TWO-SCALE DIFFERENCE EQUATION: LOCAL AND GLOBAL LINEAR INDEPENDENCE

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ABSTRACT Let ϕ be a distribution solution of the two-scale difference equation (1). First the equivalence of local and global linear independence of the integer translates of ϕ is proved and a simple characterization for global linear independence of the integer translates of ϕ is given. Second a class of functions in V_1 such that their integer translates are locally or globally linearly independent is found.

Key words: two-scale difference equation, global linear independence, local linear independence,B-spline,B-wavelet.**

1.Preliminary and Statement of Results.

The objective of this context is to study local and global linear independence of the integer translates of a distribution solution of the two-scale equation. To this end, we introduce some notations and definitions.

Let $\{c_k\}_{k=0}^N$ be a sequence such that $c_0 \neq 0, c_N \neq 0$ and $\sum_{k=0}^N c_k = 2$. Let ϕ be a unique complex-valued compactly supported distribution to satisfy a two-scale difference equation

$$\begin{cases} \phi(x) = \sum_{k=0}^{N} c_k \phi(2x-k) \\ \hat{\phi}(0) = 1, \end{cases} \tag{1}$$

where the Fourier transform $\widehat{\phi}$ of ϕ is defined by

$$\widehat{\phi}(x) = \int e^{-ix\xi}\phi(x)dx.$$

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By taking Fourier transform in both sides of (1), we get

$$\widehat{\phi}(\xi) = H(\xi/2)\widehat{\phi}(\xi/2),$$

and

$$\widehat{\phi}(\xi) = \Pi_{j=1}^{\infty} H(\xi/2^j),$$

where we denote

$$H(\xi) = \frac{1}{2}\sum_{k=0}^{N} c_k e^{ik\xi}.$$

Hereafter we will say ϕ is the solution of (1) with $H(\xi)$.

The two-scale difference equation (1) attracted much attention in recent years since the equation of type (1) arise in the construction of wavelets with compact support ([3],[6]) and in the dyadic interpolation scheme of Deslauriers and Dubuc ([8],[9]) etc. For example,the wavelet ϕ_N constructed by I. Daubechies ([6]) is the solution of (1) with ${}_NH(\xi)$, where $H_N(\xi)$ satisfies

$$|H_N(\xi)|^2 = \cos^{2N}\frac{\xi}{2}\sum_{k=0}^{N-1}\binom{N-1+k}{k}\sin^{2k}\frac{\xi}{2}$$

for $N \geq 2$, and the univariate spline function B_N is the solution of (1) with $H(\xi) = \left(\frac{1+e^{i\xi}}{2}\right)^N$ for $N \geq 1$.

We say that the integer translates of a compactly supported distribution ϕ are globally linearly independent if the condition

$$\sum_{k\in Z} c(k)\phi(x-k) = 0 \quad \text{on} \quad R$$

implies $c(k) = 0$ for all $k \in Z$. We say that the integer translates of ϕ are locally linearly independent if the conditions

$$\sum_{k\in Z} c(k)\phi(x+k) = 0 \quad \text{on} \quad A \quad \text{and} \quad \text{supp} E^k\phi \cap A \neq \emptyset. \tag{2}$$

imply $c(k) = 0$ for every open set A. Here the shift operator E^k is defined by $E^k\phi(x) = \phi(x+k)$ on R and $\emptyset$ is the empty set.

It is well known that the integer translates of B_N are locally linearly independent (see [4],[9] for box spline) and the following formula plays an important role in the proof of local linear independence

$$\frac{d}{dx}B_N(x) = B_{N-1}(x) - B_{N-1}(x-1)$$

for $N \geq 2$.

To study local linear independence of the integer translates of the solution of (1), we establish a formula as the one above. Let B_0 be the $N \times N$ dimensional matrix defined by

$$(B_0)_{ij} = c_{2i-j}$$

for $0 \leq i, j \leq N-1$ and B_1 be the $N \times N$ dimensional matrix defined by

$$(B_1)_{ij} = c_{2i-j+1}$$

for $0 \leq i, j \leq N-1$. Hereafter we assume $c_j = 0$ for $j \leq -1$ and $j \geq N+1$. Denote

$$\Phi(x) = \begin{pmatrix} \phi(x) \\ \phi(x+1) \\ \vdots \\ \phi(x+N-1) \end{pmatrix}$$

on (0,1). From the equation (1), we have the fundemental formulae

$$\begin{aligned} B_0\Phi(x) &= \Phi\left(\frac{x}{2}\right) \\ B_1\Phi(x) &= \Phi\left(\frac{x+1}{2}\right) \end{aligned} \tag{3}$$

on (0,1).

The formulae above were used by I. Daubechies and J.Lagarias ([7]) to study local and global regularity of ϕ. The corresponding formulae on high dimensions were used by A.S.Cavaretta, W.Dahmen and C.A.Micchelli ([1]) to study the relationship between regularity of ϕ and the approximating degree of quasi-interpolants. In section 2, we will use the formulae (3) to study relation between local and global linear independence.

Theorem 1. Let ϕ be the solution of (1). Then local and global linear independence of the integer translates ϕ are equivalent to each other.

The main steps to prove Theorem 1 are Lemma 1 and 3.

Denote

$$P(z) = \sum_{k=0}^{N} c_k z^k.$$

We say a polynomial $P(z)$ has symmetric root z_0 if $P(z_0) = P(-z_0) = 0$.For a compactly supported distribution ϕ we denote

$$N(\phi) = \{z \in C; \widehat{\phi}(z + 2k\pi) = 0 \quad \text{for all} \quad k \in Z\}.$$

It is proved by A.Ron ([13]) that the integer translates of ϕ is globally linearly independent if and only if $N(\phi) = \emptyset$. Naturally we hope to give a characterization for global linear independence of the integer translates of ϕ which is given in section 3.

Theorem 2. Let ϕ be the solution of (1). Then the integer translates of ϕ are globally linearly independent if and only if the following conditions hold

(i) $P(z)$ has no symmetric roots,

(ii) $P(z)$ has not the factors of the form $\Pi_{k=1}^{N-1}(z + {z_0}^{2^k})$ with ${z_0}^{2^N} = z_0$ and $z_0 \neq 1$.

After the paper was completed we know Theorem 2 were also proved by Jia and Wang ([11]) but our proof is little different with them. As observed by C.K. Chui and J-Z Wang ([3]) and P-G Lemarie ([12]),the condition in Theorem 2 is closely related to minimal support of ϕ. By a characterization in [4], we know that the condition (ii) in Theorem 2 holds if and only if $N(\phi) \cap R = \emptyset$ under the assumption $P(z)$ has no symmetric roots on $\{|z| = 1\}$. We also see from the proof of Theorem 1 (precisely Lemma 1) that the condition (i) in Theorem 2 holds if and only if B_0 and B_1 are nonsingular matrices. Therefore it suffices to use finite steps to show the conditions (i) and (ii) in Theorem 2 true.

Denote

$$V_k = \{\sum_{j\in Z} c_j \phi_{k,j}(x); \quad \{c_j\}_{j\in Z} \quad \text{is some complex-valued sequence}\},$$

where $\phi_{k,j}(x) = \phi(2^k x - j) \quad$ for $\quad k, j \in Z$. By equation (1), we have

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$$

and ϕ belongs to V_1. Now our interest turns to find the functions in V_1 such that their integer translates are locally linearly independent. The reason to consider is at least the scaling function and the wavelet function belong to V_1 when $\{V_k\}$ is a multiresolution of some space (c.f. [2],[5]). To this aim we introduce a definition. For ψ a compactly supported distribution in V_1 we call $\left[\frac{N_1}{2}, \frac{N+N_2}{2}\right]$ is the supporting interval of ψ if supp$\psi \subset \left[\frac{\widetilde{N_1}}{2}, \frac{N+\widetilde{N_2}}{2}\right]$ implies $\widetilde{N_1} \leq N_1$ and $\widetilde{N_2} \geq N_2$, where $N_i, \widetilde{N_i} \in Z, (i = 1, 2)$. In section 4 we give the following characterization.

Theorem 3. Let ϕ be the solution of (1) and $\psi \in V_1$ be as above. Assume the integer translates of ϕ be globally linearly independent. Then the local and global linear independence of the integer translates of ψ are equivalent to each other if and only if $N_2 - N_1 \leq \widetilde{N}$ provided suppψ is just the supporting interval defined above. Here we define $\widetilde{N} = N$ when $N + N_1 + N_2$ is even and $\widetilde{N} = N - 1$ when $N + N_1 + N_2$ is odd.

Also we give an simple characterization to the compactly supported distributions in V_1 such that their integer translates are globally linearly independent.

Now we give the applications of the theorems above to B-spline and B-wavelet considered by Chui and Wang in [3]. Now we assume equation (1) has L^2 solution ϕ. Let an nested sequence

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$$

be closed subspaces of $L^2 = L^2(R)$ that constitutes a multiresolution analysis of L^2 (see [6]). Let W_k be the orthogonal complement of V_k in V_{k+1}. Therefore we have the wavelet decomposition

$$L^2 = \oplus_{k\in Z} W_k.$$

We call an L^2-function ϕ the generator of the given multiresolution analysis provided that $\{E^j\phi\}_{j\in Z}$ is an unconditional basis of V_0 and ϕ satisfies the two-scale difference equation

$$\phi(x) = \sum_{k=0}^{N} c_k \phi(2x-k).$$

for some sequence $\{c_k\}_{k=0}^N$ with $c_0 \neq 0, c_N \neq 0$ and $\sum_{k=0}^N c_k = 2$. Denote by Φ the family of generator ϕ. It is known that there is an unique L^2-function $\varphi \in \Phi$ such that every $\phi \in \Phi$ is a finite linear combination of $E^j\varphi$. We call this function φ the B-spline in term of Chui and Wang. By Theorem 3.1 in [3], the characteristic polynomial $P(z)$ defined by

$$P(z) = \sum_{k=0}^{N} c_k z^k$$

has no symmetric roots and $\sum_{k\in Z} |\widehat{\varphi}(x+2k\pi)|^2$ is bounded above and below away from zero for $x \in R$. Therefore conditions (i) and (ii) in Theorem 2 hold true and the integer translates of the B-spline φ are locally linearly independent.

Theorem 4. The integer translates of any B-spline φ are locally linearly independent.

An L^2-function η is called the wavelet of the given multiresolution analysis provided that $\{E^j\eta\}_{j\in Z}$ is an unconditional basis of W_0 and

$$\eta(x) = \sum_{k=N_1}^{N_2} d_k \varphi(x-k), \tag{4}$$

for some sequence $\{d_k\}_{k=N_1}^{N_2}$ with $d_{N_1} \neq 0$, $d_{N_2} \neq 0$, where φ is the B-spline of the given multiresolution analysis. we call the wavelet η with minimum support by B-wavelet. In absence of notation, we denote the B-wavelet still by η. As we will see in section 4 that the integer translates of η are globally linearly independent. By the representation of $\mu(z)$ in [3], we know $N + N_1 + N_2$ must be an even integer. Therefore by Theorem 3 we have

Theroem 5. If suppη is just the supporting interval of η,then the integer translates of the B-wavelet η are locally linearly independent if and only if $N_2 - N_1 \le N$ in (4).

Therefore the integer translates of orthonormal wavelet constructed in [6] are locally linearly independent and the integer translates of B-wavelet of the univariate polynomial spline in [3] must not be locally linearly independent (c.f.[2] Page 184).

From now on, we always assume ϕ satisfies the two-scale difference equation (1) for $N \ge 1$ except in the last section. The reason to assume $N \ge 1$ is the unique compactly supported distrution solution of (1) is the delta distribution for which the local and global linear independence is easy to study and the results to ϕ below (except in last section) is also true for the delta distribution.

2. Local Linear Independence

It is known that local linear independence of the integer of ϕ implies its global linear independence. Therefore it suffices to prove the following slightly strong conclusion.

Theorem 6. Let $\{c_k\}_{k=0}^N$ and ϕ be as in Theorem 1. If the integer translates of ϕ are globally linearly independent, then the conditions

$$\sum_{k\in Z} c(k)\phi(x-k) = 0 \quad \text{on} \quad A \quad \text{and} \quad [k, k+N] \cap A \ne \emptyset$$

imply $c(k) = 0$ for every open set A.

Corollary 1. Assume ϕ be the solution of (1) and the integer translates of ϕ are globally linearly independent. Then supp$\phi = [0, N]$.

The procedure to prove Theorem 6 is as follows. By the definition of $B_i (i = 0, 1)$, we know the components of $(1, z, \cdots, z^{N-1})B_i(i = 0, 1) are z^j H_o(z)$ or $z^j H_e(z)$ (c.f. (6)-(9)),where $H_o(z)$ and $H_e(z)$ are the odd and even part of the characteristic polynomial $H(z) = \sum_{j=0}^N c_j z^j$, i.e.,

$$H(z) = H_e(z^2) + zH_o(z^2).$$

The first claim (Lemma 5 which was also proved implicitly in [11]) is that the global linear independence of the integer translates of ϕ implies $H_o(z)$ and $H_e(z)$ have no common zero points. Using this claim we show that $B_i(i = 0, 1)$ are nonsingular matrices (see Lemma 1), and that $W_{(0,1)} = \{0\}$ if $W_A \ne \{0\}$ for some open set $A \subset (0, 1)$,where

$$W_A = \{\sum_{k=0}^{N-1} c(k)\phi(x+k) = 0 \quad \text{on} \quad A\}$$

(see Lemma 2). Hence the matter reduces to proving $W_{(0,1)} \neq \{0\}$. Conversely if $W_{(0,1)} \neq \{0\}$, we want to find a sequence $(c_0, \cdots, c_{N-1}) \in W_{(0,1)}$ such that it can extends to $\{c_k^*\}_{k\in Z}$ such that $\sum_{k\in Z} c_k^* \phi(x+k) = 0$ on R and $c_k^* = c_k$ for $0 \le k \le N-1$, then Theorem 6 is proved since the integer translates of ϕ are globally linearly independent. When $c_k = z_0^k$ for $k = 0, \cdots, N-1$ and some $z_0 \neq 0$ an easy extension of $\{c_k\}_{k=0}^{N-1}$ to $\{c_k^*\}_{k\in Z}$ is $c_k^* = z_0^k$ for $k \in Z$. Until now we need to do the following works , the existence of z_0 such that $(1, z_0, \cdots, z_0^{N-1}) \in W_{(0,1)}$ when $W_{(0,1)} \neq \{0\}$ and $\sum_{k\in Z} {z_0'}^k \phi(x+k) = 0$. The second equation is proved by Lemma 4 which was inspired by [14]. the existence of z_0 such that $(1, z_0, \cdots, z_0^{N-1}) \in W_{(0,1)}$ when $W_{(0,1)} \neq \{0\}$ is completed in Lemma 3. We outline the proof here. Denote V be the dual of $W_{(0,1)}$ which is just linear span space of $\{\Phi(x); x \in (0,1)\}$ when $\phi(x)$ is continuous. Denotes the basis of V by e_i and $E_i(z) = (z, z^2, \cdots, z^N)e_i$ for $1 \le i \le m \le N-1$ since $\dim V \le N-1$. Hence the matter reduces to proving $\{E_i(z)\}_{i=1}^m$ have a nonzero common zero point. Denote $E(z)$ be the vector with its component $E_i(z)$, and $E_o(z)$ and $E_e(z)$ denote the odd and even part of $E(z)$, i.e., $E(z) = E_e(z^2) + z^{-1}E_o(z)$. Recall that $B_iV = V (i = 0, 1)$. Therefore $E_o(z)$ and $E_e(z)$ satisfies the equation

$$z^{-1}H_e(z)CE_e(z) + H_o(z)CE_o(z) = H_e(z)E_o(z) + H_o(z)E_e(z)$$

for some nonsingular $m \times m$ matrix C. Recall that $H_e(z)$ and $H_o(z)$ have no common zero points. We get the equation (12). Comparing the degree of the polynomials in both sides of (12), we get $\bar{P}$ in (12) is a constant vector. Then the last important equation

$$(C^2 - zI)B_m^0 E(z) = C\alpha(-H_e^2(z) + zH_o^2(z))$$

can easy obtained where B_m^0 and C are nonsingular matrices and α is a constant vector. Observe that $(C^2 - zI)$ has at most m eigenvalues and the degree of $-H_e^2(z) + zH_o^2(z)$ is exactly N. Hence $E(z_0) = 0$ for some $z_0 \neq 0$,which implies $(1, z_0, \cdots, z_0^{N-1}) \in W_{(0,1)}$ if $\dim V \le N-1$ or $W_{(0,1)} \neq \{0\}$. To prove Theorem 6, we will use the following lemmas with their proofs postponed

Lemma 1. If the integer translates of ϕ are globally linearly independent, then B_0 and B_1 are nonsingular matrices.

Lemma 2. Assume that A be an open subset in (0,1) and that B_0 and B_1 are nonsingular matrices. If there is a non-zero vector $d \in C^N$ such that $d\Phi(x) = 0$ on A, then there is a no-zero vector $d' \in C$ such that $d'\Phi(x) = 0$ on (0,1).

Denote

$$W = \{\alpha \in C^N; \quad \alpha\Phi(x) = 0 \quad \text{on} \quad (0,1)\},$$

which is just W_A for $A = (0, 1)$.

Lemma 3. Assume the integer translates of ϕ are globally linearly independent. If $W \neq \{0\}$, then there is non-zero $z_0 \in C$ such that $(1, z_0, \cdots, {z_0}^{N-1}) \in W$.

Lemma 4. If $\sum_{j\in Z} {z_0}^j\phi(x-j)=0$ on $R\backslash Z$ for some non-zero $z_0\in C$, then there is non-zero ${z_0}'\in C$ such that

$$\sum_{j\in Z} {z_0'}^j\phi(x-j)=0 \quad \text{on} \quad R.$$

For a moment, we assume the lemmas above hold true. We start to prove Theorem 6. By Lemma 1, B_0 and B_1 are nonsingular matrices. By Lemma 2 and some elementary reduction, the matter reduces to $A=(0,1)$. Observe that $\sum_{k=0}^{N-1} {z_0}^j\phi(x+k)=0$ on (0,1) implies $\sum_{k\in Z} {z_0}^k\phi(x+k)=0$ on $R\backslash Z$. By Lemma 3 and Lemma 4, the integer translates of ϕ are not globally linearly independent if $W\neq\{0\}$. Therefore $W=\{0\}$ and Theorem 6 holds true.

Before we start to prove the lemmas used in the proof of Theorem 6, we prove Corollary 1 first. Conversely if Corollary 1 is not true, then there exists an open set $A\subset[0,N]$ such that $\phi(x)=0$ on A, i.e., $\sum_{k\in Z}\delta(k)\phi(x-k)=0$ on A. Here we define $\delta(k)=1$ for $k=0$ and 0 elsewhere. Recall that $A\cap[0,N]=A\neq\emptyset$. By Theorem 6 $\delta(0)=0$, which is a contradiction. Corollary 1 is proved.

To prove Lemma 1 to Lemma 4, we will use an elementary lemma which is also proved by Jia and Wang ([11]).

Lemma 5. If there exits $z_0\in C$ such that $H(z_0)=H(-z_0)=0$, then the integer translates of ϕ are globally linearly independent. Hereafter we define the characteristic polynomial $H(z)$ by

$$H(z)=\sum_{j=0}^{N} c_j z^j$$

instead of $H(\xi)$ defined in the beginning of section 1.

Proof of Lemma 5. Recall that $H(1)=2$. Therefore $z_0\neq 1,-1$. Since $H(z_0)=H(-z_0)=0$, then we can write

$$H(z)=\frac{(z^2-z_0^2)}{1-z_0^2}H_1(z),$$

where $H_1(z)$ is a trigonometric polynomial with $H_1(1)=2$.

Let ϕ_1 be the solution of (1) with $\frac{1}{2}H_1(e^{i\xi})\frac{e^{i\xi}-z_0^2}{1-z_0^2}$. Therefore we get

$$\widehat{\phi}(\xi)=\frac{e^{i\xi}-z_0^2}{1-z_0^2}\widehat{\phi}_1(\xi)$$

and

$$\phi(x) = \frac{1}{1-z_0^2}\phi_1(x-1) - \frac{z_0^2}{1-z_0^2}\phi_1(x).$$

Hence

$$\begin{aligned}
&\sum_{j\in Z} z_0^{-2j}\phi(x-j)\\
=&\sum_{j\in Z} \frac{z_0^{-2j}}{1-z_0^2}\phi_1(x-j-1) - \sum_{j\in Z} \frac{z_0^{-2j+2}}{1-z_0^2}\phi(x-j)\\
=&0,
\end{aligned}$$

and Lemma 5 holds true.

Now we start to prove Lemma 1 to Lemma 4.

Proof of Lemma 1. We prove Lemma 1 in two cases.

Case 1. N is an odd integer.

Let B be a $(N-1)\times(N-1)$ dimensional matrix defined by

$$B_{ij} = c_{2i-j}$$

for $1 \le i, j \le N-1$. Observe that the first row of B_0 is $(c_0, 0, \cdots, 0)$ and the last row of B_1 is $(0, \cdots, 0, c_N)$. Therefore the matters reduce to the non-singularity of the matrix B.

Write

$$\begin{aligned}
H_e(z) &= \sum_i c_{2i}z^i\\
H_o(z) &= \sum_i c_{2i+1}z^i\\
Q_e(z) &= \sum_i \alpha_{2i}z^i\\
Q_o(z) &= \sum_i \alpha_{2i+1}z^i
\end{aligned}$$

for $\alpha = (\alpha_1, \cdots, \alpha_{N-1}) \in C^{N-1}$, where we assume $\alpha_i = 0$ for $i \le 0$ and $i \ge N$.

Observe that if det $B = 0$ or B is a singular matrix then there is a non-zero vector $\alpha \in C^{N-1}$ such that $(1, z, \cdots, z^{N-1})B\alpha$ is a zero polynomial about z. Also we know

$$(1, z, \cdots, z^{N-1})B\alpha = Q_o(z)H_o(z) + z^{-1}Q_e(z)H_e(z).$$

Recall from Lemma 5 and $H(z) = H_e(z^2) + zH_o(z^2)$ that $H_o(z)$ and $H_e(z)$ has no common roots. Therefore there is a polynomial $Q(z)$ such that

$$\begin{cases} Q_o(z) = -H_e(z)Q(z) \\ z^{-1}Q_e(z) = H_o(z)Q(z) \end{cases} \tag{5}$$

Recall that the degree of H_o is $\frac{N-1}{2}$ and the degree of $Q_e(z)z^{-1}$ does no exceed $\frac{N-3}{2}$. Therefore $Q(z) = 0$ and $Q_o(z) = Q_e(z) = 0$, which contradicts to $\alpha \neq 0$. Hence B_0 and B_1 are not singular matrices in Case 1.

Case 2. N is an even integer.

By the same procedure as used in Case 1, the matter reduces to $Q_o(z)H_o(z)+a^{-1}Q_e(z)H_e(z)$ being a zero polynomial only for zero vector $\alpha \in C^{N-1}$. Also we know (5) hold true. Recall that the degree of H_e is $\frac{N}{2}$ and the degree of $Q_o(z)$ does not exceed $\frac{N-2}{2}$. Therefore $Q(z) = 0$ and $Q_o(z) = Q_e(z) = 0$, which implies $\alpha = 0$. Hence B_0 and B_1 are not singular matrices in Case 2. Lemma 1 is proved.

Proof of Lemma 2. Without loss of generality we assume A is an open interval $(a, b) \subset (0, 1)$. Recall that

$$B_0\Phi(2x) = \Phi(x) \quad \text{for} \quad 0 < x < \frac{1}{2}$$
$$B_1\Phi(2x-1) = \Phi(x) \quad \text{for} \quad \frac{1}{2} < x < 1.$$

For $d \in C^N$, we denote $d_\epsilon = dB_\epsilon$ for $\epsilon = 0$ and 1. Therefore we observe that

$$d_0\Phi(x) = 0 \qquad \text{on } (2a, \min(1, 2b))$$

when $b \leq \frac{1}{2}$ or $a + b \leq 1$ and

$$d_1\Phi(x) = 0 \qquad (\max(2a-1, 0), 2b-1)$$

when $a \geq \frac{1}{2}$ or $a + b > 1$.

Recall that B_0 and B_1 are nonsingular matrices. Hence from above observation we get that when $(a, b) \subset (0, \frac{1}{2})$ or $(\frac{1}{2}, 1)$ there are an open interval (a', b') with length $2(b-a)$ and a non-zero $d' \in C^N$ such that $d'\Phi(x) = 0$ on (a', b'), and when $(a, b) \ni \frac{1}{2}$ there are an open interval $(0, b')$ and a non-zero $d'' \in C^N$ such that $d''\Phi(x) = 0$ on $(0, b')$. By the observation above we can find a non-zero $d''' \in C^N$ such that $d'''\Phi(x) = 0$ on (0,1). Lemma 2 is proved.

Proof of Lemma 3. Write

$$H_e(z) = \sum_j c_{2j} z^j$$

and

$$H_o(z) = \sum_j c_{2j+1} z^j .$$

Therefore we have

$$(1, z, \cdots, z^{N-1})B_0 = (H_e(z), zH_o(z), zH_e(z), \cdots, z^{\frac{N-1}{2}}H_o(z), z^{\frac{N-1}{2}}H_e(z)) \tag{6}$$

$$(1, z, \cdots, z^{N-1})B_1 = (H_o(z), H_e(z), \cdots, z^{\frac{N-3}{2}}H_o(z), z^{\frac{N-3}{2}}H_e(z), z^{\frac{N-1}{2}}H_o(z)) \tag{7}$$

when N is odd and

$$(1, z, \cdots, z^{N-1})B_0 = (H_e(z), zH_o(z), zH_e(z), \cdots, z^{\frac{N-2}{2}}H_o(z), z^{\frac{N-2}{2}}H_e(z), z^{\frac{N}{2}}H_o(z)) \tag{8}$$

$$(1, z, \cdots, z^{N-1})B_1 = (H_o(z), H_e(z), \cdots, z^{\frac{N-2}{2}}H_o(z), z^{\frac{N-2}{2}}H_e(z)) \tag{9}$$

when N is even.

Denote by V the dual space of W, i.e

$$V = \{e, \quad we = 0, \quad \forall w \in W\}.$$

Let

$$e_i = \begin{pmatrix} e_{i1} \\ e_{i2} \\ \vdots \\ e_{iN} \end{pmatrix}$$

$(1 \le i \le m)$ be a basis of V. Denote

$$E_i(z) = \sum_{1 \le j \le N} e_{ij} z^j$$

$$E_{ie}(z) = \sum_{1 \le 2k \le N} e_{i(2k)} z^k$$

$$E_{io}(z) = \sum_{1 \le 2k+1 \le N} e_{i(2k+1)} z^k .$$

Hence the matter reduces to $(E_1(z), \cdots, E_m(z)) = 0$ for some non-zero $z_0 \in C$ when $1 \le m \le n-1$.

Recall that $WB_0 = W$ and $WB_1 = W$. Therefore $B_0V = V$ and $B_1V = V$. In other words, there exist nonsingular matrices $B_m^\epsilon = (\lambda_{ij}^\epsilon)_{1 \le i,j \le m}$ such that

$$B_\epsilon e_i = \sum_{1 \le j \le m} \lambda_{ij}^\epsilon e_j$$

for $\epsilon = 0$ and 1. Hence from (6)-(9) we get

$$\begin{cases} H_e(z)E_{io}(z) + H_o(z)E_{ie}(z) = \sum_{1\le j\le m} \lambda_{ij}^0 E_j(z) \\ H_o(z)E_{io}(z) + z^{-1}H_e(z)E_{ie}(z) = \sum_{1\le j\le m} \lambda_{ij}^1 E_j(z) \end{cases} \tag{10}$$

for $1 \le i \le m$. Write (10) in matrix form

$$\begin{cases} H_e(z)E_0(z) + H_0(z)E_e(z) = B_m^0 E(z) \\ z^{-1}H_e(z)E_e(z) + H_0(z)E_0(z) = B_m^1 E(z), \end{cases} \tag{11}$$

where we denote

$$E_o(z) = \begin{pmatrix} E_{1o}(z) \\ E_{2o}(z) \\ \vdots \\ E_{mo}(z) \end{pmatrix},$$

$$E_e(z) = \begin{pmatrix} E_{1e}(z) \\ E_{2e}(z) \\ \vdots \\ E_{me}(z) \end{pmatrix}$$

and

$$E(z) = \begin{pmatrix} E_1(z) \\ E_2(z) \\ \vdots \\ E_m(z) \end{pmatrix}.$$

Therefore we have

$$z^{-1}H_e(z)CE_e(z) + H_o(z)CE_o(z) = H_e(z)E_o(z) + H_o(z)E_e(z),$$

where we define $C = B_m^0(B_m^1)^{-1}$.

Recall from Lemma 5 that $H_e(z)$ and $H_0(z)$ has no symmetric roots. So we have

$$\begin{aligned} z^{-1}CE_e(z) - E_o(z) &= \bar{P}(z)H_o(z) \\ CE_o(z) - E_e(z) &= -\bar{P}(z)H_e(z), \end{aligned} \tag{12}$$

where

$$\overline{P}(z) = \begin{pmatrix} P_1(z) \\ P_2(z) \\ \vdots \\ P_m(z) \end{pmatrix}.$$

and $P_i(z)(1 \le i \le m)$ are Laurent polynomials. For a Laurent polynomial $Q(z) = \sum_{k_2 \le k \le k_1} a_k z^k$ with $a_k \ne 0$ and $a_{k_2} \ne 0$, we define $d^-(Q) = k_2$ and $d^+(Q) = k_1$. For the vector $\overline{P}(z)$, we define $d^-(\overline{P}) = \min_{1 \le i \le m} d^-(P_i)$ and $D^+(\overline{P}) = \max_{1 \le i \le m} d^+(P_i)$. Observe that $d^+(E_0) \le \left[\frac{N-1}{2}\right], d^-(E_0) \ge 0$ and $d^+(E_e) \le \left[\frac{N}{2}\right], d^-(E_e) \ge 0$, where we denote $[x]$ the integer part of x.

On other hand by $c_0 \ne 0$ and $c_N \ne 0$, we have

$$d^+(H_o) = \frac{N-1}{2} \quad \text{and} \quad d^-(H_e) = 0$$

when N is odd and

$$d^+(H_e) = \frac{N}{2} \quad \text{and} \quad d^-(H_e) = 0$$

when N is even. Therefore we have $d^+(\bar{P}) \le 0$ and $d^-(\bar{P}) \ge 0$. This implies P is a constant vector, i.e., $P_i(z)$ are constant polynomials.

In other words, we can write (12) as

$$\begin{aligned} z^{-1}CE_e(z) - E_0(z) &= \alpha H_o(z) \\ CE_o(z) - E_0(z) &= -\alpha H_e(z) \end{aligned} \tag{13}$$

for some $\alpha \in C^m$. Therefore we have

$$\begin{aligned} (z^{-1}C^2 - I)E_e &= C\alpha H_o(z) - \alpha H_e(z) \\ (z^{-1}C^2 - I)E_o &= -C\alpha z^{-1} H_e(z) + \alpha H_o(z), \end{aligned} \tag{14}$$

where I is the $m \times m$ dimensional identity matrix. Taking the identity (14) into (11), we get

$$\begin{aligned} &(z^{-1}C^2 - I)B_m^0 E(z) \\ =&H_e(z)(-C\alpha z^{-1} H_e(z) + \alpha H_o(z)) + H_o(z)(C\alpha H_0(z) - \alpha H_e(z)) \\ =&C\alpha(-z^{-1} H_e^2(z) + H_o^2(z)), \end{aligned}$$

and

$$(C^2 - zI)B_m^0 E(z) = C\alpha(-H_e^2(z) + zH_o^2(z)).$$

From $c_0 \ne 0$ and $c_N \ne 0$, we have $d^+(zH_o^2(z) - H_e(z)^2) = N$ and $d^-(zH_o^2(z) - H_e(z)^2 = 0$. Therefore $zH_o^2(z) - H_e^2(z)$ has exactly N roots (with multiplicity). On other hand $(C^2 - zI)$ has exactly m eigenvalues (with multiplicity). Recall that $m \le N-1$. Therefore there exists $z_0 \in C$ such that $E(z_0) = 0$. We finish the proof of Lemma 3.

Proof of Lemma 4. Let z_0 be a non-zero complex number such that

$$\sum_{j \in Z} z_0^j \phi(x+j) = 0 \quad \text{on } R \backslash Z.$$

Let δ_k^s be the delta ditribution defined by

$$\langle \delta_k^s, f\rangle = \left(\frac{\partial}{\partial x}\right)^s f(k).$$

Therefore there exists an integer k and some $a_s \in C$ $(0 \le s \le k)$ such that

$$\sum_{j\in Z} z_0^j \phi(x+j) = \sum_{j\in Z} z_0^j (\sum_{0\le s\le k} a_s \delta_j^s). \tag{15}$$

Obviously if $a_s = 0$ for all $0 \le s \le k$ then Lemma 4 holds for $z_0' = z_0$. Now we assume $a_s (0 \le s \le k)$ are not complete zero.

Taking Fourier transform in both side of (15), we get

$$\widehat{\phi}(\theta + 2\pi k) = R(\theta + 2\pi k) \tag{16}$$

for $k \in Z$, where $\exp(\theta) = z_0$ and a polynomial $R(x) = \sum_{0\le s\le k} a_s x^s$. Recall that $\widehat{\phi}(2x) = H(x)\widehat{\phi}(x)$ and

$$\widehat{\phi}(\theta + 2^m k\pi) = \Pi_{1\le j\le m-1} H(2^{-j}\theta)\widehat{\phi}(2^{-m+1}\theta + 2k\pi).$$

Therefore by the continuity of $\widehat{\phi}$ we have

$$\lim_{m\to\infty} \widehat{\phi}(\theta + 2^m k\pi) = \widehat{\phi}(\theta)\widehat{\phi}(2k\pi) \tag{17}$$

for all $k \in Z$. On other hand if the degree of R is not zero then $R(\theta + 2^m k\pi)$ tends to infinite as m tends to infinite. Hence the degree of R must be zero. By (16) and (17), we have

$$\widehat{\phi}(\theta)\widehat{\phi}(2k\pi) = a_0 \ne 0.$$

and

$$\widehat{\phi}(2k\pi) = \widehat{\phi}(2k'\pi)$$

for all $k, k' \in Z$ Recall that H is a periodic function with period 2π. Therefore we can inductively prove that

$$\widehat{\phi}\left(\frac{m}{2^s}\pi\right) = \widehat{\phi}\left(\frac{m'}{2^s}\pi\right),$$

when $\frac{m-m'}{2^{s+1}}$ is an integer. By the continuity of $\widehat{\phi}$, we have

$$\widehat{\phi}(y) = \widehat{\phi}(y + 2\pi)$$

for all $y \in R$, i.e., $\widehat{\phi}$ is a periodic function with period 2π. Hence we have

$$\text{supp } \phi \subset \{0, 1, \cdots, N\},$$

and we can write

$$\phi(x) = \sum_{k=0}^{N} d(k)\delta_k^0.$$

Recall that $c_0 \neq 0$ and $c_N \neq 0$. Therefore there exists k_1 and k_2 such that $k_1 \neq k_2$, $d(k_1) \neq 0$ and $d(k_2) \neq 0$. By the algebraic fundemental theorem, there exists non-zero complex number z_0' such that $\sum_{k=0}^{N} d(k){z_0'}^k = 0$. Hence

$$\begin{aligned}\sum_{j\in Z} {z_0'}^j \phi(x+j) &= \sum_{j\in Z} {z_0'}^j \sum_{k=0}^{N} d(k)\delta_{k-j}^0 \\ &= \sum_{j\in Z} \delta_m^0 {z_0'}^{-m} \quad (\sum_{k=0}^{N} d(k){z_0'}^k) \\ &= 0.\end{aligned}$$

and Lemma 4 holds true.

3. Global Linear Independence

In this section we use a method in [4] to prove Theorem 2 which is also proved by Jia and Wang ([11]).

First the necessity. By Lemma 5, we have $P(z)$ has no symmetric roots on $C\backslash\{0\}$. Recall that $P(0) = c_0 \neq 0$. Therefore (i) holds. Conversely we assume (ii) do not hold. Therefore there is $x \in R$ such that $\widehat{\phi}(x + 2\pi k) = 0$ for all $k \in Z$ by Theorem 1 in [4]. Therefore $x \in N(\phi)$, which contradicts to $N(\phi) = \emptyset$ by Theorem 1.1 in [13]. Hence (i) and (ii) hold.

Second the sufficiency. By the assumption (ii) and (i) for $|z| = 1$, we have $N(\phi) \cap R = \emptyset$. Conversely we assume there exists $z_0 \in C$ with non-zero imaginary part such that $\widehat{\phi}(z_0 + 2k\pi) = 0$ for all $k \in Z$. Observe that

$$2^{-j}(z_0 + 2k\pi) \neq 2^{-j'}(z_0 + 2k'\pi) \pmod{2\pi}$$

for $j \neq j'$ and $k, k' \in Z$. Here we say $x \neq y \pmod{2\pi}$ if $\frac{x-y}{2\pi}$ is not an integer. Denote

$$D_j = \{k; H(2^{-j}(z_0 + 2k\pi)) = 0\}.$$

Therefore $k \in D_j$ if and only if $k + 2^j \in D_j$. Recall that H is a trigonometric polynomial. Therefore there exist $M \in Z$ such that $D_j = \emptyset$ for all $j \geq M + 1$ and $\cup_{1\leq j\leq M} D_j = Z$. On other hand (i) implies $|H(\xi)|^2 + |H(\xi+\pi)|^2 \neq 0$ for $\xi \in C\backslash R$. Therefore $k \in D_j$ implies $k + 2^{j-1} \notin D_j$ for $1 \le j \le M$. Denote

$$B_j = Z\backslash U_{1\le s\le j} D_s$$

for $1 \le j \le M$. Obviously B_1 is not empty set. Inductively we assume B_{s-1} is not empty set where $s \le M$. Therefore there exists $k \in B_{s-1}$. Recall that k and $k+2^{s-1} \in B_{s-1}$ and that at most one of k and $k+2^{s-1}$ is contained in D_s. Therefore B_s is not empty set and B_M is also not empty set by induction, which contradicts $B_M = \emptyset$. Hence $N(\phi) = \emptyset$ and the sufficiency of Theorem 2 is proved by Theorem 1.1 in [13].

4. Compactly Supported Distributions In V_1

Before we start to prove Theorem 3, we give a charaterization to global linear independence. Recall that $\psi \in V_1$. We write

$$\psi(x) = \sum_{k\in Z} d_k \phi(2x-k).$$

To prove $\{d_k\}_{k\in Z}$ is a finite sequence, i.e., there exists $\widetilde{N}$ such that $d_k = 0$ for $|k| > \widetilde{N}$, we will use the following lemma.

Lemma 6 ([14]) Assume that the integer translates of a compactly supported distribution ϕ are globally linearly independent. Then there exists a bounded set A such that the conditions

$$\sum_{k\in Z} c(k)\phi(x+k) = 0 \quad \text{on} \quad A \quad \text{and} \quad \text{supp}E^k\phi \cap A \neq \emptyset$$

imply $c(k) = 0$.

Recall that ψ is compactly supported distribution and the integer translates of ϕ are globally linearly independent. Therefore there exists $\overline{N}$ such that $d_k = 0$ for $|k| \ge \overline{N}$. Now we can write

$$\psi(x) = \sum_{k=N_1}^{N_2} d_k \phi(2x-k), \tag{18}$$

where $d_{N_1} \neq 0$ and $d_{N_2} \neq 0$. Recall from Corollary 1 that $\text{supp}\phi = [0, N]$. Therefore the supporting interval of ψ is just $\left[\frac{N_1}{2}, \frac{N+N_2}{2}\right]$.

Theorem 7. The integer translates of ψ are globally linearly independent if and only if $\mu(z)$ has no symmetric roots, where the "symbol" polynomial is defined by

$$\mu(z) = \sum_{k=N_1}^{N_2} d_k z^k.$$

Proof of Theorem 7. The necessity. Assume $\mu(z_0)=\mu(-z_0)=0$ for some non-zero $z_0\in C$. Therefore

$$\begin{aligned}
&\sum_{k\in Z} {z_0}^{-2k}\psi(x-k)\\
=&\sum_{k\in Z} {z_0}^{-2k}\sum_{j=N_1}^{N_2} d_j\phi(2x-2k-j)\\
=&\sum_{k\in Z}\phi(2x-k)\Big(\sum_{\substack{N_1\le j\le N_2\\ k-j \text{ even}}} {z_0}^j d_j\Big)\cdot {z_0}^{-k}\\
=&\frac{1}{2}\sum_{k\in Z} {z_0}^{-k}\phi(2x-k)\cdot(\mu(z_0)+\mu(-z_0)(-1)^k)\\
=&0.
\end{aligned}$$

and the integer translates of ψ are not globally linearly independent, which is a contradiction. The necessity is proved.

The sufficiency. Conversely if the sufficiency is not true, then by Theorem 1.1 in [13] there is $\theta\in C$ such that

$$\sum_{k\in Z} e^{2i\theta k}\psi(x-k)=0 \quad \text{on} \quad R.$$

On other hand by (18) we have

$$\begin{aligned}
&\sum_{k\in Z} e^{2i\theta k}\psi(x-k)\\
=&\sum_{k\in Z}\phi(2x-h)\sum_{\substack{N_1\le j\le N_2\\ h-j \text{ even}}} d_j e^{-ij\theta} \quad \text{on} \quad R.
\end{aligned}$$

By Theorem 1 and 2 we have

$$\sum_{\substack{N_1\le j\le N_2\\ j \text{ even}}} d_j e^{-ij\theta}=0$$

and

$$\sum_{\substack{N_1\le j\le N_2\\ j \text{ odd}}} d_j e^{-ij\theta}=0.$$

Hence $\mu(e^{-i\theta})=\mu(-e^{-i\theta})=0$ and $\mu(z)$ has a symmetric root $e^{-i\theta}$, which is a contradiction. Hence the sufficiency is proved and Theorem 7 is proved.

Proof of Theorem 3 Without loss of generality we assume $N_2=0$ or 1, otherwise we replaced ψ be shifted distribution $E^{-[\frac{N_1}{2}]}\psi$. We divide two cases to prove Theorem 3.

Case 1. $N+N_2+N_1$ is even integer.

Case 1.1. N_1=0

Denote $m=\frac{N+N_2}{2}$. By (4), supp$\psi\subset[0,m]$. By some simple reduction, the matter reduces to $A\subset(0,1)$. Let $\{\psi_k\}_{k=-m+1}^{0}$ be a sequence such that

$$\sum_{-m+1\le k\le 0} a_k\psi(x-k)=0 \quad \text{on} \quad A, \tag{19}$$

i.e.,

$$\begin{aligned}\sum_{-m+1\le k\le 0} a_k\sum_{j\in Z} d_j\phi(2x-2k-j) &= \sum_{h\in Z}\phi(2x-h)\sum_{-m+1\le k\le 0} d_{h-2k}a_k\\ &=0. \qquad \text{on} \quad A.\end{aligned}$$

Hereafter we assume $d_j=0$ for $j\le N_1-1$ and for $j\ge N_2+1$. Recall by Theorem 1 and 2 that the integer translates of ϕ are locally linearly independent. Therefore we have

$$\sum_{-m+1\le k\le 0} d_{h-2k}a_k=0 \tag{20}$$

for $-N+1\le h\le 0$ when $A\cap(0,\frac{1}{2})\ne\emptyset$ and for $-N\le h\le -1$ when $A\cap(0,\frac{1}{2})=\emptyset$. In matrix notation we write (20) as

$$D_AF=0. \tag{21}$$

Here we define the vector

$$F=\begin{pmatrix} a_0\\ a_{-1}\\ \vdots\\ a_{-m+1}\end{pmatrix}$$

and the $N\times m$ matrix D_A by

$$(D_A)_{ij}=d_{i-2j}$$

for $-N+1\le i\le 0$ and $-m+1\le j\le 0$ when $A\cap(0,\frac{1}{2})\ne\emptyset$ and for $-N+2\le i\le 1$ and $-m+1\le j\le 0$ when $A\cap(0,\frac{1}{2})=\emptyset$.

First the necessity. Denote by $r(D_A)$ the rank of the matrix D_A. Obviously it suffices to show there exists a no-zero sequence $\{a_k\}_{-m+1}^{0}$ such that (19) holds on $(0,\frac{1}{2})$ for every

$N_2 \geq N+1$. Observe that $r(D_A) \leq N$ and $m \geq N+1$. Therefore there exists a non-zero vector F or non-zero sequence $\{a_k\}_{k=-m+1}^{0}$ such that $D_A F = 0$. Therefore

$$\begin{aligned} &\sum_{-m+1\leq k\leq 0} a_k \psi(x-k) \\ &\qquad = \sum_{h\in Z} \phi(2x-h) \sum_{-m+1\leq k\leq 0} d_{h-2k} a_k \\ &\qquad = 0 \qquad\qquad \text{on} \quad (0, \frac{1}{2}), \end{aligned}$$

which contradicts local linear independence of the integer translates of ψ since $\mathrm{supp}\psi = [\frac{N_1}{2}, \frac{N+N_2}{2}]$ and $\mathrm{supp}\psi(\cdot - k) \cap (0, \frac{1}{2}) \neq \emptyset$ for $-m+1 \leq k \leq 0$.

The necessity in Case 1.1 is proved.

Second the sufficiency. Obviously it suffices to prove $D_A F = 0$ holds only for $F = 0$, or to prove $r(D_A) = m \leq N$ when $N_2 \leq N$. Let $\widetilde{D_A}$ be a $N \times m$ matrix defined by

$$(\widetilde{D_A})_{ij} = \tilde{d}_{2j-i}$$

when $A \cap (0, \frac{1}{2}) \neq \emptyset$ and

$$(\widetilde{D_A})_{ij} = \tilde{d}_{2j-i-1}$$

when $A \cap (0, \frac{1}{2}) = \emptyset$, where $1 \leq i \leq N$ and $1 \leq j \leq m$. Hereafter we denote $\tilde{d}_j = d_{N_2 - j}$ for $j \in Z$. Obviously $r(\widetilde{D_A}) = r(D_A)$. Denote the transpose of $\widetilde{D}_A$ by D_A^*. Hence the matter reduces to the construction of nonsingular m dimensional submatrix E of D_A^*. We divide two cases to construct E explicitly. The construction E when $A \cap (0, \frac{1}{2}) = \emptyset$ is similar to the one when $A \cap (0, \frac{1}{2}) \neq \emptyset$. We only construct E explicitly when $A \cap (0, \frac{1}{2}) \neq \emptyset$ here.

Case 1.1.a. N_2 is an even integer

Write

$$D_A^* = \begin{pmatrix} E_1 & E_2 \\ 0 & E_3 \end{pmatrix},$$

where $N_2 \times N_2$ matrix E_1 is defined by

$$(E_1)_{ij} = \tilde{d}_{2i-j}$$

for $1 \leq i, j \leq N_2$ and 0 is the zero matrix.

We construct E in Case 1.1.a as follows. Let the k-row of E be the k-row of D_A^* for $1 \leq k \leq N_2$ and be the $(2k - N_2)$-row of D_A^* for $N_2 + 1 \leq k \leq m$. Recall that $2m - N_2 = N$. So our construction of E is convenient. Furthermore we can write

$$E = \begin{pmatrix} E_1 & E_2' \\ 0 & E_3' \end{pmatrix},$$

where E_3' is a $(m-N_2)$ dimensional upper triangular matrix with diagonal elements $\widetilde{d}_N \neq 0$ identically. By the proof of Lemma 1 and Theorem 7, E_1 is nonsingular matrix or $r(E_1) = N_2$. Therefore $r(E) = m$ and the construction of E in Case 1.1.a is finished.

Case 1.1.b. N_2 is an odd integer.

Write

$$D_A^* = \begin{pmatrix} E_4 & 0 \\ E_5 & E_6 \end{pmatrix},$$

where E_6 is a N_2 dimensional matrix defined by

$$(E_6)_{ij} = \widetilde{d}_{2i-j}$$

for $1 \leq i, j \leq N_2$.

We construct E as follows. Let the k-row of E be the $(2k)$-row of D_A^* for $1 \leq k \leq \frac{N-N_2}{2}$ and the $(k + \frac{N-N_2}{2})$ row of D_A^* for $\frac{N-N_2}{2} + 1 \leq k \leq m$. Therefore we can write

$$E = \begin{pmatrix} E_4' & 0 \\ E_5' & E_6 \end{pmatrix},$$

where E_4' is a $(m - N_2)$ dimensional lower triangular matrix with diagonal element $\widetilde{d}_0 \neq 0$ identically. By the proof of Lemma 1 and Theorem 7, E_6 is a nonsingular matrix. Therefore $r(E) = m$ and the construction of E Case 1.1.b is finished.

We finish the proof of the sufficiency in Case 1.1.

Case 1.2 $N_1 = 1$.

Obviously supp$\psi \subset [\frac{1}{2}, \frac{N+N_2}{2}]$. Denote $m' = \frac{N+N_2-1}{2}$. Observe that the set $\{j, [\frac{1}{2} + j; \frac{N+N_2}{2} + j] \cap A \neq \emptyset\}$ is $\{j, -m' \leq j \leq -1\}$ for $A \subset (0, \frac{1}{2}), \{j, -m' + 1 \leq j \leq 0\}$ for $A \subset (\frac{1}{2}, 1)$ and $\{j; -m' \leq j \leq 0\}$ for $\frac{1}{2} \in A$. It is easy to see that the conclusion for $\frac{1}{2} \in A$ would follow from the conclusions for $A \subset (0, \frac{1}{2})$ and $A \subset (\frac{1}{2}, 1)$ because $A = (A \cap (0, \frac{1}{2})) \cup (A \cap (\frac{1}{2}, 1)) \cup \{\frac{1}{2}\}$. Hence the matter reduces to the two cases $A \subset (0, \frac{1}{2})$ and $A \subset (\frac{1}{2}, 1)$. Define $N \times m'$ matrix by

$$(D_A)_{ij} = d_{i-2j}$$

for $-N + 1 \leq i \leq 0$ and $-m' \leq j \leq -1$ when $A \subset (0, \frac{1}{2})$ and for $-N \leq i \leq -1$ and $-m' + 1 \leq j \leq 0$ when $A \subset (\frac{1}{2}, 1)$. Denote

$$F_A = \begin{pmatrix} a_\varepsilon \\ a_{\varepsilon-1} \\ \vdots \\ a_{\varepsilon-m'+1} \end{pmatrix},$$

where $\varepsilon = 0$ for $A \subset (\frac{1}{2}, 1)$ and $\varepsilon = -1$ for $A \subset (0, \frac{1}{2})$. Therefore we establish the equation corresponding to (21)

$$D_A F_A = 0. \tag{22}$$

By the procedure used in Case 1.1 and (22), we prove Theorem 3 in Case 1.2. We finish the proof of Theorem 3 in Case 1.

Case 2. $N + N_2 + N_1$ is an odd integer

Case 2.1 $N_1 = 0$

First we have supp$\psi \subset [0, \frac{N+N_2}{2}]$. As in Case 1.2, it suffices to consider the two case $A \subset (0, \frac{1}{2})$ and $A \subset (\frac{1}{2}, 1)$. Denote $m'' = \frac{N+N_2-1}{2}$. Observe that $\{j, [j, j + \frac{N+N_2}{2}] \cap A \neq \emptyset\}$ is $\{j; -m'' \leq j \leq 0\}$ for $A \subset (0, \frac{1}{2})$ and $\{j; -m'' + 1 \leq j \leq 0\}$ for $A \subset (\frac{1}{2}, 1)$. Define the matrix D_A by

$$(D_A)_{ij} = d_{i-2j}$$

for $-N + 1 \leq i \leq 0$ and $-m'' \leq j \leq 0$ for $A \subset (0, \frac{1}{2})$ and for $-N \leq i \leq -1$ and $-m'' + 1 \leq j \leq 0$ for $A \subset (\frac{1}{2}, 1)$. Here we should point out D_A is $N \times (m'' + 1)$ matrix for $A \subset (0, \frac{1}{2})$. Similarly we define

$$F_A = \begin{pmatrix} a_0 \\ a_{-1} \\ \vdots \\ a_{-m'} \end{pmatrix}$$

when $A \subset (0, \frac{1}{2})$ and

$$F_A = \begin{pmatrix} a_{-1} \\ \vdots \\ a_{-m'} \end{pmatrix}$$

when $A \subset (\frac{1}{2}, 1)$. Then we establish the equation corresponding to (21)

$$D_A F_A = 0. \tag{23}$$

By the procedure used in Case 1.1 we can prove Theorem 3 in Case 2.1.

Case 2.2 $N_1 = 1$.

We can also establish an equation corresponding to (22). By the procedure used in Case 1.2 we can prove Theorem 3 in Case 2.2. We omit the details here. The proof of Theorem 3 is fininshed.

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