$$A\|f\|_p \leq \|f * G\|_p \leq B\|f\|_p \quad \forall\ f \in V, \tag{1.1}$$

that is, the Calderon convolution C defined by

$$C : V \ni f \longmapsto (f * g_1, \ldots, f * g_s)^T \in (L^p)^{(s)} \tag{1.2}$$

is an isomorphism between V and its image.

The Calderon convolution for a shift-invariant space, which has properties analogous to the famous Calderon reproducing formula in L^2 but without dilation involved, was introduced in [4] for $p = 2$. A characterization for Calderon convolutions for a finitely generated shift-invariant space is established in [4].

In this paper, we introduce and study the localization of Calderon convolution for a finitely generated shift-invariant space in the Fourier domain, using similar techniques as in [14], where semi-convolution and the frame operator in finitely generated shift-invariant spaces are localized in the Fourier domain. Define the *Fourier transform* $\hat{f}$ of an integrable function f by $\hat{f}(\xi) = \int_{\mathbf{R}^d} e^{-ix\xi} f(x)dx$, and extend the definition to that of a tempered distribution as usual. For a shift-invariant space V and a measurable subset E of $\mathbf{R}^d$, we let

$$V_E = \{f \in V : \ \hat{f} \text{ is supported in } E + 2\pi\mathbf{Z}^d\}.$$

If $V = V_p(F)$ for some generator F and $p \in [1, \infty]$, we use $V_{p,E}(F)$ to denote V_E.

Definition 1.2. *We say that G is a stable Calderon convolutor for V at the frequency $\xi_0 \in \mathbf{R}^d$ if the estimate (1.1) holds for all $f \in V_{B(\xi_0,\delta)}$, where $\delta > 0$ and $B(\xi_0, \delta)$ is the ball with center ξ_0 and radius δ.*

In this paper, stable Calderon convolutors for a finitely generated shift-invariant space at a frequency (Theorem 2.1) are characterized. In Calderon reconstruction formula in L^2, the convolution with dilation and shifts of some dual function is used ([6, 11]), while in Theorem 2.1, we use different procedure in the reconstruction from the Calderon convolution for a finitely generated shift-invariant space. Precisely, the procedure includes the following three steps: convoluting with some dual function; sampling on the integer set to obtain sequences; and using the semi-convolution to recover the original function from the convolution.

In applications where a finitely generated shift-invariant space is used as the model space, the sensors (the convolutors) to collect data can be modeled as compactly supported functions. In this paper, we show that there are only two possibilities when both the generator F and the convolutor G are compactly supported: either G is an unstable Calderon convolutor for $V_p(F)$ at all frequencies, or G is a stable Calderon convolutor for $V_p(F)$ at almost all frequencies (Theorem 2.4).

Under certain decay assumption on the generator F and the convolutor G at infinity and also the closedness assumption of the shift-invariant space $V_p(F)$, it is proved that G is a stable Calderon convolutor for $V_p(F)$ if and only if it is a stable Calderon convolutor for $V_p(F)$ at any frequency (Theorem 3.1). So we may consider the stable Calderon convolutor at a certain frequency as the localization of stable Calderon convolutor in the Fourier domain. Also it indicates that the localization of Calderon convolution in the Fourier domain gives more information than the Calderon convolution.

§2. Characterization

In this section, we give a characterization of stable Calderon convolutors on a finitely generated shift-invariant space $V_p(F), 1 \le p \le \infty$, at some frequency.

To state our results, we recall the definitions of some function spaces, where the generators of the shift-invariant space and the dual functions in the reconstruction are chosen from. Let $\mathcal{L}^p, 1 \le p \le \infty$, be the space of all functions f with finite $\|f\|_{\mathcal{L}^p} := \left\| \sum_{j \in \mathbf{Z}^d} |f(\cdot + j)| \right\|_{L^p([0,1)^d)}$. Here $L^p(K), 1 \le p \le \infty$, is the space of all p-integrable functions on a measurable set K, and $\| \cdot \|_{L^p(K)}$ is the usual $L^p(K)$-norm. Let $W(L^p, \ell^1)$ be the space of all functions f so that $\|f\|_{W(L^p,\ell^1)} := \sum_{k \in \mathbf{Z}^d} \|f\|_{L^p(k+[0,1)^d)}$ is finite. Clearly $W(L^p, \ell^1) \subset \mathcal{L}^p$ for $1 \le p \le \infty$. For any $D \in (\ell^p)^{(r)}$ and $F \in \mathcal{L}^p$, it is shown in [9] that

$$\|F *' D\|_p \le \|D\|_{\ell^p} \|F\|_{\mathcal{L}^p}. \tag{2.1}$$

So in Theorem 2.1, we assume that the generator F of the shift-invariant space belongs to $\mathcal{L}^p$ for $1 \le p < \infty$ and it belongs to $W(L^\infty, \ell^1)$ for $p = \infty$. In that case $V_p(F)$ is a linear subspace of L^p by (2.1).

We say that a function f is a *C^∞-function with ℓ^1-decay* if the partial derivative $D^n f$ satisfies

$$\sum_{k \in \mathbf{Z}^d} \|D^n f\|_{L^\infty(k+[0,1)^d)} < \infty \text{ for any } n \in (\mathbb{Z}_+)^d$$

(see [12, 14]). A Schwartz function is a C^∞-function with ℓ^1-decay, and so are linear combinations of the integer shifts of a Schwartz function using ℓ^1 coefficients. In Theorem 2.1, the dual functions in the reconstruction from Calderon convolutions are C^∞-functions with ℓ^1-decay.

Now we state the characterization of a stable Calderon convolutor for a finitely generated shift-invariant space at a certain frequency.

Theorem 2.1. *Let $1 \le p \le \infty$, $\xi_0 \in \mathbf{R}^d$, $F = (f_1, \ldots, f_r)^T \in (\mathcal{L}^p)^{(r)}$ for $1 \le p < \infty$ and $F \in W(L^\infty, \ell^1)$ for $p = \infty$, and let $G = (g_1, \ldots, g_s)^T \in (L^1)^{(s)}$. Assume that the rank of the $r \times \mathbf{Z}^d$ matrix $(\widehat{F}(\xi + 2k\pi))_{k\in\mathbf{Z}^d}$ is independent of ξ in a small neighborhood of ξ_0. Then the following three statements are equivalent.*

(i) *G is a stable Calderon convolutor for $V_p(F)$ at the frequency ξ_0.*

(ii) *The rank of the $r \times (s \times \mathbf{Z}^d)$ matrix*

$$\Big(\widehat{F}(\xi_0 + 2k\pi)\overline{\widehat{g}_1(\xi_0 + 2k\pi)}, \ldots, \widehat{F}(\xi_0 + 2k\pi)\overline{\widehat{g}_s(\xi_0 + 2k\pi)}\Big)_{k\in\mathbf{Z}^d} \tag{2.2}$$

equals that of $(\widehat{F}(\xi_0 + 2k\pi))_{k\in\mathbf{Z}^d}$.

(iii) *There exist $\delta' \in (0, \delta_0)$ and C^∞-functions $\tilde{g}_{l,i}, 1 \le l \le s, 1 \le i \le r$, with ℓ^1-decay such that*

$$f = \sum_{i=1}^{r}\sum_{l=1}^{s}\sum_{k\in\mathbf{Z}^d} \langle f * g_l, \tilde{g}_{l,i}(\cdot - k)\rangle f_i(\cdot - k) \quad \forall\, f \in V_{p,B(\xi_0,\delta')}(F). \tag{2.3}$$

We remark that the rank assumption about $(\widehat{F}(\xi+2k\pi))_{k\in\mathbf{Z}^d}$ in Theorem 2.1 is closely related to the localization of p-frame property in the Fourier domain (see [14] or Lemma 4.3).

To prove Theorem 2.1, we need to recall some results about the localization of stable shifts in the Fourier domain. For any sequence $D = \{D(k)\}$ with polynomial growth, that is, $|D(k)| \le p(k)$, $k \in \mathbf{Z}^d$, for some polynomial p, its Fourier series $\mathcal{F}(D)$, to be defined by $\mathcal{F}(D)(\xi) := \sum_{k\in\mathbf{Z}^d} D(k)e^{-ik\xi}$, is a periodic tempered distribution. For any measurable subset E of $\mathbf{R}^d$, we let $\ell^p_E, 1 \le p \le \infty$, be the set of all ℓ^p sequences whose Fourier series are supported in $E + 2\pi\mathbf{Z}^d$.

Definition 2.2. *We say that F has ℓ^p stable shifts at the frequency $\xi_0 \in \mathbf{R}^d$ if there exist positive constants C and δ such that*

$$C^{-1}\|D\|_{\ell^p} \le \|F *' D\|_p \le C\|D\|_{\ell^p} \quad \forall\, D \in (\ell^p_{B(\xi_0,\delta)})^{(r)}, \tag{2.4}$$

and that F has ℓ^p stable shifts if (2.4) holds for all $D \in (\ell^p)^{(r)}$.

(See for instance [7, 8, 13, 15]) and the references therein for applications of ℓ^p stable shifts in the approximation by shift-invariant spaces, the regularity of scaling functions, and the convergence of cascade algorithms). The property of ℓ^p stable shifts at a certain frequency is the localization of the usual ℓ^p stable shifts in the Fourier domain. In [14], we establish the following characterization.

Lemma 2.3. *Let $\xi_0 \in \mathbf{R}^d$, and let $F = (f_1, \ldots, f_r)^T \in (\mathcal{L}^p)^{(r)}$ for $1 \le p < \infty$ and $F \in (W(L^\infty, \ell^1))^{(r)}$ for $p = \infty$. Then the following three statements are equivalent to each other.*

(i) F has ℓ^p stable shifts at the frequency ξ_0.

(ii) The $r \times \mathbf{Z}^d$ matrix $(\widehat{F}(\xi_0 + 2k\pi))_{k \in \mathbf{Z}^d}$ is of full rank.

(iii) There exist C^∞-functions $h_1, \ldots, h_r$ with ℓ^1-decay and a positive constant δ such that

$$d_i(k) = \langle F *' D, h_i(\cdot - k)\rangle, \ 1 \le i \le r, \ k \in \mathbf{Z}^d,$$

for any ℓ^p sequences $D := \{(d_1(k), \ldots, d_r(k))^T\}$, whose Fourier series $\mathcal{F}(D)$ are supported in $B(\xi_0, \delta) + 2\pi\mathbf{Z}^d$.

Now we start to prove Theorem 2.1.

Proof of Theorem 2.1 We divide the proof into the following steps: (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i). First we prove (i) $\Longrightarrow$ (ii). Suppose on the contrary that the assertion (ii) does not hold. Then there exists a nonzero (complex-valued) vector v so that

$$v^T \widehat{F}(\xi_0 + 2k_0\pi) \neq 0 \quad \text{for some } k_0 \in \mathbf{Z}^d, \tag{2.5}$$

and

$$v^T \widehat{F}(\xi_0 + 2k\pi)\widehat{g_l}(\xi_0 + 2k\pi) = 0 \quad \forall\, k \in \mathbf{Z}^d \text{ and } 1 \le l \le s. \tag{2.6}$$

Define f_0 by $\widehat{f_0} = v^T \widehat{F}$. For any $D \in \ell^p$, whose Fourier series $\mathcal{F}(D)$ is supported in $B(\xi_0, \delta) + 2\pi\mathbf{Z}^d$ for some sufficiently small $\delta > 0$, it follows from (1.1), (2.5), and Lemma 2.3 that

$$\|D\|_{\ell^p} \le C_1 \|f_0 *' D\|_p \le C_2 \sum_{l=1}^{s} \|(f_0 * g_l) *' D\|_p, \tag{2.7}$$

where C_1, C_2 are two positive constants independent of D. Similarly, for any $\epsilon > 0$ and any $\delta_0 > 0$, it follows from (1.1), (2.6) and Lemma 2.3 that there exists a nonzero sequence $D_{\epsilon,\delta_0} \in \ell^p$ whose Fourier series $\mathcal{F}(D)$ is supported in $B(\xi_0, \delta_0) + 2\pi\mathbf{Z}^d$ such that

$$\sum_{l=1}^{s} \|(f_0 * g_l) *' D_{\epsilon,\delta_0}\|_p \le \epsilon \|D_{\epsilon,\delta_0}\|_{\ell^p}. \tag{2.8}$$

Combining (2.7) and (2.8) leads to a contradiction since $\epsilon > 0$ can be chosen arbitrarily.

Next we prove (ii) $\Longrightarrow$ (iii). By our assumption, the rank of the $r\times \mathbf{Z}^d$ matrix $(\widehat{F}(\xi+2k\pi))_{k\in\mathbf{Z}^d}$ is $r_0\le r$ for all ξ in a small neighborhood of ξ_0. So we may assume that F has stable shifts at the frequency ξ_0 by Lemma 2.3, otherwise we replace F by the function $\tilde F$ defined by $\widehat{\tilde F}(\xi)=A(\xi)\widehat{F}(\xi)$, where the $r_0\times r$ matrix $A(\xi)$ with entries in Wiener class is so chosen that $\big(\widehat{\tilde F}(\xi+2k\pi)\big)_{k\in\mathbf{Z}^d}$ is of rank r_0 in a small neighborhood of ξ_0. For any $f=F*'D$ for some $D=(D_1,\ldots,D_r)^T\in(\ell^p)^{(r)}$ with $\mathcal{F}(D)$ supported in $B(\xi_0,\delta)+2\pi\mathbf{Z}^d$, taking Fourier transform on both sides of (2.3) leads to

$$\sum_{i=1}^r \mathcal{F}(D_i)(\xi)\widehat{f_i}(\xi)=\sum_{i,i'=1}^r\sum_{l=1}^s \mathcal{F}(D_i)(\xi)$$
$$\times\sum_{k\in\mathbf{Z}^d}\widehat{f_i}(\xi+2k\pi)\widehat{g_l}(\xi+2k\pi)\overline{\widehat{\tilde g}_{l,i'}(\xi+2k\pi)}\widehat{f_{i'}}(\xi).$$

By the arbitrariness of sequences $D_i\in(\ell^p)^{(r)}, 1\le i\le r$, and stable shifts of F at the frequency ξ_0, it suffices to find C^∞-functions $\tilde g_{l,i}$, $1\le l\le s, 1\le i\le r$, with ℓ^1-decay so that

$$\sum_{l=1}^s\sum_{k\in\mathbf{Z}^d}\widehat{f_i}(\xi+2k\pi)\widehat{g_l}(\xi+2k\pi)\overline{\widehat{\tilde g}_{l,i'}(\xi+2k\pi)}=\delta_{ii'},\ \ 1\le i,i'\le r, \tag{2.9}$$

for all $\xi\in B(\xi_0,\delta)+2\pi\mathbf{Z}^d$. By the assumption (ii), there exists $k_1,\ldots,k_T\in \mathbf{Z}^d$ so that $(\widehat{F}(\xi_0+2k_t\pi)\widehat{g_l}(\xi_0+2k_t\pi))_{(l,t)\in\{1,\ldots,s\}\times\{1,\ldots,T\}}$ has rank r. Let the Schwartz functions $h_l, 1\le l\le s$, be so chosen that $\widehat{h_l}$ is supported in $\cup_{t=1}^T B(\xi_0+2k_t\pi,\delta)$, and $\overline{\widehat{h_l}(\xi_0+2k_t\pi)}=\widehat{g_l}(\xi_0+2k_t\pi)$. Therefore we see that $\sum_{l=1}^s\widehat{g_l}(\xi_0+2k_t\pi)\widehat{h_l}(\xi_0+2k_t\pi)\ge 0$ for any $1\le t\le T$, and the inequality becomes an equality if and only if $\widehat{g_l}(\xi_0+2k_t\pi)=0$ for all $1\le l\le s$. Therefore the $r\times T$ matrix $\big(\widehat{F}(\xi_0+2k_t\pi)\sum_{l=1}^s\widehat{g_l}(\xi_0+2k_t\pi)\overline{\widehat{h_l}(\xi_0+2k_t\pi)}\big)_{1\le t\le T}$ has rank r, which implies that the function $\tilde F$ defined by $\widehat{\tilde F}(\xi)=\widehat{F}(\xi)\sum_{l=1}^s\widehat{g_l}(\xi)\overline{\widehat{h_l}(\xi)}$ has stable shifts at the frequency ξ_0. By Lemma 2.3, there exist C^∞-functions $\tilde g_i, 1\le i\le r$, with ℓ^1-decay so that

$$\sum_{k\in\mathbf{Z}^d}\widehat{\tilde f}_i(\xi+2k\pi)\overline{\widehat{\tilde g}_{i'}(\xi+2k\pi)}=\delta_{ii'},\ \ 1\le i,i'\le r,$$

for all $\xi\in B(\xi_0,\delta)+2\pi\mathbf{Z}^d$ for some $\delta>0$, where $\tilde F=(\tilde f_1,\ldots,\tilde f_r)^T$. This proves (2.9) and hence the assertion (iii) when we let $g_{l,i}$ be defined by $\widehat{g}_{l,i}(\xi)=\widehat{h_l}(\xi)\widehat{\tilde g}_i(\xi)$.

Finally we prove (iii) $\Longrightarrow$ (i). Recall that

$$\sum_{i=1}^r\|\{\langle f,f_i(\cdot-k)\rangle\}\|_{\ell^p}\le\sum_{i=1}^r\|f_i\|_{W(L^{p/(p-1)},\ell^1)}\|f\|_p, \tag{2.10}$$

for any $F = (f_1, \ldots, f_r)^T \in (W(L^{p/(p-1)}, \ell^1))^{(r)}$ and $f \in L^p$ ([14]). This together with (2.1) and (2.3) imply

$$\begin{aligned}\|f\|_p \leq & \|F\|_{\mathcal{L}^p} \sum_{i=1}^{r} \Big\| \Big\{ \sum_{l=1}^{s} \langle f * g_l, \tilde{g}_{l,i}(\cdot - k)\rangle \Big\} \Big\|_{\ell^p} \\ \leq & \|F\|_{\mathcal{L}^p} \sum_{i=1}^{r} \sum_{l=1}^{s} \|f * g_l\|_p \|\tilde{g}_{l,i}\|_{W(L^{p/(p-1)},\ell^1)} \quad \forall\, f \in V_{p,B(\xi_0,\delta)}(F).\end{aligned}$$

This proves (i). $\square$

Applying Theorem 2.1, we obtain the following result for the case that both the generator of the shift-invariant space and the Calderon convolutor are compactly supported.

Theorem 2.4. *Let $F = (f_1, \ldots, f_r)^T$ and $G = (g_1, \ldots, g_s)^T$ be vector-valued compactly supported L^p and L^1 functions respectively. Then either G is an unstable Calderon convolutor for $V_p(F)$ at all frequencies, or G is a stable Calderon convolutor for $V_p(F)$ at almost all frequencies.*

Proof: Let k_1 and k_2 be the maximum of the rank of the $r \times \mathbf{Z}^d$ matrix $A_1(\xi) := (\widehat{F}(\xi + 2k\pi))_{k\in\mathbf{Z}^d}, \xi \in \mathbf{R}^d$, and the $r \times (s \times \mathbf{Z}^d)$ matrix $A_2(\xi) := (\widehat{F}(\xi+2k\pi)\widehat{g}_1(\xi+2k\pi), \cdots, \widehat{F}(\xi+2k\pi)\widehat{g}_s(\xi+2k\pi))_{k\in\mathbf{Z}^d}, \xi \in \mathbf{R}^d$ respectively. Recall that $\widehat{f}_1, \ldots, \widehat{f}_r$ and $\widehat{g}_1, \ldots, \widehat{g}_s$ are analytic functions. Then there exists an open set O whose complement has zero Lebesgue measure so that the ranks of $A_1(\xi)$ and $A_2(\xi), \xi \in O$, are k_1 and k_2 respectively. Therefore the conclusion follows directly from Theorem 2.1. $\square$

§3. Calderon convolution and its localization

In this section, we consider the connection between Calderon convolution and its localization in the Fourier domain. In particular, we have

Theorem 3.1. *Let $1 \leq p \leq \infty$, $F = (f_1, \ldots, f_r)^T \in (\mathcal{L}^p)^{(r)}$ for $1 \leq p < \infty$ and $F \in W(L^\infty, \ell^1)$ for $p = \infty$, and let $G = (g_1, \ldots, g_s)^T \in (L^1)^{(s)}$. Assume that the rank of the $r \times \mathbf{Z}^d$ matrix $(\widehat{F}(\xi+2k\pi))_{k\in\mathbf{Z}^d}$ is independent of ξ. Then G is a stable Calderon convolutor for $V_p(F)$ if and only if G is a stable Calderon convolutor for $V_p(F)$ at every frequency.*

For the function F in Theorem 3.1, it is shown in [2, 14] that the rank $(\widehat{F}(\xi+2k\pi))_{k\in\mathbf{Z}^d}$ is independent of ξ if and only if the shift-invariant space $V_p(F)$ is closed in L^p. So applying Theorem 3.1, we see that under certain closedness assumption on the finitely generated shift-invariant system, the verification whether a Calderon convolutor is stable or not can be done pointwise in the Fourier domain. We do not know whether the above criterion can be established without the closedness assumption on that shift-invariant system.

Proof of Theorem 3.1 Clearly a stable Calderon convolutor for a shift-invariant space is a stable Calderon convolutor for that space at every frequency. Conversely suppose that $G \in (L^1)^{(s)}$ is a stable Calderon convolutor for the shift-invariant space $V_p(F)$ at every frequency. Thus for any $\xi_0 \in [-\pi,\pi]^d$ there exist a positive number $\delta(\xi_0)$ and C^∞-functions $\tilde{g}_{l,i,\xi_0}$ with ℓ^1-decay and by Theorem 2.1 so that

$$f = \sum_{i=1}^{r}\sum_{l=1}^{s}\sum_{k\in\mathbf{Z}^d} \langle f*g_l, \tilde{g}_{l,i,\xi_0}(\cdot-k)\rangle f_i(\cdot-k) \quad \forall\, f \in V_{p,B(\xi_0,\delta(\xi_0))}(F). \tag{3.1}$$

By the compactness of $[-\pi,\pi]^d$, there is a finite covering $\{B(\xi_1,\delta(\xi_1)/2)+2\pi\mathbf{Z}^d,\ldots,B(\xi_t,\delta(\xi_t)/2)+2\pi\mathbf{Z}^d\}$ of $\mathbf{R}^d$. Let $h_1,\ldots,h_t$ be C^∞ periodic functions that form a partition of unity associated with the above covering, that is, $\sum_{m=1}^t h_m(\xi) = 1$ for all $\xi \in \mathbf{R}^d$ and h_m is supported in $B(\xi_t,\delta(\xi_t)/2)+2\pi\mathbf{Z}^d$ for all $1 \le m \le t$. Then it follows from (3.1) that

$$\begin{aligned} f = \sum_{m=1}^{t} f_m &= \sum_{m=1}^{t}\sum_{i=1}^{r}\sum_{l=1}^{s}\sum_{k\in\mathbf{Z}^d} \langle f_m * g_l, \tilde{g}_{l,i,\xi_m}(\cdot-k)\rangle f_i(\cdot-k) \\ &= \sum_{i=1}^{r}\sum_{l=1}^{s}\sum_{k\in\mathbf{Z}^d} \langle f * g_l, \tilde{g}_{l,i}(\cdot-k)\rangle f_i(\cdot-k) \quad \forall\, f \in V_p(F), \end{aligned} \tag{3.2}$$

where $\widehat{f_m} = \hat{f} h_m$ and $\widehat{\tilde{g}}_{l,i} = \sum_{m=1}^t h_m \widehat{\tilde{g}}_{l,i,\xi_t}$. This gives a reconstruction formula of any function in $V_p(F)$ from its convolution $f * g_l, 1 \le l \le s$. Combining (2.1), (2.10) and (3.2) leads to

$$\|f\|_p \le C \sum_{l=1}^{s} \|f * g_l\|_p$$

for some positive constant C. Hence F is a stable Calderon convolutor for $V_p(F)$. □

§4. Calderon convolution and p-frame

Definition 4.1. *We say that F generates a p-frame at the frequency $\xi_0 \in \mathbf{R}^d$ if there exist positive constants δ, A and B such that*

$$A\|f\|_p \le \sum_{i=1}^{r} \|\{\langle f, f_i(\cdot-k)\rangle\}\|_{\ell^p} \le B\|f\|_p \tag{4.1}$$

holds for all $f \in V_{p,B(\xi_0,\delta)}(F)$*, and that* F *generates a* p*-frame if (4.1) holds for all* $f \in V_p(F)$*.*

The concept of p-frame at a certain frequency was introduced in [14] as the localization of the p-frame in the Fourier domain, while the p-frame property was introduced and characterized in [2]. As shown in [14], a vector-function F with certain decay at infinity, generates a p-frame if and only if it generates a p-frame at every frequency.

The p-frame property is related to Calderon convolution in the situation that the convolutor is the same as the generator of the shift-invariant space. In particular, we have

Theorem 4.2. *Let* $\xi_0 \in \mathbf{R}^d$*, and let* $F = (f_1, \ldots, f_r)^T \in (\mathcal{L}^p)^{(r)} \cap (W(L^{p/(p-1)}, \ell^1))^{(r)}$ *for* $1 \le p < \infty$ *and* $F \in (W(L^\infty, \ell^1))^{(r)}$ *for* $p = \infty$*. If* F *generates a* p*-frame at the frequency* ξ_0*, then* F *is a stable Calderon convolutor of the shift-invariant space* $V_p(F)$ *at the frequency* ξ_0*.*

To prove Theorem 4.2, we recall the characterization in [14] of p-frame at a frequency ξ_0:

Lemma 4.3. *Let* $F = (f_1, \ldots, f_r)^T \in (\mathcal{L}^p)^{(r)} \cap (W(L^{p/(p-1)}, \ell^1))^{(r)}$ *for* $1 \le p < \infty$ *and* $F \in (W(L^\infty, \ell^1))^{(r)}$ *for* $p = \infty$*. Then for any* $\xi_0 \in \mathbf{R}^d$ *the following statements are equivalent to each other.*

(i) F *generates a* p*-frame at the frequency* ξ_0*.*

(ii) The space $V_{p,B(\xi_0,\delta)}(F)$ *is a closed subspace of* L^p *for sufficiently small* $\delta > 0$*.*

(iii) The rank of the $r \times \mathbf{Z}^d$ *matrix* $(\widehat{F}(\xi + 2k\pi))_{k\in\mathbf{Z}^d}$ *is independent of* ξ *in a small neighborhood of* ξ_0*.*

Proof of Theorem 4.2. Note that for $F = (f_1, \ldots, f_r)^T \in (L^1)^{(r)}$, the rank of the $r \times (r \times \mathbf{Z}^d)$ matrix

$$\Big(\widehat{F}(\xi + 2k\pi)\overline{\widehat{f_1}(\xi + 2k\pi)}, \ldots, \widehat{F}(\xi + 2k\pi)\overline{\widehat{f_r}(\xi + 2k\pi)}\Big)_{k\in\mathbf{Z}^d}$$

equals that of $(\widehat{F}(\xi + 2k\pi))_{k\in\mathbf{Z}^d}$ for all $\xi \in \mathbf{R}^d$. Then the assertion follows from Theorem 2.1 and Lemma 4.3. □

The converse of Theorem 4.2 is not true, as demonstrated in the following example.

Example 4.4. *Let* h *be a* C^∞*-function so chosen that* $h(x) = 1$ *for all* $x \in [-3/8, 3/8]$*, and* $h(x) = 0$ *for all* $x \in \mathbb{R}\backslash[-1/2, 1/2]$*. Define* f_1 *and* f_2 *by*

$$\widehat{f_1}(\xi) = h\Big(\frac{\xi - \pi}{4\pi}\Big) \quad \text{and} \quad \widehat{f_2}(\xi) = (1 - e^{-i\xi})h\Big(\frac{\xi}{\pi}\Big).$$

By direct computation, the rank of $(\widehat{F}(\xi+2k\pi))$ *is equal to* 1 *for* $\xi=0$, *and it equals* 2 *for* $\xi\in[-3\pi/8,3\pi/8]\backslash\{0\}$. *Thus* $F=(f_1,f_2)^T$ *does not generate a* p*-frame at the frequency* 0 *by Lemma 4.3. Note that any* $f\in V_{p,B(0,\delta)}(F)$ *with* $0<\delta<\pi/8$ *can be written as* $f=F*'D$ *for some* $D=(D_1,D_2)^T$ *with* $D_1,D_2\in\ell^p_{B(0,2\delta)}$. *Then*

$$\begin{aligned}\widehat{f*f_1}(\xi)=&\mathcal{F}(D_1)(\xi)\big(h((\xi-\pi)/4\pi)\big)^2\\&\qquad+(1-e^{-i\xi})\mathcal{F}(D_2)(\xi)h(\xi/\pi)h\big((\xi-\pi)/4\pi\big)\\=&\mathcal{F}(D_1)(\xi)h\big((\xi-\pi)/4\pi\big)+\mathcal{F}(D_2)(\xi)h\big(\xi/\pi\big)=\hat{f}(\xi).\end{aligned}$$

Therefore $\|f*f_1\|_p=\|f\|_p$ *for any* $f\in V_{p,B(0,\delta)}(F)$ *with* $0<\delta<\pi/8$, *and hence* F *is a stable Calderon convolutor for the shift-invariant space* $V_p(F)$ *at the frequency* 0.

Acknowledgments. Research of the first author is supported in part by NSF grants DMS-0103104 and DMS-0139740. Research of the third author is supported in part by the Academic Research Fund No. R-146-000-025-112, National University of Singapore.

References

1. A. Aldroubi and K. Gröchenig, Nonuniform sampling and reconstruction in shift-invariant spaces, *SIAM Rev.*, 43(2001), 585–620.
2. A. Aldroubi, Q. Sun, and W.-S. Tang, p-frames and shift-invariant spaces of L^p, *J. Fourier Anal. Appl.*, 7(2001), 1–21.
3. A. Aldroubi, Q. Sun, and W.-S. Tang, Non-uniform average sampling and reconstruction in multiply generated shift-invariant spaces, *Constr. Approx.*, to appear.
4. A. Aldroubi, Q. Sun, and W.-S. Tang, Convolution, average sampling, and a Calderon resolution of the identity for shift-invariant spaces, In preparation.
5. C. K. Chui, *An Introduction to Wavelets*, Academic Press, 1992.
6. I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia, 1992.
7. G. Jetter and G. Plonka, A survey on L^2 approximation orders from shift-invariant spaces, In *Multivariate Approximation and Applications* (N. Dyn, D. Leviatan, D. Levin, A. Pinkus, eds.), Cambridge University Press, 2001, pp. 73–111.
8. R. Q. Jia, Subdivision schemes in L^p spaces, *Adv. Comput. Math.*, 3(1995), 309–341.
9. R. Q. Jia and C. A. Micchelli, Using the refinement equation for the construction of pre-wavelets II: powers of two, In *Curve and Surfaces*

(P. J. Laurent, A. Le Méhauté and L. L. Schumaker eds.), Academic Press, New York 1991, pp. 209–246.

10. S. Mallat, *A Wavelet Tour of Signal Processing*, Academic Press, 1998.
11. Y. Meyer, *Ondelettes et Operateurs*, Hermann, Paris 1990.
12. Q. Sun, Stability of the shifts of global supported distributions, *J. Math. Anal. Appl.*, 261(2001), 113–125.
13. Q. Sun, Convergence of cascade algorithm and smoothness of refinable distributions, *Chinese Ann. Math.*, 24(2003), 367-386.
14. Q. Sun, Localization of stability and p-frame in the Fourier domain, In *Proceeding of Special Session on Wavelets, Frames and Operator Theory* (C. Heil, P. Jorgensen and D. Larson eds.), AMS Contemporary Mathematics Series, to appear.
15. L. F. Villemoes, Wavelet analysis of refinement equations, *SIAM J. Math. Anal.*, 25(1994), 1433–1460.

Akram Aldroubi: Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA

Qiyu Sun: Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA

Wai-Shing Tang: Department of Mathematics, National University of Singapore, Singapore 117543, SINGAPORE