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INTERPOLATING FILTERS WITH PRESCRIBED ZEROS AND THEIR REFINABLE FUNCTIONS

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Abstract. In this paper, we study the minimally supported interpolating filters with prescribed zeros and their corresponding refinable functions.

1. **Introduction.** In signal processing, it is usual for filters to be constructed according to various filter design specifications, [23]. However, in the theory of orthonormal wavelets of compact support there has been a preference for filters of maximal flatness to generate refinable functions and wavelets which have a prescribed regularity [1, 7, 16, 21, 24]. Perhaps, this is due to the belief that maximally flat filters will lead to maximally smooth wavelets even though there seems to be no concrete evidence to support that hypothesis. Therefore, in this paper we are led to explore the possibility of specifying zeros of the filter freely as a device to improve their properties and those of the associated refinable functions. Our finding is in no way definitive and there remains many intriguing issues that demand clarification.

For the sake of generality we focus on *symmetric interpolating* filters Q, that is, those 2π-periodic functions which satisfy the equations, $Q(\xi)+Q(\xi+\pi)=1$, $Q(-\xi)=Q(\xi)$, $\xi\in\mathbb{R}$ and $Q(0)=1$. This choice of terminology shall become clear as we amplify on our point of view. Let us emphasize here that we are not concerned with *conjugate quadrature* filters. However, both notions are related and the latter can be constructed from the former by a Riesz factorization, if the filter is *nonnegative*. The possibility of this additional condition on interpolating filters leads to some challenging and important questions that have yet to be satisfactorily resolved, see [11, 12, 18, 19, 20] for recent progress on the matter.

Given any positive integer N, the minimally supported interpolating filter with the flatness order $2N$ at the frequency π was discovered by Hermann in [13] and later used by Daubechies, [7]. Recall that Q_N is given *explicitly* by the formula

$$Q_N(\xi)=\cos^{2N}\frac{\xi}{2}\,P_N\Big(\sin^2\frac{\xi}{2}\Big),\quad \xi\in\mathbb{R} \tag{1.1}$$

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where

$$P_N(z) = \sum_{l \in \mathbb{Z}_{N-1}} \binom{N-1+l}{l} z^l, \quad z \in \mathbb{Z} \tag{1.2}$$

and $\mathbb{Z}_N := \{0, 1, \ldots, N\}$.

The polynomial P_N is the *unique* polynomial of degree at most $N-1$ which satisfies the Bézout equation

$$(1-z)^N P_N(z) + z^N P_N(1-z) = 1, \quad z \in \mathbb{C} \tag{1.3}$$

and it is fortuitous that it is nonnegative on $[0,1]$. Indeed, it does not fall below one on that interval. Therefore, the filter Q_N vanishes only at $\xi = \pi$ and it may be expressed as the modulus squared of a polynomial on the unit circle, thereby leading to orthonormal refinable functions, [7].

Let I be the ideal filter, defined for all $\xi \in \mathbb{R}$ as

$$I(\xi) = \sum_{j \in \mathbb{Z}} \chi_{[-\pi/2,\pi/2]}(\xi + j),$$

where χ_E is the characteristic function of a set E. The filter Q_N tends to the ideal filter I on $[-\pi,\pi]\backslash\{\pi/2, -\pi/2\}$ pointwise, that is, for all $\xi \in \mathbb{R}\backslash(\pi/2 + \pi\mathbb{Z})$ we have that

$$\lim_{N\to\infty} Q_N(\xi) = I(\xi). \tag{1.4}$$

Moreover, the limit above holds uniformly on any compact subset of $[-\pi,\pi]\backslash\{\pi/2, -\pi/2\}$. As we shall see later, this fact follows from Corollary 4.3.

For a Hölder continuous filter Q with a prescribed flatness constraint at the frequency π, so that $Q(0) = 1$, we may define the Fourier transform of the corresponding *refinable function* ϕ by the functional equation,

$$\widehat{\phi}(\xi) = Q(\xi/2)\widehat{\phi}(\xi/2), \quad \widehat{\phi}(0) = 1, \quad \xi \in \mathbb{R}. \tag{1.5}$$

Here, $\hat{f}$ is the Fourier transform of an integrable function f defined by the equation $\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi}dx,\ \xi \in \mathbb{R}$ and for tempered distribution it is understood in the usual sense. The refinable function ϕ is uniquely determined by the function Q. Indeed, we have that

$$\hat{\phi}(\xi) = \prod_{l \in \mathbb{N}} Q(2^{-l}\xi),\ \xi \in \mathbb{R}. \tag{1.6}$$

The symbol associated with this refinable function is the function A defined such that $A(e^{i\xi}) = 2Q(\xi),\ \xi \in \mathbb{R}$. The Fourier coefficients of Q will determine the refinement equation for ϕ, [3].

For the interpolating filter Q_N, the corresponding refinable function, denoted by Ψ_N, has interpolating property,

$$\Psi_N(l) = \delta_l,\ l \in \mathbb{Z}, \tag{1.7}$$

where $\delta = (\delta_l : l \in \mathbb{Z})$ is the delta sequence. This function was introduced in [8] and it is the autocorrelation of the orthonormal refinable function studied in [7]. The Fourier exponent $s_p(\Psi_N)$ of the function Ψ_N satisfies the limit relation,

$$\lim_{N\to\infty} \frac{s_p(\Psi_N)}{N} = 2 - \frac{\ln 3}{\ln 2}, \tag{1.8}$$

[7, 17, 24, 26]. Recall that the Fourier exponent $s_p(f), 0 < p \le \infty$, of a function f with measurable Fourier transform is defined by the formula

$$s_p(f) = \sup\{s : (1 + |\cdot|)^s \hat{f} \in L^p\},$$

where L^p is the usual space of all p-integrable functions on $\mathbb{R}$.

One may easily verify the useful fact that

$$s_p(f) - 1/p \geq s_{p'}(f) - 1/p', \quad \text{if} \quad 0 < p < p' \leq \infty. \tag{1.9}$$

There has been a great deal of interest in estimating of the Fourier exponent of a refinable function, see for instance [4, 10, 14, 24, 25, 26] and references therein.

As we have mentioned earlier the only zero of the interpolating filter Q_N is at the frequency π. In this paper, we are interested in the minimally supported interpolating filter with flatness constraints at the prescribed frequencies $\xi_1, \ldots, \xi_s, \pi$ and properties of the corresponding refinable functions.

Let $\xi_1, \ldots, \xi_s, \pi$ be ordered so that $0 < \xi_1 < \xi_2 < \ldots < \xi_s < \pi$, and let $k_1, k_2, \ldots, k_s, N$ be positive integers. We introduce the set $E = \{\xi_j : j \in \mathbb{N}_s\}$ and s-tuple of positive integers $K = (k_l : l \in \mathbb{N}_s)$ where we have set $\mathbb{N}_s := \{1, \ldots, s\}$. First, we show in Theorem 4.1 that a necessary and sufficient condition for the existence of an interpolating filter Q having the flatness order k_j at the frequency $\xi_j, j \in \mathbb{N}_s$ and the flatness order $2N$ at the frequency π is that $E \cap (\pi - E) = \emptyset$. We denote this minimal supported interpolating filter having the above flatness constraints by $Q_{N,E,K}$. Our notation ensures that $Q_{N,E,K} = Q_N$ when E is an empty set and generally $Q_{N,E,K}$ is a trigonometric polynomial of degree $2N + 2|K| - 1$, where we define $|K| := \sum_{j \in \mathbb{N}_s} k_j$. In this paper, we establish similar limit properties as (1.4) for $Q_{N,E,K}$, when $E \subset (\pi/2, \pi)$. Specifically, we shall show in Corollary 4.3, for any $\xi \in \mathbb{R} \backslash (\pi/2 + \pi\mathbb{Z})$ that

$$\lim_{N \to \infty} Q_{N,E,K}(\xi) = I(\xi).$$

However, this limit property no longer holds true when $\xi_1 \in (0, \pi/2)$. In fact, we shall show in Corollary 4.5 whenever $\xi \in (-\xi_1, \xi_1) + \pi\mathbb{Z}$ that

$$\lim_{N \to \infty} Q_{N,E,K}(\xi) = I(\xi)$$

while for $\xi_1 \in (0, \pi/2)$ and $\xi \notin ([-\xi_1, \xi_1] \cup E \cup (-E) \cup \{\pi/2\}) + \pi\mathbb{Z}$, it follows that

$$\lim_{N \to \infty} Q_{N,E,K}(\xi) = \infty.$$

We denote the refinable function with filter $Q_{N,E,K}$ by $\Phi_{N,E,K}$. For the case that $\xi_1 \in (\pi/2, \pi)$, Theorem 5.1 provides a lower bound estimate of Fourier exponent of the refinable function $\Phi_{N,E,K}$ similar to those for Ψ_N.

As mentioned early the upper bound of the filter $Q_{N,E,K}$ tends to infinity when $\xi_1 \in (0, \pi/2)$. Due to the decay property of the refinable function Ψ_N, the study of smoothness of the refinable functions $\Phi_{N,E,K}$ when $\xi_1 \in (0, \pi/2)$ is possible. In particular, Theorem 5.1 ensures for $\xi_1 \in (\pi/3, \pi/2)$, that the Fourier exponent $s_p(\Phi_{N,E,K})$ is still proportional to cN for some positive constant c and hence $\Phi_{N,E,K}$ is certainly continuous in that case for sufficiently large N. However, Theorem 5.1 also shows that the situation deteriorates when $\xi_1 \in (0, \pi/3)$. Although, the Fourier exponent $s_p(\Phi_{N,E,K})$ is still proportional to cN the constant c in this case is *negative* and hence $\Phi_{N,E,K}$ is *discontinuous* for sufficiently large N .

In section two we begin our presentation with some facts pertaining to *iterative interpolation* as studied by Deslauries and Dubuc in [8] and in section three we treat the case of a *single* zero. This allows us to explain the method of analysis we employ with clarity and precision before the general case is treated in sections four and five.

We leave for another time the challenging problem of the type studied here when the number of zeros of $Q_{N,E,K}$, different from π, also depend upon N.

2. **Laurent polynomials and iterative interpolation.** We begin this section by recalling the method of Deslauriers and Dubuc [8] of using *local* Lagrange interpolation to *iteratively* construct a continuous function $f \in C(\mathbb{R})$ which interpolates given data $\{y_j : j \in \mathbb{Z}\}$ at *all* integers, that is, $f(j) = y_j,\ j \in \mathbb{Z}$. To this end, for a given nonnegative integer N, we denote by $L_j,\ j \in \mathbb{J}_N := -N + 1 + \mathbb{Z}_{2N-1}$, the Lagrange polynomials of degree $2N - 1$ defined by the requirement that

$$L_j(l) = \delta_{jl}, \quad j, l \in \mathbb{J}_N. \tag{2.1}$$

The first step of their method is to set

$$f(\frac{1}{2}) := \sum_{j \in \mathbb{J}_N} L_j(\frac{1}{2}) f(j), \tag{2.2}$$

that is, to *define* $f(\frac{1}{2})$ as the value of the polynomial which interpolates the *finite* set of data $\{y_j : j \in \mathbb{J}_N\}$ at $\frac{1}{2}$. Generally, the values of f on the grid $\frac{1}{2} + \mathbb{Z}$ are given by the formula

$$f\Big(j + \frac{1}{2}\Big) := \sum_{l \in \mathbb{J}_N} L_l(\frac{1}{2}) f(l + j), \quad j \in \mathbb{Z}. \tag{2.3}$$

By this procedure we have specified f on the *fine* grid $\mathbb{Z}/2$ from it values on the *coarse* grid $\mathbb{Z}$. The process is repeated iteratively to obtain f on $\mathbb{Z}/2^r,\ r \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. It is proved in [8] that this iterative process converges as $r \to \infty$ to a continuous function f which interpolates the original data set $\{y_j : j \in \mathbb{Z}\}$ and has the form $f = \sum_{j \in \mathbb{Z}} f(j) \Psi_N(\cdot - j)$ where the function Ψ_N has the property that $\Psi_N(j) = \delta_j, \quad j \in \mathbb{Z}$. Equation (1.8) mentioned earlier provides the Fourier exponent of the function Ψ_N. The case $N = 2$ was first considered by Dubuc in [9].

The function Ψ_N is *refinable* and satisfies the *refinement equation*

$$\Psi_N = \sum_{j \in \mathbb{Z}} d_j \Psi_N(2 \cdot - j), \tag{2.4}$$

where the *mask sequence* $\{d_j : j \in \mathbb{Z}\}$ is defined by the equation

$$d_{l-2j} = \begin{cases} L_j\left(\dfrac{l}{2}\right), & j \in \mathbb{J}_N, \\ 0, & \text{otherwise}, \end{cases} \qquad l \in \{0, 1\}. \tag{2.5}$$

The symbol of the mask given by the Laurent polynomial

$$D_N(z) := \sum_{j \in \mathbb{Z}} d_j z^j, \quad z \in \mathbb{C} \backslash \{0\} \tag{2.6}$$

is *nonnegative* for $z \in \Delta :=$ unit circle, as can be seen from the formula

$$D_N(e^{i\xi}) = 2 \frac{\int_\pi^\xi (\sin t)^{2N-1}\, dt}{\int_\pi^{2\pi} (\sin t)^{2N-1}\, dt}, \quad \xi \in [-\pi, \pi] \tag{2.7}$$

and vanishes *only* at $z = -1$. In particular, we have that $D_1(z) = 1 + z$. In general, we have from (2.7) that

$$D_N(e^{i\xi}) = \frac{(2N-1)!}{2^{2N-2}[(N-1)!]^2} \int_\xi^\pi (\sin t)^{2N-1}\, dt, \quad \xi \in [-\pi, \pi]. \tag{2.8}$$

Moreover, it is known that

$$D_N(e^{i\xi}) = 2Q_N(\xi), \quad \xi \in \mathbb{R}, \tag{2.9}$$

a result which follows because each side of the equation is uniquely defined by its properties, see below. The prevailing terminology is that a *symbol* is connected to an associated *filter* by such a relation.

The key to the convergence of the iterative scheme described above is the fact that the filter Q_N is nonnegative on Δ and vanishes *only* at $\xi = \pi$. In fact, *any* subdivision scheme with a symbol whose corresponding filter has this property converges to a continuous function (actually, only positivity on the interval $(\pi/2, \pi)$ is needed for the validity of this conclusion). All the properties mentioned above and extensions can be found in [20].

To construct other filters with properties similar to Q_N we shall isolate *four* essential conditions which determine this Laurent polynomial *uniquely*. First, it is *symmetric*. Recall that a Laurent polynomial A given as

$$A(z) := \sum_{j\in\mathbb{Z}} a_j z^j, \quad z \in \mathbb{C}\backslash\{0\} \tag{2.10}$$

is symmetric provided that

$$A(z) = A(z^{-1}), \quad z \in \mathbb{C}\backslash\{0\}. \tag{2.11}$$

Second, it is of *degree* at most $M := 2N-1$, that is, its coefficient sequence $\{a_j : j \in \mathbb{Z}\}$ vanishes for $j \notin \mathbb{K}_M := -M + \mathbb{Z}_{2M}$. We denote this class by $\mathcal{L}_M$. Next, since the iterative scheme described above always leaves the data on the coarse grid $\mathbb{Z}$ unaltered, that is, it is an interpolatory subdivision scheme the Laurent polynomial must satisfy the equation

$$A(z) + A(-z) = 2, \quad z \in \mathbb{C}\backslash\{0\}. \tag{2.12}$$

Finally, the scheme preserves all polynomials of degree $2N-1$, in the sense that for every $p \in \pi_{2N-1}$ and $l \in \mathbb{Z}$ we have that

$$p(\frac{l}{2}) = \sum_{j\in\mathbb{Z}} d_{l-2j}\, p(j). \tag{2.13}$$

This is equivalent to the fact that the Laurent polynomial A has a zero of order at least $2N$ at -1, that is

$$A^{(j)}(-1) = 0, \quad j \in \mathbb{Z}_{2N-1}, \tag{2.14}$$

[20]. This observation motivates the following definition.

Definition 2.1. For any $N \in \mathbb{N}$ and $M \in \mathbb{Z}_+$, we denote by $\mathcal{A}_{N,M}$ the class of all Laurent polynomial $A \in \mathcal{L}_{2(N+M)-1}$ which satisfies (2.12) and (2.14).

Let us note the following representation for the class $\mathcal{A}_{N,M}$.

Theorem 2.2. *$A \in \mathcal{A}_{N,M}$ if and only if there exist real constants $t_j, j \in \mathbb{Z}_M$ with $\sum_{j\in\mathbb{Z}_M} t_j = 1$ such that*

$$A = \sum_{j\in\mathbb{Z}_M} t_j\, D_{N+j}. \tag{2.15}$$

Proof. By (2.11) and (2.12), we may uniquely associate with each element $A \in \mathcal{A}_{N,M}$ the coefficient vector $(a_1, a_3, \dots, a_{2(N+M)-1})$ formed from some of its coefficients and the polynomial reproduction property (2.13) is equivalent to the following conditions on this vector

$$\sum_{r\in\mathbb{Z}_{N+M-1}} a_{2r+1}(r+\frac{1}{2})^{2l} = \begin{cases} 0, & l \in \mathbb{Z}_{N-1}\backslash\{0\}, \\ \frac{1}{2}, & l = 0. \end{cases} \tag{2.16}$$

These linear relations are independent and so the first $N-1$ equations imply that every $A \in \mathcal{A}_{N,M}$ is in unique correspondence with an element in a linear space S of dimension $k+1$. The coefficient sequences associated with the Laurent polynomials D_{N+j}, $j \in \mathbb{Z}_M$ are in S and are linearly independent which is a consequence of the fact that they are of *exact* degree $2N+j$. Consequently, for any $A \in \mathcal{A}_{N,M}$ there exist unique real numbers $t_j, j \in \mathbb{Z}_M$ so that

$$A = \sum_{j\in\mathbb{Z}_M} t_j \, D_{N+j},$$

and the last equation in (2.15) implies that $\sum_{j\in\mathbb{Z}_M} t_j = 1$. □

Note that as a special case of the above fact when $M = 0$ implies that the class $\mathcal{A}_{N,0}$ consists only of the polynomial D_N.

The zero at -1 is known to affect the regularity of the refinable function associated with A, see, for example, [3] and, as already mentioned, its positivity on the interval $(\pi/2, \pi)$ leads to convergence of the subdivision scheme. Thus, finding nonnegative members of the class $\mathcal{A}_{N,M}$ is especially pleasant and desirable as they will also lead to orthonormal wavelets and the associated subdivision scheme will converge, [20, 22].

The class $\mathcal{A}_{N,M}$ has M degrees of freedom which can be used by specifying zeros of A. We shall do this in the next section. But as preparation for this we look at the class $\mathcal{A}_{N,1}$ in some detail. To this end, for $N \in \mathbb{N}$ and $t \in \mathbb{R}$ we define the polynomial $A_N(\cdot|t)$ at $z \in \mathbb{C}$ as

$$A_N(z|t) = (1-t)D_N(z) + tD_{N+1}(z),$$

and recall that

$$A_N(-1|t) = 0 \text{ and } A_N(1|t) = 2, \quad t \in \mathbb{R}.$$

We use $Q_N(\cdot|t)$ for the associated filter, that is, $A_N(e^{i\xi}|t) = 2Q_N(\xi|t)$, $\xi \in \mathbb{R}$. Clearly, $Q_N(\cdot|t)$ is nonnegative on Δ for $t \in [0,1]$. More information of this type is provided next.

Proposition 2.3. *For any positive integer N, the filter $Q_N(\cdot|t)$ is strictly positive in $(0,\pi)$ for $t \in [-2N, 1]$, while it has a unique zero in $(\pi/2, \pi)$ for $t \in (1, \infty)$.*

Proof. A direct computation using formula (2.8) yields the formula for the derivative of the filter $Q_N(\xi|t)$. Specifically, we have that

$$Q_N'(\xi|t) = \frac{1}{2^{2N-1}} \binom{2N-1}{N} (\sin\xi)^{2N+1} \left(t(2N+1)\cos^2\xi - (t+2N)\right).$$

Recall, by definition that $2Q_N(\xi|0) = D_N(e^{i\xi})$ and $2Q_N(\xi|1) = D_{N+1}(e^{i\xi})$. And so, they are both positive for $\xi \in (-\pi, \pi)$. So, the filter $Q_N(\cdot|t)$ is strictly positive in $(0,\pi)$ for $t \in [0,1]$.

For $t \in [-2N, 0]$ it follows from our formula for the derivative of the filter that $Q_N'(\xi|t) < 0$ for $\xi \in (0,\pi)$. Moreover, since $Q_N(\cdot|t)$ is an interpolatory filter it necessarily follows for all $t \in \mathbb{R}$ that $Q_N(\pi/2|t) = 1/2$. Hence, since it vanishes at $\xi = \pi$ we conclude that $Q_N(\xi|t) > 0$ for all $\xi \in (0, \pi)$, otherwise, by Rolle's theorem, its derivative would have a sign change in $(0, \pi)$.

For $t > 1$, $Q'_N(\cdot|t)$ has exactly two zeros in $(0, \pi)$. One is given by $\cos\xi_1 = \sqrt{\frac{t+2N}{2t(N+1)}}$ which is in the interval $(0, \pi/2)$, while the other is $\cos\xi_2 = -\sqrt{\frac{t+2N}{2t(N+1)}}$ which is in the interval $(\pi/2, \pi)$. From our formula for the derivative of $Q_N(\cdot|t)$ we have for all $t \in \mathbb{R}$ that $\operatorname{sgn} Q'_N(0^+|t) = \operatorname{sgn}(t-1)$ and $\operatorname{sgn} Q'_N(\pi^-|t) = \operatorname{sgn}(t-1)$. Therefore, we conclude that $\operatorname{sgn} Q'_N(\pi^-|t) < 0$. Thus, since $Q_N(\pi|t) = 0$ and $Q_N(\pi/2|t) = 1/2$, $Q_N(\cdot|t)$ must have at least one zero in $(\pi/2, \pi)$. However, since $Q'_N(\cdot|t)$ has exactly one zero in that interval we conclude that $Q_N(\cdot|t)$ has exactly one zero in $(\pi/2, \pi)$. □

3. **The case of one zero.** In this section we shall explicitly construct an interpolatory filter of minimal degree with *one* prescribed zero and develop some of its properties. Thus, the problem we consider is to find a polynomial P_{N,ξ_1} of least degree which satisfies the equation $(1-z)^N P_{N,\xi_1}(z) + z^N P_{N,\xi_1}(1-z) = 1$, $z \in \mathbb{C}$ with the property that $P_{N,\xi_1}(1-t_1) = 0$ where $t_1 := \cos^2\frac{\xi_1}{2}$ and $\xi_1 \in (0, \pi)$. Therefore, the associated filter Q_{N,ξ_1} defined for $\xi \in \mathbb{R}$ by the equation $Q_{N,\xi_1}(\xi) = \cos^{2N}\frac{\xi}{2} P_{N,\xi_1}(\sin^2\frac{\xi}{2})$ vanishes at ξ_1.

When $\xi_1 = \pi/2$ there is clearly no solution. So, we assume that $\xi_1 \in (0, \pi)\backslash\{\pi/2\}$ and then recall that the least degree solution of the equation which we wish to solve *without* the demand that it vanish at $1 - t_1$ is the polynomial

$$P_N(z) = \sum_{l \in \mathbb{Z}_{N-1}} \binom{N-1+l}{l} z^l, \quad z \in \mathbb{C}. \tag{3.1}$$

Since this polynomial is positive on $[0, 1]$ the polynomial P_{N,ξ_1} which we seek is of at least of degree N and is obtained by appropriately modifying P_N. Indeed, it can be verified directly that P_{N,ξ_1} is given uniquely at $z \in \mathbb{C}$ by the formula

$$P_{N,\xi_1}(z) = P_N(z) + z^N(1-2z)P_N(1-t_1)(1-t_1)^{-N}(1-2t_1)^{-1}. \tag{3.2}$$

Let us now provide estimates for this polynomial which will lead us to properties of the refinable function corresponding to the filter Q_{N,ξ_1}. We begin with the following estimates for the polynomial P_N.

Proposition 3.1.

$$P_N(t) \le \begin{cases} (1-t)^{-N}, & t \in [0, 1/2], \\ 2^{2(N-1)}t^{N-1}, & t \in [1/2, 1], \end{cases} \tag{3.3}$$

and

$$P_N(t) \ge \begin{cases} (1-t)^{-N}/2, & t \in [0, 1/2], \\ N^{-1}2^{2(N-1)}t^{N-1}, & t \in [1/2, 1]. \end{cases} \tag{3.4}$$

Proof. Since P_N satisfies $(1-t)^N P_N(t) + t^N P_N(1-t) = 1$ and $P_N(t) \ge 0$ for all $t \in [0, 1]$, we have for any $t \in [0, 1]$ that

$$P_N(t) \le (1-t)^{-N}. \tag{3.5}$$

Substituting $t = 1/2$ into the Bézout identity for P_N, we obtain that $P_N(1/2) = 2^{N-1}$. Therefore, for $t \in [1/2, 1]$, we conclude that

$$\begin{aligned} P_N(t) &= \sum_{l \in \mathbb{Z}_{N-1}} \binom{N-1+l}{l} (2t)^l 2^{-l} \\ &\leq (2t)^{N-1} P_N(1/2) = 2^{2(N-1)} t^{N-1}. \end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6) leads to the estimate (3.3).

For $t \in [0, 1/2]$, we have that

$$(1-t)^N P_N(t) = 1 - t^N P_N(1-t) \geq 1 - t^N 2^{2(N-1)} (1-t)^{N-1}.$$

Since for $N \in \mathbb{N}$, the expression $t^N(1-t)^{N-1}$ is an increasing function of t on $[0, 1/2]$, we conclude that

$$P_N(t) \geq (1-t)^{-N}/2.$$

Therefore, the first estimate in (3.4) follows.

To prove the second claim we use the inequality

$$\binom{2l}{l} \geq \frac{2^{2l}}{l+1}$$

valid for all $l \in \mathbb{N}$ which may be proved by induction on l. This together with the lower bound

$$P_N(t) = \sum_{l \in \mathbb{Z}_{N-1}} \binom{N-1+l}{l} t^l \geq \binom{2N-2}{N-1} t^{N-1}$$

leads to the second conclusion of (3.4). □

Using the above estimate for P_N we have the following estimate for P_{N,ξ_1}. Below, we always have that $t_1 = \cos^2 \frac{\xi_1}{2}$.

Proposition 3.2. *If* $t_1 \in [0, 1/2)$, *then*

$$|P_{N,\xi_1}(t)| \leq (1 + (4(1-t_1)(1-2t_1))^{-1}) \begin{cases} (1-t)^{-N}, & t \in [0, 1/2], \\ 2^{2N} t^N, & t \in [1/2, 1], \end{cases} \tag{3.7}$$

while for $t_1 \in [1/2, 1)$

$$|P_{N,\xi_1}(t)| \leq (1 + (4t_1(1-t_1))^{-N} |1-2t_1|^{-1}) \begin{cases} (1-t)^{-N}, & t \in [0, 1/2], \\ 2^{2N} t^N, & t \in [1/2, 1]. \end{cases} \tag{3.8}$$

Proof. For $t_1 \in (0, 1/2)$ and $t \in [0, 1/2]$, we have that

$$\begin{aligned} |P_{N,\xi_1}(t)| &\leq (1-t)^{-N} + t^N 2^{2(N-1)} (1-t_1)^{N-1} (1-t_1)^{-N} (1-2t_1)^{-1} \\ &\leq (1 + (4(1-t_1)(1-2t_1))^{-1})(1-t)^{-N}. \end{aligned} \tag{3.9}$$

For $t_1 \in (0, 1/2)$ and $t \in [1/2, 1]$, we obtain that

$$\begin{aligned} |P_{N,\xi_1}(t)| &\leq 2^{2(N-1)} t^{N-1} + t^N 2^{2(N-1)} (1-t_1)^{N-1} (1-t_1)^{-N} (1-2t_1)^{-1} \\ &\leq (1 + (4(1-t_1)(1-2t_1))^{-1}) 2^{2N} t^N. \end{aligned} \tag{3.10}$$

Combining inequalities (3.9) and (3.10) proves (3.7).

For $t_1 \in [1/2, 1)$ and $t \in [0, 1/2]$,

$$\begin{aligned} |P_{N,\xi_1}(t)| &\leq (1-t)^{-N} + t^N t_1^{-N} (1-t_1)^{-N} |1-2t_1|^{-1} \\ &\leq (1-t)^{-N} (1 + (4t_1(1-t_1))^{-N} |1-2t_1|^{-1}), \end{aligned} \tag{3.11}$$

while for $t_1 \in [1/2, 1)$ and $t \in [1/2, 1]$,

$$\begin{aligned} |P_{N,\xi_1}(t)| &\le 2^{2(N-1)} t^{N-1} + t^N t_1^{-N}(1-t_1)^{-N}|1-2t_1|^{-1} \\ &\le (1 + (4t_1(1-t_1))^{-N}|1-2t_1|^{-1}) 2^{2N} t^N. \end{aligned} \tag{3.12}$$

Combining inequalities (3.11) and (3.12) leads to (3.8). □

To study the smoothness of the refinable function associated with the interpolating filter Q_{N,ξ_1}, we shall also require the estimates which we present next.

Proposition 3.3. *Let P_{N,ξ_1} be defined as above. Then*

$$|P_{N,\xi_1}(3/4)| \ge \begin{cases} \dfrac{(1-4t_1)(5-4t_1)}{24N(1-2t_1)(1-t_1)} 3^N, & \text{if } t_1 \in (0, 1/4), \\ \dfrac{4t_1 - 1}{8N(1-t_1)(1-2t_1)} 3^N, & \text{if } t_1 \in (1/4, 1/2), \\ \dfrac{1}{4(2t_1-1)} \left(\dfrac{3}{4t_1(1-t_1)} \right)^N, & \text{if } t_1 \in (1/2, 1). \end{cases} \tag{3.13}$$

Proof. For $t_1 \in (0, 1/4)$, we get that

$$\begin{aligned} &\left| P_{N,\xi_1}\left(\frac{3}{4}\right) \right| \\ = &\sum_{l \in \mathbb{Z}_{N-1}} \binom{N-1+l}{l} \left(\left(\frac{3}{4}\right)^l - \frac{1}{2(1-2t_1)} \left(\frac{3}{4}\right)^N (1-t_1)^{l-N} \right) \\ = &\sum_{l \in \mathbb{Z}_{N-1}} \binom{N-1+l}{l} \left(\frac{3}{4}\right)^l \left(1 - \frac{1}{2(1-2t_1)} \left(\frac{3}{4(1-t_1)} \right)^{N-l} \right) \\ \ge &\sum_{l \in \mathbb{Z}_{N-1}} \binom{N-1+l}{l} \left(\frac{3}{4}\right)^l \left(1 - \frac{3}{8(1-2t_1)(1-t_1)} \right) \\ \ge &N^{-1} 3^{N-1} \left(1 - \frac{3}{8(1-2t_1)(1-t_1)} \right), \end{aligned}$$

where we have used the facts that $2(1-2t_1) > 1$ and $3 < 4(1-t_1)$ for $t_1 \in (0, 1/4)$. Consequently, the first inequality follows and likewise invoking Proposition 3.1 so too does the last inequality, thereby establishing the first estimate in (3.13).

For $t_1 \in (1/4, 1/2)$, we have that $2(1-2t_1) < 1$ and $3 > 4(1-t_1)$. Therefore, we obtain that

$$\begin{aligned} &\left| P_{N,\xi_1}\left(\frac{3}{4}\right) \right| \\ = &\left| \sum_{l \in \mathbb{Z}_{N-1}} \binom{N-l+l}{l} \left(\left(\frac{3}{4}\right)^l - \frac{1}{2(1-2t_1)} \left(\frac{3}{4}\right)^N (1-t_1)^{l-N} \right) \right| \\ = &\sum_{l \in \mathbb{Z}_{N-1}} \binom{N-1+l}{l} \left(\frac{3}{4}\right)^l \left(\frac{1}{2(1-2t_1)} \left(\frac{3}{4(1-t_1)} \right)^{N-l} - 1 \right) \\ \ge &\sum_{l \in \mathbb{Z}_{N-1}} \binom{N-1+l}{l} (1-t_1)^l \left(\frac{3}{4(1-t_1)} \right)^N \left(\frac{1}{2(1-2t_1)} - 1 \right) \\ \ge &\frac{4t_1 - 1}{8N(1-t_1)(1-2t_1)} 3^N, \end{aligned}$$

where we have used the inequality $st - 1 \geq (s-1)t$, valid for any $t \geq 1$, to obtain the first inequality and the estimate of $P_N(1-t_1)$ in Proposition 3.1 to obtain the last inequality. This proves the second estimate in (3.13).

For $t_1 \in (1/2, 1)$, we compute

$$\begin{aligned} P_{N,\xi_1}(3/4) &= P_N(3/4) + (3/4)^N(-1/2)P_N(1-t_1)(1-t_1)^{-N}(1-2t_1)^{-1} \\ &\geq (3/4)^N(-1/2)P_N(1-t_1)(1-t_1)^{-N}(1-2t_1)^{-1} \\ &\geq \frac{1}{4|1-2t_1|}\left(\frac{3}{4}\right)^N t_1^{-N}(1-t_1)^{-N} \\ &= \frac{1}{4|1-2t_1|}\left(\frac{3}{4t_1(1-t_1)}\right)^N, \end{aligned} \tag{3.14}$$

which proves the third estimate in (3.13). □

We are now ready to estimate the regularity of the refinable function ϕ_{N,ξ_1} associated with the filter Q_{N,ξ_1}. To this end, we recall from [7] the estimate for the Fourier transform of refinable functions.

Lemma 3.4. *If ϕ is a refinable function associated with a trigonometric polynomial Q, that is,*

$$\widehat{\phi}(2\xi) = Q(\xi)\widehat{\phi}(\xi), \quad \xi \in \mathbb{R},$$

and Q has the form

$$Q(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N V(\xi),\ \xi \in \mathbb{R}$$

for some integer N and trigonometric polynomial V such that for some positive constant q,

$$\begin{cases} |V(\xi)| \leq q, & |\xi| \leq 2\pi/3, \\ |V(\xi)V(2\xi)| \leq q^2, & 2\pi/3 \leq |\xi| \leq \pi, \end{cases}$$

then there exists a positive constant c such that for all $\xi \in \mathbb{R}$

$$|\widehat{\phi}(\xi)| \leq c(1+|\xi|)^{-N+\ln q/\ln 2}.$$

To use the above estimate to study the regularity of the refinable function with filter Q_{N,ξ_1} we need the following result.

Lemma 3.5. *If a is the 2π periodic function defined by*

$$a(\xi) = \begin{cases} (\cos\xi/2)^{-2}, & |\xi| \leq \pi/2, \\ 4(\sin\xi/2)^2, & \pi/2 \leq |\xi| \leq \pi, \end{cases}$$

then

$$\begin{cases} |a(\xi)| \leq 3, & |\xi| \leq 2\pi/3, \\ |a(\xi)a(2\xi)| \leq 9, & 2\pi/3 \leq |\xi| \leq \pi. \end{cases}$$

Proof. The first inequality is obvious. For $2\pi/3 \leq \xi \leq 3\pi/4$, we verify that

$$a(\xi)a(2\xi) = 64\sin^4\frac{\xi}{2}\Big(1 - \sin^2\frac{\xi}{2}\Big).$$

Note that $t^2 - t^3$ is a decreasing function of t on $(2/3, 1)$, and so we conclude that

$$a(\xi)a(2\xi) \leq a(2\pi/3)a(4\pi/3) = 9.$$

For $3\pi/4 \le \xi \le \pi$, we use the formula

$$a(\xi)a(2\xi) = 4\sin^2\frac{\xi}{2}\left(1-2\sin^2\frac{\xi}{2}\right)^{-2}.$$

Since $t/(1-2t)^2$ is also decreasing function of t on $(1/2,1)$, we conclude that

$$a(\xi)a(2\xi) \le a(3\pi/4)a(3\pi/2) \le a(2\pi/3)a(4\pi/3) = 9.$$

This completes the proof. □

Theorem 3.6. *If ϕ_{N,ξ_1} is the refinable function with the filter Q_{N,ξ_1} then there is a positive constant c such that for all $t \in (0,1/2)\backslash\{1/2\}$, we have the estimate*

$$|\widehat{\phi}_{N,\xi_1}(\xi)| \le c(1+|\xi|)^{(-N\ln(4/3)+\ln\alpha_N(\cos^2\xi_1/2))/\ln 2},$$

where

$$\alpha_N(t) := \begin{cases} 1+(4t(1-2t))^{-1}, & t\in(0,1/2),\\ 1+(4t(1-t))^{-N}|1-2t|^{-1}, & t\in[1/2,1). \end{cases}$$

Proof. The above result is a direct consequence of Lemmas 3.4 and 3.5, and the estimates in Proposition 3.2. Specifically, we use the Lemma 3.4 with q replaced by $\alpha_N(t_1)3^N$, N by $2N$, and V by the function satisfying $|V(\xi)| = |P_{N,\xi_1}(\cos^2(\xi/2))|$ for all $\xi\in\mathbb{R}$. □

Note that

$$1+(4t_1(1-t_1))^{-N}|1-2t_1|^{-1} \le 2(4t_1(1-t_1))^{-N}|1-2t_1|^{-1}$$

for $t_1\in(1/2,3/4)$. Therefore, as a consequence of Theorem 3.6, for $\xi_1\in(\frac{\pi}{3},\pi)\backslash\{\frac{\pi}{2}\}$ and sufficiently large N, the Fourier transform of ϕ_{N,ξ_1} is integrable. In particular, we have the following estimate for N which guarantees the continuity of ϕ_{N,ξ_1}.

Corollary 3.7. *If either*

$$\xi_1\in(\pi/2,\pi) \quad \text{and} \quad N \ge \frac{\ln(2+2\sin^{-2}\xi_1)}{\ln(4/3)},$$

or

$$\xi_1\in(\pi/3,\pi/2) \quad \text{and} \quad N \ge \frac{\ln(4|\cos\xi_1|^{-1})}{\ln(4/3\sin^2\xi_1)},$$

then ϕ_{N,ξ_1} is continuous.

We shall now provide estimates of the Sobolev exponent of the refinable function corresponding to the filter Q_{N,ξ_1}. To this end, we recall the following two results, see [7, 24].

Lemma 3.8. *If ϕ is a refinable function with $\widehat{\phi}$ continuous and continuous filter Q such that $Q(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N V(\xi),\ \xi\in\mathbb{R}$ for some continuous 2π-periodic function V where for some positive constant q there holds*

$$\begin{cases} |V(\xi)| \le q, & |\xi| \le 2\pi/3,\\ |V(\xi)V(2\xi)| \le q^2, & 2\pi/3 \le |\xi| \le \pi, \end{cases}$$

then for $0<p\le\infty$ we have that

$$s_p(\phi) \ge N - \frac{\ln q}{\ln 2} - \frac{1}{p}. \tag{3.15}$$

Lemma 3.9. *If ϕ be a refinable function with $\widehat{\phi}$ continuous and continuous filter Q such that $Q(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N V(\xi),\ \xi \in \mathbb{R}$ for some continuous 2π-periodic function V and that $\widehat{\phi}(2\pi/3 + 2l\pi) \neq 0$ for some integer l then for $0 < p \le \infty$, we have that*

$$s_p(\phi) \le N - \frac{\ln |V(2\pi/3)|}{\ln 2}. \tag{3.16}$$

Using Lemma 3.8, Lemma 3.9 and the estimate in Proposition 3.3, we have the following result for the Sobolev exponent of ϕ_{N,ξ_1}.

Theorem 3.10. *If $\xi_1 \in (0,\pi)\backslash\{\pi/2, 2\pi/3\}$ and for some integer $l \in \mathbb{Z}$ $\widehat{\phi}_{N,\xi_1}(2\pi/3 + 2l\pi) \neq 0$ then there exists a positive constant c such that for all positive integers N and $\xi_1 \in (\pi/2, \pi)$ there holds the bound*

$$\left| s_p(\phi_{N,\xi_1}) - N\frac{\ln 4/3}{\ln 2} \right| \le c \ln N,$$

while for $\xi_1 \in (0, \pi/2)$

$$\left| s_p(\phi_{N,\xi_1}) - N\frac{\ln((4\sin^2\xi_1)/3)}{\ln 2} \right| \le c.$$

As a consequence of Theorem 3.10, we have the following result.

Corollary 3.11. *If $\xi_1 \in (0, \pi/3)$ and for some $l \in \mathbb{Z}$ that $\widehat{\phi}_{N,\xi_1}(2\pi/3 + 2l\pi) \neq 0$, then ϕ_{N,ξ_1} is discontinuous for all*

$$N \ge \frac{\ln(4|\cos\xi_1|)}{\ln(3/4\sin^2\xi_1)}.$$

4. **Minimally supported filters with prescribed zeros.** In this section, we consider the existence and uniqueness of minimally supported interpolating filters with prescribed zeros and their asymptotic behavior as the order of zero only at the frequency π tends to infinity. First, we present the following existence result.

Theorem 4.1. *Let N be a positive integer, $K = (k_j : j \in \mathbb{N}_s)$ be a s-tuple of positive integers, and $E = \{\xi_j : j \in \mathbb{N}_s\}$ be a subset of $(0,\pi)$ such that $0 < \xi_1 < \xi_2 < \cdots < \xi_s < \pi$. A necessary and sufficient condition for the existence of a trigonometric polynomial Q which has a zero of order $2N$ at π and zeros at $\xi_1, \ldots, \xi_s$ of orders $k_1, \ldots, k_s$, respectively, and satisfies the equations*

$$Q(\xi) + Q(\xi + \pi) = 1, \quad \xi \in \mathbb{R} \tag{4.1}$$

$$Q(\xi) = Q(-\xi), \quad \xi \in \mathbb{R} \quad \text{and} \quad Q(0) = 1, \tag{4.2}$$

is that for all $j, l \in \mathbb{N}_s$ we have that

$$\xi_j + \xi_l \neq \pi. \tag{4.3}$$

This result even holds when E contains complex numbers (in that case we do not order the elements of E), although we do not take advantage of this generality here. Nonetheless, when $E \subseteq \mathbb{C}\backslash[0,1]$ the interesting question arises as to whether or not the polynomial P defined so that $\cos^{2N}\frac{\xi}{2}P(\sin^2\frac{\xi}{2}) = Q(\xi)$, $\xi \in \mathbb{R}$ is nonnegative on $[0,1]$ at least for *some* choice of Q, if not for the minimal degree choice. Of course, if E is the empty set it is indeed true for the minimal degree solution. For progress on understanding this question see [11, 12, 18, 19, 20]. In fact, if Q is the least degree solution for a P required to vanish at some prescribed zeros in the set $(1,\infty)$ then Q is nonnegative on $[0,1]$. This filter has two applications. Given a

Sobolov subspace of L^2 the zeros of P can be chosen to obtain orthonormal wavelets of compact support and arbitrary regularity in that subspace, [21]. Alternatively, this filter can be generated by local interpolation with exponential functions.This lead to orthonormal wavelets on L^2 of compact support whose Fourier transform has prescribed zeros on the imaginary axis, [20].

Generally, the method of local interpolation will lead to interpolating filters and when exponentials are used zeros of the filter will emerge. Alternatively, spline functions maybe be used and this has been investigated to a limited degree in [6].

Returning to Theorem 4.1 we denote the trigonometric polynomial of minimal degree described in it by $Q_{N,E,K}$. Whenever we talk about this filter we always assume that the conditions of Theorem 4.1 hold. By (4.8), the degree of $Q_{N,E,K}$ is $2N+2|K|$ and it is low pass filter, since $Q_{N,E,K}(0)=1$ and $Q_{N,E,K}(\pi)=0$. Moreover, we have the following upper bound estimates.

Theorem 4.2. *Given K and E as above there exist a positive constant c such that for all positive integers N we have for $\xi_1 \in (\pi/2, \pi)$ that*

$$|Q_{N,E,K}(\xi)| \le \begin{cases} c, & \xi \in [0,\pi/2], \\ c\sin^{2N}\xi, & \xi \in [\pi/2,\pi], \end{cases} \tag{4.4}$$

while for $\xi_1 \in (0,\pi/2)$ there holds the inequality

$$|Q_{N,E,K}(\xi)| \le \begin{cases} cN^{k_1-1}, & \xi \in [0,\xi_1], \\ cN^{k_1-1}\left(\dfrac{\sin\xi}{\sin\xi_1}\right)^{2N}, & \xi \in [\xi_1,\pi]. \end{cases} \tag{4.5}$$

For $\xi_1 \in (\pi/2,\pi)$, a consequence of Theorem 4.2 is that the filter $Q_{N,E,K}$ converges to the ideal filter exponentially fast outside the transition band as N tends to infinity, see [2] for an quantitative measurement for the approximation to the ideal filter. We state this fact next.

Corollary 4.3. *Given K and E as above with $E \subset (\pi/2,\pi)$ and for any $\delta \in (0,\pi/2)$ there exist a positive constant c and an $r \in (0,1)$ such that all positive integers N and $\xi \in [0,\pi/2-\delta] \cup [\pi/2+\delta,\pi]$ there holds the inequality*

$$|Q_{N,E,K}(\xi) - I(\xi)| \le cr^N. \tag{4.6}$$

For $\xi_1 \in (0,\pi/2)$, Theorem 4.2 says that the filter $Q_{N,E,K}$ converges to the ideal filter when $\xi \in (0,\xi_1) \cup (\pi-\xi_1,\pi)$ as N tends to infinity. However, for $\xi \in (\xi_1,\pi-\xi_1)\backslash(E \cup (\pi-E))$, $Q_{N,E,K}$ *diverges* as N tends to infinity. This is a consequence of the next theorem which also provides a lower bound estimate.

Theorem 4.4. *Let K and E be as above with $\xi_1 \in (0,\pi/2)$. If J is a compact subset of $(\pi/2,\pi)\backslash(E \cup (\pi-E) \cup \{\pi/2\})$ then there exist a positive constant c and a positive integer M such that for all $\xi \in J$ and $N \ge M$ there holds the estimate*

$$|Q_{N,E,K}(\xi)| \ge cN^{k_1-1}\left(\frac{\sin\xi}{\sin\xi_1}\right)^{2N}. \tag{4.7}$$

Theorems 4.2 and 4.4 lead to the following asymptotic property for the filter $Q_{N,E,K}$ as N tends to infinity.

Corollary 4.5. *Let E and K be as above with $\xi_1 \in (0, \pi/2)$. For any compact subsets J_1 of $(-\xi_1, \xi_1) + \pi\mathbb{Z}$ and J_2 of $([-\pi, \pi]\backslash([-\xi_1, \xi_1] \cup E \cup (-E))) + \pi\mathbb{Z}$, there exist two positive constants c and $r \in (0, 1)$, such that for $\xi \in J_1$ we have that*

$$|Q_{N,E,K}(\xi) - I(\xi)| \le cr^N$$

while for $\xi \in J_2$ it follows that

$$|Q_{N,E,K}(\xi) - I(\xi)| \ge c^{-1} r^{-N}.$$

4.1. **Proof of Theorem 4.1.** As usual we use (4.2) to introduce the polynomial P defined by the requirement for all $\xi \in \mathbb{R}$ that $P(\sin^2(\xi/2)) = Q(\xi)$ and transform equations (4.1) and (4.2) to

$$P(z) + P(1 - z) = 1, \; z \in \mathbb{C} \tag{4.8}$$

where

$$P(0) = 1. \tag{4.9}$$

From this version of the problem the result follows from the solvability of the Bézout identity (4.8). □

4.2. **Proof of Theorem 4.2.** We prepare for the proof of Theorem 4.2 with two lemmas. The first concerns estimates for Q_N in (1.1) which follows from Proposition 3.1.

Lemma 4.6.

$$|Q_N(\xi)| \le \begin{cases} 1, & \xi \in [0, \pi/2], \\ \cos^2 \dfrac{\xi}{2} \sin^{2N-2} \xi, & \xi \in [\pi/2, \pi], \end{cases} \tag{4.10}$$

and

$$|Q_N(\xi)| \ge \begin{cases} \dfrac{1}{2}, & \xi \in [0, \pi/2], \\ \dfrac{1}{N} \cos^2 \dfrac{\xi}{2} \sin^{2N-2} \xi, & \xi \in [\pi/2, \pi]. \end{cases} \tag{4.11}$$

In the next lemma we estimate the derivative of the function R_N defined for $t \in \mathbb{R}$ by the equation $R_N(t) := t^{-N} P_N(t)$.

Lemma 4.7. *If $\delta \in (0, 1/4)$ and n is a nonnegative integer then there exists a positive constant c such that for any $N \in \mathbb{N}$ and $t \in (1/2 + \delta, 1]$ there holds the estimate*

$$c^{-1} N^{-1/2} 2^{2N} \le |R_N^{(n)}(t)| \le c N^{-1/2} 2^{2N}, \tag{4.12}$$

while for $t \in (\delta, 1/2 - \delta)$ we have that

$$c^{-1} N^n (t(1-t))^{-N} \le |R_N^{(n)}(t)| \le c N^n (t(1-t))^{-N}. \tag{4.13}$$

Proof. By the definition of the polynomial P_N, we have that

$$\begin{aligned} R_N^{(n)}(t) &= \sum_{l \in \mathbb{Z}_{N-1}} \begin{pmatrix} N-1+l \\ l \end{pmatrix} \prod_{r \in \mathbb{Z}_{n-1}} (l - N - r) t^{l-N-n} \\ &=: t^{-N-n} \sum_{l \in \mathbb{Z}_{N-1}} a_l(t) \prod_{r \in \mathbb{Z}_{n-1}} (l - N - r). \end{aligned} \tag{4.14}$$

For $t \in (1/2 + \delta, 1]$, we observe that $a_l(t), l \in \mathbb{Z}_{N-1}$, is an increasing sequence. Moreover, there exists $\lambda \in (0, 1)$ such that for all $l \in \mathbb{Z}_{N-2}$ and $t \in (1/2 + \delta, 1)$

there holds the inequality $a_l(t)/a_{l+1}(t) \leq \lambda$. Therefore, by (4.14) and the Stirling formula there is a positive constant c such that for all $N \in \mathbb{N}$ we get that

$$\begin{aligned} |R_N^{(n)}(t)| &\leq t^{-N-n} a_{N-1}(t) \sum_{l\in\mathbb{Z}_{N-1}} \prod_{r\in\mathbb{Z}_{n-1}} (N+r-l)\lambda^{N-l} \\ &\leq c \begin{pmatrix} 2N-2 \\ N-1 \end{pmatrix} \leq cN^{-1/2}2^{2N}, \end{aligned} \tag{4.15}$$

and

$$|R_N^{(n)}(t)| \geq \begin{pmatrix} 2N-2 \\ N-1 \end{pmatrix} \prod_{r\in\mathbb{Z}_{n-1}} (1+r)t^{-1-n} \geq c^{-1}N^{-1/2}2^{2N}. \tag{4.16}$$

Hence, we conclude that (4.12) follows from (4.15) and (4.16).

Similarly, for $t \in (\delta, 1/2-\delta)$, we have by (4.14) that

$$|R_N^{(n)}(t)| \leq cN^n R_N(t),$$

which, together with the estimate in Lemma 4.6, lead to the inequality

$$|R_N^{(n)}(t)| \leq cN^n (t(1-t))^{-N}. \tag{4.17}$$

On the other hand, applying (4.14) again gives the estimate,

$$|R_N^{(n)}(t)| \geq cN^n t^{-N} \sum_{N\delta/(2-2\delta)\leq l\leq N/(1+2\delta)} \begin{pmatrix} N-1+l \\ l \end{pmatrix} t^l. \tag{4.18}$$

Since, for $l \in \mathbb{Z}_{N-2}$, $\frac{a_{l+1}(t)}{a_l(t)} = t(N-l)/(l+1)$ we see that $a_l(t), l \in \mathbb{Z}_{N-1}$ is an increasing sequence when $l \leq b$ and a decreasing sequence when $l \geq b+1$, where b is the integral part of $tN/(1-t)$. Therefore, there exists a $\lambda \in (0,1)$ such that for $t \in (\delta, 1/2-\delta)$, N sufficiently large and either $l \leq N\delta/(2-2\delta)$ or $l \geq N/(1+2\delta)$, we have that

$$a_l(t) \leq r^N (a_b(t) + a_{b+1}(t)).$$

This observation, together with Lemma 4.6, implies that

$$\begin{aligned} &\sum_{N\delta/(2-2\delta)\leq l\leq N/(1+2\delta)} \begin{pmatrix} N-1+l \\ l \end{pmatrix} t^l \\ &\qquad \geq c\sum_{l=0}^{N-1} \begin{pmatrix} N-1+l \\ l \end{pmatrix} t^l \geq c(1-t)^{-N}. \end{aligned} \tag{4.19}$$

Combining the estimates in (4.17), (4.18) and (4.19) proves (4.13). □

Proof of Theorem 4.2. We define a polynomial $P_{N,E,K}$ so that

$$P_{N,E,K}(\sin^2 \xi/2) = Q_{N,E,K}(\xi),\ \xi \in \mathbb{R},$$

from which it follows that $P_{N,E,K}(0) = 1$, and for $z \in \mathbb{C}$

$$P_{N,E,K}(z) + P_{N,E,K}(1-z) = 1. \tag{4.20}$$

We set

$$P_{N,E,K}(z) = (1-z)^N \prod_{j\in\mathbb{N}_s} (z-t_j)^{k_j} U_{N,E,K}(z), \tag{4.21}$$

where $U_{N,E,K}$ is a polynomial of degree $|K|-1$ and $t_j := \sin^2 \xi_j/2, j \in \mathbb{N}_s$. By (4.20) and (4.21), we get that

$$\prod_{j\in\mathbb{N}_s}(z-t_j)^{k_j} U_{N,E,K}(z) = P_N(z) + z^N(1-2z)W(z) \tag{4.22}$$

for some polynomial W of degree $2|K|-2$ which satisfies the symmetry relation $W = W(1-\cdot)$. For each $l \in \mathbb{Z}_{|K|-1}$, we introduce the polynomial p_l defined for all $z \in \mathbb{C}$ as $p_l(z) = (z-1/2)^{2l}$, write

$$W = \sum_{l\in\mathbb{Z}_{|K|-1}} c_l p_l$$

for some constants $c_l, l \in \mathbb{Z}_{|K|-1}$, introduce the function h_N defined for $z \in \mathbb{C}$ by $h_N(z) = (1-2z)^{-1} z^{-N} P_N(z)$, and for $i \in \mathbb{N}_s, j \in \mathbb{Z}_{k_i-1}$ and $l \in \mathbb{Z}_{|K|-1}$ denote the values of the j-th derivative of functions p_l and h_N at t_i by b_{ij} and $a_{ij,l}$ respectively. Therefore, the coefficients $c_l, l \in \mathbb{Z}_{|K|-1}$, are the solution of the following linear system

$$\sum_{l\in\mathbb{Z}_{|K|-1}} a_{ij,l} c_l = b_{ij},\ i \in \mathbb{N}_s, j \in \mathbb{Z}_{k_i-1}. \tag{4.23}$$

Note that the matrix with entries $a_{ij,l}$ is a nonsingular matrix, as it is similar in form to Vandermonde matrix. Therefore, the linear system in (4.23) has a unique solution.

By Lemma 4.7, we conclude that there exists a positive constant c such that for all $i \in \mathbb{N}_s$, $j \in \mathbb{Z}_{k_i-1}$ and $t_i \in (0,1/2)$ the quantity b_{ij} is dominated by $cN^j(t_i(1-t_i))^{-N}$ and by $cN^{-1/2}2^{2N}$ when $t_i \in (1/2,1)$. Hence, there is a positive constant c such that the quantities $b_{ij}, i \in \mathbb{N}_s, j \in \mathbb{Z}_{k_i-1}$, are dominated by $cN^{k_1-1}(t_1(1-t_1))^{-N}$ when $\xi_1 \in (0,\pi/2)$ and by $cN^{-1/2}2^{2N}$ when $\xi_1 \in (\pi/2,\pi)$. These facts, together with (4.23), yield for all $l \in \mathbb{Z}_{|K|-1}$ the inequality

$$|c_l| \le \begin{cases} cN^{k_1-1}(t_1(1-t_1))^{-N}, & \xi_1 \in (0,\pi/2), \\ cN^{-1/2}2^{2N}, & \xi_1 \in (\pi/2,\pi). \end{cases} \tag{4.24}$$

Therefore, from the inequality,

$$|Q_{N,E,K}(\xi)| \le (1-t)^N \big(P_N(t) + ct^N \max\{|c_l| : l \in \mathbb{Z}_{|K|-1}\}\big). \tag{4.25}$$

where $t = \sin^2 \xi/2$, the estimates (4.4) and (4.5) follow from (4.24) and (4.25) and the estimates in Lemma 4.6. □

Embodied in the proof above is the following fact.

Proposition 4.8. *The family of functions $\{p_l : l \in \mathbb{Z}_n\}$ form a Chebyshev system on any interval which does not contain $1/2$ in it interior.*

Proof. The proof is by induction on n and the induction is advanced by differentiating and using Rolle's theorem. □

4.3. **Proof of Theorem 4.4.** Let $b_{ij}, i \in \mathbb{N}_s, j \in \mathbb{Z}_{k_i-1}$, be defined as in the proof of Theorem 4.2. By Lemma 4.7, there exists a positive constant c so that

$$|b_{ij}| \le cN^{k_1-2}(t_1(1-t_1))^{-N} \tag{4.26}$$

for either $i \ge 2$ or $i = 1$ and $j \le k_1 - 2$, and

$$|b_{1(k_1-1)}| \ge c^{-1}N^{k_1-1}(t_1(1-t_1))^{-N}. \tag{4.27}$$

Combining (4.23) and (4.26) yields

$$\Big|\sum_{l\in\mathbb{Z}_{|K|-1}} c_l(t-1/2)^{2l} - b_{1(k_1-1)}\frac{\det A(t)}{\det A}\Big| \le cN^{k_1-2}(t_1(1-t_1))^{-N},$$

where A is the matrix with entries $a_{ij,l}$ and $A(t)$ is the matrix A with $a_{1(k_1-1),l}$ replaced by $(t-1/2)^l$. Note that both A and $A(t)$ are matrices of Vandermonde type, which implies that $\det A(t)\neq 0$ for all $\xi\in J$, where $t=\sin^2\xi/2$. This, together with (4.27), show that

$$|R(t)|\ge cN^{k_1-1}(t_1(1-t_1))^{-N},\ \xi\in J \tag{4.28}$$

for sufficiently large N, where $t=\sin^2\xi/2$. So for sufficiently large N and $\xi\in(\pi/2,\pi-\xi_1)\cap K$ we conclude that

$$\begin{aligned}|Q_{N,E,K}(\xi)| &\ge c((1-t)t)^N|1-2t|N^{k_1-1}(t_1(1-t_1))^{-N}-1\\ &\ge cN^{k_1-1}(\sin\xi/\sin\xi_1)^{2N},\end{aligned} \tag{4.29}$$

and for sufficiently large N and $\xi\in(\pi-\xi_1,\pi)\cap K$,

$$\begin{aligned}|Q_{N,E,K}(\xi)| &\ge c((1-t)t)^N|1-2t|N^{k_1-1}(t_1(1-t_1))^{-N}-(4t(1-t))^N\\ &\ge cN^{k_1-1}(\sin\xi/\sin\xi_1)^{2N}.\end{aligned} \tag{4.30}$$

Therefore, (4.7) follows from (4.29) and (4.30). □

5. **Regularity of refinable functions.** Let $\Phi_{N,E,K}$ be the refinable function with the filter $Q_{N,E,K}$, that is, $\widehat{\Phi}_{N,E,K}(0)=1$ and

$$\widehat{\Phi}_{N,E,K}(\xi)=Q_{N,E,K}(\xi/2)\widehat{\Phi}_{N,E,K}(\xi/2),\ \xi\in\mathbb{R}. \tag{5.1}$$

In this section, we study the Fourier exponent of the function $\Phi_{N,E,K}$.

Theorem 5.1. *If $\Phi_{N,E,K}$ is the refinable function with filter $Q_{N,E,K}$, then for any $0<p\le\infty$, there is a positive constant c such that for all N we have that*

$$s_p(\Phi_{N,E,K})\ge\begin{cases}\dfrac{\ln 4-\ln(3\sin^2\xi_1)}{\ln 2}N-\dfrac{(k_1-1)\ln N}{\ln 2}-c, & \text{if } \xi_1\in(0,\pi/2),\\ \big(2-\dfrac{\ln 3}{\ln 2}\big)N-c, & \text{if } \xi_1\in(\pi/2,\pi).\end{cases} \tag{5.2}$$

Conversely, if $\xi_1\in(0,\pi/2)$ and $\pi/3,2\pi/3\notin E$ then there is a positive constant d such that for all N we have the following upper bound estimate of the Fourier exponent of $\Phi_{N,E,K}$,

$$s_p(\Phi_{N,E,K})\le\Big(2-\frac{\ln 3}{\ln 2}+\frac{\ln\sin^2\xi_1}{\ln 2}\Big)N-\frac{(k_1-1)\ln N}{\ln 2}+d. \tag{5.3}$$

Proof. By Theorem 4.2, we have for $\xi\in\mathbb{R}$ that

$$Q_{N,E,K}(\xi)=\cos^{2N}\frac{\xi}{2}V(\xi) \tag{5.4}$$

for some trigonometric polynomial V which has the property that

$$|V(\xi)|\le\begin{cases}ca(\xi)^N, & \text{if } \xi_1\in(\pi/2,\pi),\\ cN^{k_1-1}(\sin\xi_1)^{-2N}a(\xi)^N, & \text{if } \xi_1\in(0,\pi/2),\end{cases} \tag{5.5}$$

where a is the function defined as in Lemma 3.5 and c is a positive constant independent of N. Applying (5.4) and (5.5) and using Lemmas 3.8 and 3.5 yield the lower bound estimate of $s_p(\Phi_{N,E,K})$ in (5.2).

By Theorem 4.4 and our assumption on E, it follows that, for sufficiently large N,

$$|V(2\pi/3)| \geq cN^{k_1-1}(\sin\xi_1)^{-2N}3^N, \tag{5.6}$$

and

$$|Q_{N,E,K}(\pi/3)| \neq 0 \quad \text{and} \quad |Q_{N,E,K}(2\pi/3)| \neq 0.$$

Since for all $\xi \in \mathbb{R}$ we have that $|Q_{N,E,K}(\xi)| + |Q_{N,E,K}(\xi+\pi)| \neq 0$ the techniques used in [5, 15] leads to conclusion that for some integer k

$$\widehat{\Phi}_{N,E,K}(2\pi/3 + 2k\pi) \neq 0. \tag{5.7}$$

Therefore, the upper bound estimate (5.3) follows from (3.8), (5.7) and Lemma 3.9. □

As an easy consequence of Theorem 5.1, we have the following corollary.

Corollary 5.2. *If $\xi_1 \in (\pi/3, \pi)$ and N is sufficiently large then $\Phi_{N,E,K}$ is continuous while if $\xi \in (0, \pi/3)$ it is discontinuous for N sufficiently large.*

In summary, the interpolating filter $Q_{N,E,K}$ of minimal degree with prescribed zeros in $E \cup \{\pi\}$ and the associated refinable functions $\Phi_{N,E,K}$ do not behave well, as ξ_1 moves from the origin to $\pi/3$, in terms of approximation to the ideal filter and the regularity. When the zero ξ_1 lies in the interval $(\pi/3, \pi/2)$, the filter $Q_{N,E,K}$ still does not improve its performance but the refinable function $\Phi_{N,E,K}$ is quite acceptable. However, when ξ_1 lies in the interval $(\pi/2, \pi)$, both the filter and the associated refinable function have satisfactory behavior. The above observation is confirmed by the Figure 1 of the interpolating filters $Q_{N,E,K}$ and the associated refinable functions $\Phi_{N,E,K}$.

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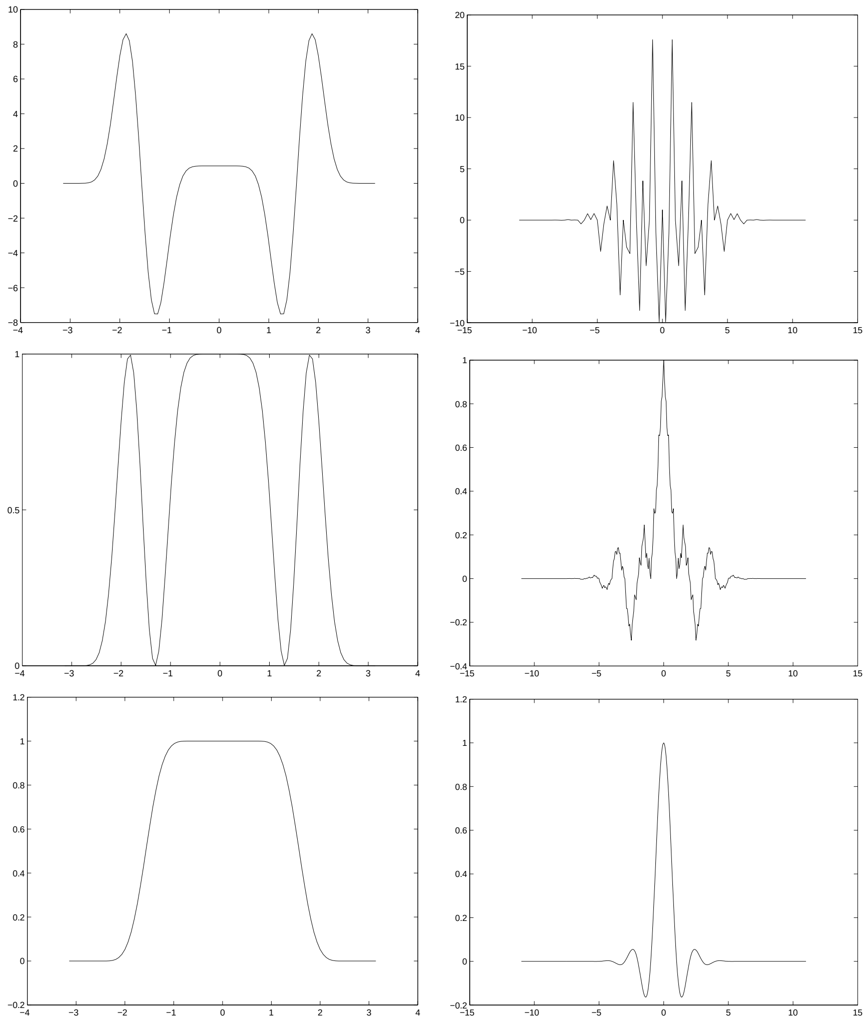

FIGURE 1. The left column is for "interpolating filter" image, whereas the right column is for "refinable function" image. The parameters N, E, K are $5, \{\pi/4\}, \{1\}$; $5, \{5\pi/12\}, \{1\}$; and $5, \{3\pi/4\}, \{1\}$ respectively (from top to bottom), while the right-hand side are the corresponding refinable functions.

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